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On a Transversality over Local Global Rings

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ABSTRACT. The purpose of this paper prove the following property; Suppose A has many units (local global ring) and $|A/m| > 5$ for every maximal ideal $m \subsetneq A$. Let$(E,q) \in Q(A)$ and $E = E_1 \perp \cdots \perp E_t$ be an orthogonl decomposition of E with $t \geq 2$ and $rk(E_i) \geq 1$, for $i = 1, \cdots, t$. Let $x \in E$ be a primitive vector. Then there exists $\sigma \in O(q)$ such that $\sigma(x)$ is transversal to this decomposition.

1. Introduction

We show all of these abstract theories can be applied to some quadratic forms over a ring with many units with $2 \in A^*$, where A^* : the multiplicative group of the ring A. That is, if every polynomial over A with local unit values has unit values, we call A a ring with many units or a local global ring . This work was studied by Dc-Donald and kirkwood in [7], and many properties of the new ring have been shown to parallel in many respects that of semi local ring.[1].[8].

We need some words and notations. Denote by $\mathfrak{F}(A)$ the class of all finitely generate free A-modules, a quadratic space over A is a pair (E,q), where $E \in \mathfrak{F}(A)$, and $Q(A)$ the class of all guadratic spaces. Let $(E,q) \in Q(A), x \in E$, we say x is isotropic if $q(x,x) = 0$ and x is primitive if there exists $f \in \hat{E}$, where $\hat{E} = \mathrm{Hom}_A(E,A)$ such that $f(x) = 1$ forthermore an element $x \in E$ is said to be anisotropic if $q(x,x) \in A^*$. Note that any element in an orthogonal basis is necessarily anisotropic, clearly any istropic vetor in E is primitive.

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It is well known that $e \in E$ is anistropic, then $E = Ae \oplus (Ae)^2$, where $(Ae)^2 = \{x \in E | q(x,e) = 0\}$. Let $E = E_1 \perp \cdots \perp E_t, t \geq 2$, be an orthogonal decomposition of (E,q), we say x is transversal to this decomposiition if $x = x_1 + \cdots + x_t, x_i \in E_i$ is isotropic for $i = 1, \cdots, t$ If $e_1, \cdots, e_n$ is an orthogonal basis for E and x is transversal to the decomposition $Ae_1 \perp \cdots \perp Ae_n$, then we say x is transversal to the basis $e_1, \cdots, e_n$.

The purpose of this paper is to prove the proposition 6, explicitly.

2. Some preliminary lemmas and result

Let $\gamma_z : E \to E$ by the A-isomorphism given by $\gamma_z(z) = -z$, $\gamma_z(y) = y$,for all $y \in (Az)^\perp$. So for any $x \in E$, $\gamma_z(x) = x - \frac{2q(x,x)}{q(z,z)} z$. γ_z is called the reflection determined by z and is an isometry of $[E,q]$.

Next Lemmas are trivial.

LEMMA 1. *i) Let F be a field and $(E,q) \in Q(F)$. Let $x, y \in E$ be anisotropic vectors such that $q(x,x) = q(y,y)$. Then there exists a reflection $\gamma_z : E \to E$ such that $\gamma_z(x) = y$.*

ii) Let $x \in E$ be anisotropic and $w \in (F_x)^\perp$ such that $q(w,w) \neq -q(x,x)$. Let $b \in F, b \neq 0$. Then there exists $a, c \in F, c \neq 0$ such that $q(ax + cw, ax + cw) = b^2\ q(x,x)$[10].

LEMMA 2. *Suppose $(E,q) \in Q(A)$ with dimension $n \geq 3$. Let $x, e \in E$ be anisotropic. Then exists a reflection $\gamma_z : E \to E$ such that $\gamma_z(x) = ae + u, a \in A$ and $u \in (A_e)^\perp$ is anisotropic[10].*

LEMMA 3. *Suppose A is a field with $|A| > 5$. Then for each $a \in A^*$, there exists $b, c \in A^*$ such that $a = b^2 - c^2$.*

PROOF. It is simple. Since $|A| > 5, |A^{*^2}| \geq 3$ and hence, there exist $d \in A^*$ such that $d^2 \neq \pm 1$. Since $d^2 = (\frac{d^2+1}{2})^2 - (\frac{d^2-1}{2})^2$, thus $1 = b^2 - c^2$ for some $b, c \in A^*$ and $-1 = c^2 - b^2$. Suppose $a \in A^*$, $a \neq \pm 1$. Then $a = (\frac{a+1}{2})^2 - (\frac{a-1}{2})^2$ so $a = b^2 - c^2$ also $b, c \in A^*$ [8].

LEMMA 4. *Suppose A is a field with $|A| > 5$. Let $(E, q) \in Q(A)$ with $rk(E) \geq 2$and $x, e \in E$ be anisotropic vectors. Then there exists a reflection $\gamma_z : E \to E$ such that $\gamma_z(x) = ae + u$, where $a \in A^*$and $u \in (Ae)^{\perp}$ is anisotropic.*

PROOF. Put $rk(E) = 2$ and $e = e_1, e_2$ be an orthogonal basis of E and let $a_i = q(e_i, e_i)$ for $i = 1, 2$. Suppose there exists $y_1, y_2 \in A^*$ such that $q(x, x) = x_1^2 a_1 + x_2^2 a_2 = y_1^2 a_1 + y_2^2 a^2 = q(y_1 e_1 + y_2 e_2, y_1 e_1 + y_2 e_2)$. Then , by Lemma 1, there exist a reflection $\gamma_z : E \to E$ such that $\gamma_z(x) = \pm(y_1 e_1 + y_2 e_2)$, so taking $a = \pm y_1, u = \pm y_2 e_2$,we are done.

Thus it is sufficient to show that there exists $y_1, y_2 \in A^*$ such that $x_1^2 a_1 + x_2^2 a_2 = y_1^2 a_1 + y_2^2 a_2$. If $x_i \neq 0$, for $i = 1, 2$ take $y_1 = x_2$, $y_2 = x_2$ and we are done .Otherwise one of the x_i's is zero, we may assume $x_2 = 0$, then we show $y_1, y_2 \in A^*$such that $x_1^2 a_1 = y_1^2 a_1 + y_2^2 a_2$, that is, $(\frac{x_1}{y_2})^2 - (\frac{y_1}{y_2})^2 = \frac{a_2}{a_1}$. By above Lemma, there exist $b, c \in A^*$ such that $\frac{a_2}{a_1} = b^2 - c^2$, so define y_1, y_2 by $\frac{x_1}{y_2} = b$, $\frac{y_1}{y_2} = c$.

Now assume $rk(E) \geq 3$. By Lemma 2, there exists a reflection $\gamma : E \to E$ such that $\gamma(x) = be + v$, $b \in A$ and $v \in (Ae)^{\perp}$ is anisotropic. Let $F = A_e \oplus Av, y = \gamma(x) \in F$. Since y is anisotropic, and $rk(F) = 2$, there exists a reflection $\tau : F \to F$ such that $\tau(y) = ce + w, c \in A^*$ and $w \in Av$ is anisotropic. Extend τ to a reflection of E so $\tau \circ \gamma(x) = ce \pm w$. By Lemma 1, there exists a reflection $\gamma_z : E \to E$ such that $\gamma_z(x) = \pm(ce + w)$ so take $a = \pm c \in A^*$ and $u = \pm w \in Av \subseteq (Ae)^{\perp}$ [12].

LEMMA 5. *Suppose A is a field and $(E, q) \in Q(A)$ with $rk(E) \geq 2$. Let $x \in E$ be a primitive isotropic vector and $e \in E$ be any anisotropic vector. Then there exists a reflection $\gamma_z(x) : E \to E$ such that $\gamma_z(x) = ae + u, a \in A^*, u \in (Ae)^{\perp}$ is anisotropic.*

PROOF. Put $x = be + t, b \in A, t \in (Ae)^{\perp}$.

If $b \neq 0$ then $0 = q(x,x) = b^2 q(e,e) + q(t,t)$ so $q(t,t) \neq 0$, thus, we can take $z = e, a = -b$, and $u = t$ and we are done. So assume $b = 0$, that is ,$x \in (Ae)^{\perp}$, there exist $y \in (Ae)^{\perp}$ such that $e + y$ is isotropic and $e + x + y$ is anisotropic.Then, if $z = e + x + y, \gamma_z(x) = -(e+y)$ so taking $a = -1$ and $u = -y$, we are done. Indeed $z' = x - (e+y) \in (A_z)^{\perp}$. So $x = \frac{1}{2}(z+z'), e+y = \frac{1}{2}(z-z')$ and $\gamma_z(x) = -(e+y)$. Since $q(e,e) \neq 0$, and $0 = q(e+y, e+y) = q(e,e) + q(y+y), q(y,y) \neq 0$ so y is anisotropic.

We now show that such a y exist. That is, we want to find $y \in (Ae)^{\perp}$ such that $0 = q(e+y,e+y) = q(e,e) + q(y,y)$ and $q(e+x+y, e+x+y) = q(e,e) + q(x,x) + q(y,y) + 2q(x,y) = 2q(x,y) \neq 0$. Since $x \in (Ae)^{\perp}$ is primitive and $q|_{(Ae)^{\perp}}$ is non-degenerate, there exist $u \in (Ae)^{\perp}$ such that $q(u,x) = 1$ .We try for $y \in (Ae)^{\perp}$ of the form $y = cx + u, c \in A$. Then, $q(x,y) = cq(x,x) + q(x,u) = 1 \neq 0$ and $q(e,e)+q(y,y) = q(e,e)+c^2q(x,x)+2cq(u,x)+q(u,u) = q(e,e)+q(u,u)+2c$. Since we want this to be 0, take $c = -\frac{1}{2}(q(e,e)+q(u,u))$. We are well done [9], [12].

Next prosition is well known in case where every residue field of A is infinite [7] and similar result holds for semi local ring [1].

The purpose of this paper is to prove the transversality theorem over a local global ring explicity[11].

PROPOSITION 6. *Suppose A has many units (local global ring) and $|A/m| > 5$ for every maximal ideal $m \subseteq A$. Let $(E,q) \in Q(A)$ and $E = E_1 \perp \cdots \perp E_t$ be an orthogonal decomposition of E with $t \geq 2$ and $rk(E_i) \geq 1$, for $i = 1, \cdots, t$. Let $x \in E$ be a primitive vector.*

Then there exists $\sigma \in O(q)$ such that $\sigma(x)$ is transveral to this decomposition.

PROOF. Clearly it is sufficient to consider the case $t = 2$. Let $E = E_1 + E_2$ be an orthogonal decomposition of E. We go on by induction on $k = \gamma k(E_1)$.

Suppose $k = 1, E_1 = Ae_1$, for some anisotropic $e_1 \in E$. Complete e_1 to an orthogonal basis $e_1, \cdots, e_n$ and, let $\alpha_i = q(e_i, e_i)$ for $i = 1, \cdots, n$. We can write $x = \sum x_i e_i$ and suppose $z = \sum z_i e_i \in E$ is anisotropic. Then

$$\gamma_z(x) = \frac{1}{q(z,z)} \sum_{i=1}^{n} u_i(z_1, \cdots, z_n) e_i$$

where $u_i(z_1 \cdots, z_n) = q(z,z)e_i - 2q(x,z)z_i$, for $i = 1, \cdots, n$. Let

$$a = \frac{u_1}{q(z,z)} \in A, \quad u = \frac{1}{q(z,z,)} \sum_{i=2}^{n} u_i e_i \in (Ae_1)^{\perp} = E_2.$$

So we want $a \in A^*$ and $q(u,u) = \frac{1}{q(z,z)^2} \sum_{i=2}^{n} u_i^2 \alpha_i \in A^*$. That is, we want the polynomial

$$f(z_1, \cdots, z_n) = q(z,z)\ u_1\ (z_1, \cdots, z_n) \sum_{i=2}^{n} u_i(z_1, \cdots, z_n)^2 \alpha_i$$

to have local units values. Let $m \subsetneq A$ be a maximal ideal . Then $\bar{x} \in E/mE$ is primitive , and $\bar{e} \in E/mE$ isanisotropic. If $\bar{x}$ is isotropic, then ,by Lemma 5, there exists $z_1, \cdots, z_n \in A$ with $f(z_1, \cdots, z_n) \notin m$. If $\bar{x}$ is anisotropic,then, since $|A/m| > 5$, there exists $z_1, \cdots, z_n \in A$ with $f(z_1, \cdots, z_n) \notin m$ by Lemma 4. Hence,f has local unit values and therefore, there exists $z = \Sigma z_i e_i \in E$ such that γ_z is the required isometry.

Now suppose $k \geq 2$. Let $e \in E_1$ be anisotropic and $E_1 = Ae \perp F$ be an orthogonal decomposition of E_1. By the above, there exists a reflection $\gamma : E \to E$ such that $\gamma(x) = ae + u$,where $a \in A^*$ and $u \in F \oplus E_2$ is anisotropic. Since $rk(F) = k - 1$, by induction, there exists an isometry $\tau : F \oplus E_2 \to F \oplus E_2$ such that $\tau(u) = u_1 + u_2$, where $u_1 \in F \subseteq E_1$ and $u_2 \in E_2$ are both anisotrpic. Extend τ to an isometry of E so that $\tau(e) = e$. Let $v_1 = ae + u_1 \in E$, and $v_2 = u_2 \in E_2$. Then to $\tau \circ \gamma(x) = \tau(ae+u) = ae+u_1+u_2 = v_1+v_2 \in Av_2 \oplus E_1$. Since $\tau \circ \gamma(x)$ is primtive and $rk(Av_2) = 1$, there exists a reflection $\gamma' : Av_2 \oplus E_1 \to Av_2 \oplus E_1$ such that $\gamma'(v_1 + v_2) = bv_2 + w$, where $b \in A^*$ and $w \in E_1$ is anisotropic. Extend γ' to a reflection of E. Then $\gamma' \circ \tau \circ \gamma(x) = w + v$,where $w \in E_1$ and $v = bv_2 \in E_2$ are both anisotropic. We give the following corollory.

COROLLORY 7. *Assume $|A/m| > 5$ for all maximal ideals $m \subseteq A$. for $q \in Q(A)$, the following are equivalent :*

1) q is isotrpic.

2) E contains a primitive isotropic vector.

PROOF. 1) $\Longrightarrow$ 2). This is clear.

2) $\Longrightarrow$ 1). Suppose $x \in E$ is primitive and isotropic. Note that $rk(E) \geq 2$. Let $e \in E$ be anisotropic. Then, by the above propostion, there exists $\sigma \in O_{(q)}$ such that $\sigma(x) = u_1 + u_2, u_1 \in Ae, u_2 \in (Ae)^\perp$ are both anisotropic.

$$\begin{aligned}\text{Then, } 0 = q(x,x) &= q(\sigma_{(x)}, \sigma_{(x)}) \\ &= q(u_1 + u_2, u_1 + u_2) \\ &= q(u_1, u_2) + q(u_1, u_2).\end{aligned}$$

Thus ,if $a = q(u_1, u_1), < a, -a > \cong < 1, -1 >$ is a orthogonal summand of q, hence, q is isotropic.

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