JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 7, June 1994

A Note on Derivations of Banach Algebras

GWANG-HUI KIM

ABSTRACT. Let A be a (complex) Banach algebra. The object of the this paper shall be remove the continuity of the derivation in the recently theorems. We prove that every derivation D on A satisfying $[D(a), a] \in \text{Prad}(A)$ for all $a \in A$ maps into the radical of A. Also if $\alpha D^3 + D^2$ is a derivation for some $\alpha \in C$ and all minimal prime ideals are closed, then D maps into its radical.

1. Introduction

In 1955 Singer and Wermer [12] proved that every bounded derivation on a commutative Banach algebra maps into its radical. They conjectured that the continuity of the derivation in their theorem can be removed. In 1988 Thomas [13] proved their conjecture ; Every derivation on a commutative Banach algebra maps into its radical. For noncommutative versions, in 1984 B.Yood [15] proved that the continuous derivations on Banach algebras satisfing $[D(a), b] \in \text{Rad}(A)$ for all $a, b \in A$ have the radical range,where $[a, b]$ will be denote the commutator $ab - ba$. In 1990 M.Bresar and J.Vukman [1] have generalized Yood's result, that is, the continuous linear Jordan derivation on Banach algebra that satisfies $[D(a), a] \in \text{Rad}(A)$ for all $a \in A$ has the radical range. In next year Mathieu and Murphy [5] proved that every bounded centralizing derivation on Banach algebras has its image in the radical. In year after next Mathieu and Runde [6] removed the boundedness of that.

Received by the editors on April 30, 1994.
1980 *Mathematics subject classifications*: Primary 46H05.

A *derivation* on an algebra is a linear mapping $D : A \to A$ that satisfies $D(ab) = aD(b) + D(a)b$ for all $a, b \in A$. A mapping F of a ring R is said to be *commuting* if $[F(x), x] = 0$ for all $x \in R$, and is said to be *centralizing* on R if $[F(x), x] \in Z(R)$ holds for all $x \in R$, where $Z(R)$ is the center of a ring R. Throughout this paper, radical (prime radical) of A will be denoted by Rad(A) (Prad(A)). Note that Rad (A) (Prad(A)) is the intersection of all primitive ideals (prime ideals) of A. A is said to be *semisimple* or *semiprime* if Rad(A) $= 0$ or Prad(A) $= 0$ respectively.

Let X, Y be Banach spaces, and $T : X \to Y$ a linear mapping.Then

$$S(T) = \{y| \text{ there is a sequence } \{x_n\} \textit{ in } X \text{ with } x_n \to 0, \text{and } Tx_n \to y\}$$

is said to be the *separating space* of T. It is a closed linear subspace of Y. The separating space of a derivation on a Banach algebra is a *separating ideal*, and $S(T) = 0$ iff T is continuous. The detail proofs and definitions will be seen in A.M. Sinclair [11] and J. Cusack [4].

The following theorem has been proved in [4], and for commutativity we can find in [3]

THEOREM A. *Let A be a noncommutative Banach algebra, $S \subset A$ a seperating ideal, and $P \subset A$ a minimal prime ideal which does not contain S. Then P is closed.*

The following theorem has been proved in [6], and for a continuous derivation we can find in Theorem 2.2[10].

THEOREM B. *Let D be a derivation on a ring R. Then D fixes each minimal prime ideal P of R such that R/P is torsion free.*

2. The Results

The proofs of the results rest heavily on the Theorem B of Mathieu and Runde[6], and they conjectured that the assumption of continuity may be removable.

REMARK. Theorem B implies that the minimal prime ideal P in a Banach algebra are invariant under derivations,and so we can 'drop'the derivation $D : A \to A$ to the derivation $D_P : A/P \to A/P$, defined by

$$(1) \qquad D_P(a + P) = Da + P \quad (a + P \in A/P).$$

Then the algebra $A/P = A_P$ is a prime algebra, and if P is closed, then A/P is a prime Banach algebra. The method of proofs in results will follow the this proceeding.

As an immediate consequence of the Remark, we obtain that $D(\mathrm{Prad}(A)) \subseteq \mathrm{Prad}(A)$ for every derivation D on A, which means that there is no loss of generality in assuming that A is semiprime.

THEOREM 1. *Let D be a derivation on a Banach algebra A. If $[D(a), a] \in Prad(A)$ for all $a \in A$, then D maps A into $Rad(A)$.*

PROOF. For each minimal prime ideal P we perform as the Remark. Suppose first that P is closed. Then A/P is a prime Banach algebra. So we observe that $[D_P(x), x] = 0$ for all $x \in A/P$. Hence D_P is commuting on A/P, so it is centralizing. By Mathieu and Runde Theorem[6] $D_P(A/P) \subseteq \mathrm{Rad}(A/P) \subseteq Q/P$, where Q is a primitive ideal of A. Hence $D(A) \subseteq Q$, and so $D(A) \subseteq \mathrm{Rad}(A)$. If P is not closed, then $S(D) \subseteq P$ by Theorem A. We define the cannonical epimorphism π from A onto $A_{\overline{P}}$ by Sinclair[11].Then we have

$$S(\pi \circ D) = \overline{\pi(S(D))} = \{0\}$$

and so $\pi \circ D$ is continuous. As a result, $(\pi \circ D)\overline{P} = \{0\}$, that is, $D(\overline{P}) \subseteq \overline{P}$. Performing as the Remark for each $\overline{P}$, we observe that $[D_{\overline{P}}(x), x] = 0$ for all $x \in A/\overline{P}$, where $D_{\overline{P}}$ is a centralizing derivation on $A/\overline{P}$ defined by (1). Then Mathieu and Runde Theorem [6] implies that $D_{\overline{P}}(A_{\overline{P}}) \subseteq \mathrm{Rad}(A_{\overline{P}}) \subseteq Q_{\overline{P}}$ for all primitive ideals Q of A. Therefore $D(A) \subseteq Q$,and so $D(A) \subseteq \mathrm{Rad}(A)$. The proof of the theorem is complete.

COROLLARY 2. *Let D be a derivation of a semisimple Banach algebra A, and $[D(a), a] \in$ Prad(A) for all $a \in A$, then D is continuous.*

THEOREM 3. *Let D be a derivation on a Banach algebra A. If $[D(a), a]^2 \in$ Prad(A) for all $a \in A$, and all minimal prime ideals are closed, then D maps A into Rad(A).*

PROOF. Let Q be a primitive ideal of A. Using Zorn's lemma, we find a minimal prime ideal $P \subseteq Q$,which is D-invariant as the Remark. We observe that $[D_P(x), x]^2 = 0$ for all $x \in A/P$, where A/P is a prime Banach algebra. If A/P is commutative, then Thomas Theorem implies that $D_P(A/P) \subseteq \mathrm{Rad}(A/P) \subseteq Q/P$ for all primitive ideals Q of A, and so $D(A) \subseteq \mathrm{Rad}(A)$. We consider the case that A/P is noncommutative. We see that there is no loss of generality in assuming that A is prime, noncommutative, and $[D(x), x]^2 = 0$ for all $x \in A$. Therefore we get $D = 0$ throughout the same proceeding of Theorem 3[2]. Namely, $D(A) \subseteq P \subseteq Q$ for every primitive ideal Q. The proof of the theorem is complete.

THEOREM 4. *Let D and G be a derivation on a Banach algebra A such that $[D^2(a) + G(a), a] \in$ Prad(A) for all $a \in A$, and all minimal prime ideals are closed. Then both D and G map A into Rad(A).*

PROOF. For each D and G, as in the proof of Theorem 3 we observe that $[{D_P}^2(x) + G_P(x), x] = 0$ for all $x \in A/P$. If A/P is a commutative, then $D_P(A/P) \subseteq \text{Rad}(A/P) \subseteq Q/P$ and $G_P(A/P) \subseteq \text{Rad}(A/P) \subseteq Q/P$ for every primitive ideal Q of A by Thomas Theorem, and so $D(A) \subseteq \text{Rad}(A)$ and $G(A) \subseteq \text{Rad}(A)$. If A/P is noncommutative,then we see that there is no loss of generality in assuming that A is prime, noncommutative and that $[D^2(x) + G(x), x] = 0$ for all $x \in A$. Therefore we get $D = G = 0$ throughout the same proceeding of Theorem 1[2]. Namely, $D(A) \subseteq P \subseteq Q$ and $G(A) \subseteq P \subseteq Q$ for every primitive ideal Q of A. So the proof of the theorem is complete.

COROLLRAY 5. *Let D and G be derivations of a semisimple Banach algebra A such that $[D^2(a) + G(a), a] \in$ Prad(A) for all $a \in A$,and all minimal prime ideals are closed. Then both D and G equal zero.*

THEOREM 6. *Let D be a derivation on a Banach algebra A such that $D(a)^2 \in$ Prad(A) for all $a \in A$. Then D maps A into $\cap\overline{P}$, where P runs over all minimal prime ideals of A.*

PROOF. For each minimal prime ideal P, we perform as the Remark. In the first case assume that P is closed. Observe that $D_P(x)^2 = 0$ for all $x \in A/P$. Then we see that there is no loss of generality in assuming that A is prime and $D(x)^2 = 0$ for all $x \in A$. We must show that $D = 0$. For all $x \in A$ we have

$$\begin{aligned} D^2x^2 = D(xD(x) + D(x)x) &= xD^2(x) + 2D(x)^2 + D^2(x)x \\ &= xD^2(x) + D^2(x)x. \end{aligned}$$

Hence D^2 is a Jordan derivation,and therefore by a result of Herstein, it is actually a derivation.Thus

$$D^2(xy) = xD^2(y) + D^2(x)y.$$

However we also have

$$D^2(xy) = xD^2(y) + 2D(x)D(y) + D^2(x)y.$$

Hence $D(x)D(y) = 0$ for all $x, y \in A$. Replacing x by xz we have

$$0 = D(xz)D(y) = xD(z)D(y) + D(x)zD(y),$$

from which $D(x)AD(y) = 0$ follows. By the primeness of A, we conclude that $D(x) = 0$ for all $x \in A$. So $D(A) \subseteq P$. In second case assume that P is not closed. By the Theorem B, $S(D) \subseteq P$. Such as the second half of Theorem 1,we observe that $D_{\overline{P}}(x)^2 = 0$ for all $x \in A/\overline{P}$ where $D_{\overline{P}}$ is an induced derivation on $A/\overline{P}$ defined by (1), and so $D_{\overline{P}} = 0$. In any case $D(A) \subseteq \overline{P}$, so $D(A) \subseteq \cap\overline{P}$, where P run over all minimal prime ideal of A. The proof of the theorem is complete.

THEOREM 7. *Suppose there exists a derivation D on a Banach algebra A such that $\alpha D^3 + D^2$ is a derivation for some $\alpha \in C$, and all minimal prime ideals are closed. In this case, D maps A into Rad(A).*

PROOF. Let Q be a primitive ideal of A. Using Zorn's lemma, we find a minimal prime ideal $P \subseteq Q$, which is D-invariant as the Remark. Then A/P is a prime Banach algebra. If A/P is commutative, then Thomas Theorem implies that $D_P(A/P) \subseteq \mathrm{Rad}(A/P) \subseteq Q/P$, and so $D(A) \subseteq \mathrm{Rad}(A)$. We consider the case that A/P is noncommutative. The assumption of the theorem that $\alpha D^3 + D^2$ is a derivation gives that $\alpha {D_P}^3 + {D_P}^2$ is a derivation. Let us first assume that $\alpha = 0$. In this case we have ${D_P}^2$ is a derivation, and since A/P is prime , similarly in the proof of Theorem 6, $D_P = 0$. In case $\alpha \neq 0$, all the assumpitions of Theorem 2[14] are fulfilled (note that D_P stands for D_1 and D_P/α for D_2). Thus we have $D_P = 0$ or $D_P/\alpha = 0$. In any case $D_P = 0$. In other words, $D(A) \subseteq \mathrm{Rad}(A)$. So the proof of the theorem is complete.

COROLLARY 8. *Suppose there exists a derivation D on a semisimple Banach algebra A such that $\alpha D^3 + D^2$ is a derivation for some $\alpha \in C$, and all minimal prime ideals are is closed. In this case, $D = 0$.*

REFERENCES

1. M.Bresar and J.Vukman, *On left derivations and related mapping*, Proc. Amer.Math.Soc. **110** (1990), 7-16.
2. M.Bresar and J.Vukman, *Derivations of noncommutative Banach algebras*, Arch.Math. **59** (1992), 363-370.
3. P.C.Curtis, *Derivations on commutative Banach algebras , Lecture Notes in Math*, vol. 975, Springer, 1983, pp. 328-333.
4. J.Cusack, *Automatic continuity and Topollogically simple radical Banach algebras*, J.London Math.Soc. **16**(2) (1977), 493-500.
5. M.Mathieu and G.J.Murphy, *Derivations mapping into the radical*, Arch. Math. **57** (1991), 469-474.
6. M.Mathieu and Runde, *Derivations mapping into the radical*II, Bull.London Math.Soc. **24** (1992), 485-487.
7. E.C.Posner, *Derivations in prime rings*, Proc.Amer.Math.Soc. **8** (1957), 1093-1100.
8. V.Ptak, *Commutators in Banach Algebras*, Proc.Edinburgh Math. Soc. **22** (1979), 207-211.
9. V.Runde, *Automatic continuity of derivations and epimorphisms*, Pacific of Math. **147**(2) (1991), 365-374.
10. A.M.Sinclair, *Continuous derivations on Banach algebras*, Proc. Amer. Math.Soc **20** (1969), 166-170.
11. A.M.Sinclair, *Automatic continuity of linear operators*, London Math. Soc. Cambrige Univ. (1976).
12. I.M.Singer and J.Wermer, *Derivations on commutative normed algebras*, Math.Ann. **129** (1955), 260-264.
13. M.Thomas, *The image of a derivation is contained in the radical*, Ann of Math. **128** (1988), 435-460.
14. J.Vukman, *A result concerning derivations in Banach algebras*, Proc.Amer. Math.Soc. **116** (4) (1992), 971-975.

15. B. Yood, *Continuous homomorphisms and derivations on Banach algebras*, Contemp. Math. **32** (1984), 279-284.

Department of Mathematics
University of Kangnam
Suwan, 449-702, Korea.