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On a Construction of Subobject Classifiers

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ABSTRACT. In this paper, we consider the higher-order type theoretic language L and construct a type model D of complete partial ordered set. We show that the complete partial ordered set for the language L gives rise to a category and generates a topos.

1. Introduction

The language L under consideration is called type-theoretic because its syntax is based on Russell's simple theory of types. L will contain both constants and variables in every syntactic category, and it will allow quantification over variables of any category. Thus, L will have not only variables ranging over individuals which is characteristic of first-order languages, and variables ranging over predicates too, as does a second-order language, but variables ranging over every category defined in the type theory. Thus the language is known as a higher order language. We recall the concept of categories in L.

1. The category of *terms* of L will be designated by the symbol e.
2. The category of *formulas* of L will be designated by the symbol t.
3. The category of *one-place predicates* of L will be designated by the symbol $< e, t >$.
4. The category of *two-place predicates* of L will be designated by $< e, < e, t >>$.

Now we can give the formal definitions of the syntax and semantics of L.

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2. Syntax of L

(1) The set of types of L is defined recursivly as the following:[2]

(a) e is a type.

(b) t is a type.

(c) It a and b are any types, then $< a, b >$ is a type.

(d) Nothing else is a type.

(2) The basic expressions of L consist of non-logical constants and variables:

(a) For each type a, the set of non-logical constants of type a, denoted Con_a, contains constants $C_{n,a}$ for each natural number n.

(b) For each type a, the set of variables of type a, denoted Var_a, contains variables $V_{n,a}$ for each natural number n.

(3) Syntactic rules of L.

The set of meaningful expressions of type a, denoted "ME_a", for any type a is defined recursively as follows :

(a) For each type a, every variable and every non-logical constant of type a is a member of ME_a.

(b) For any types a and b, if $\beta \in ME_{<a,b>}$ and $\alpha \in ME_a$, then $\beta(\alpha) \in ME_b$.

(c) - (g) If ϕ and ψ are in ME_t, then so are each of the following : $\neg\phi$, $[\phi \wedge \psi]$, $[\phi \vee \psi]$, $[\phi \rightarrow \psi]$, $[\phi \leftrightarrow \psi]$.

(h) If $\phi \in ME_t$ and u is a variable (of any type), then $\forall u\phi \in ME_t$.

(i) If $\phi \in ME_t$ and u is a variable (of any type), then $\exists u\phi \in ME_t$.

3. Semantics of L

A *model* for L is then an ordered pair $< A, F >$ such that A is the domain of individuals or entities and F is a function assigning a denotation to each non-logical constant of L of type a from the set D_a.

An *assignment of values to variables* (*variable assignment*) g is a function assigning to each variable to $V_{n,a}$ a denotation from the set D_a, for each type a and natural number n.

The *denotation of an expression* of L relative to a *model* M and *variable assignment* g is defined recursively as follows :

(1) (a) If α is a non-logical constant, then $[\![\alpha]\!]^{M,g} = F(\alpha)$.

(b) If α is a variable, then $[\![\alpha]\!]^{M,g} = g(\alpha)$.

(2) If $\alpha \in ME_{<a,b>}$ and $\beta \in ME_a$, then $[\![\alpha(\beta)]\!]^{M,g} = [\![\alpha]\!]^{M,g}([\![\beta]\!]^{M,g})$.

(3)-(7) If ϕ and ψ are in ME_t, then $[\![\neg\phi]\!]^{M,g}$, $[\![\phi \wedge \psi]\!]^{M,g}$, $[\![\phi \vee \psi]\!]^{M,g}$, $[\![\phi \rightarrow \psi]\!]^{M,g}$ and $[\![\phi \leftrightarrow \psi]\!]^{M,g}$ are as specified for the first-order predicate. If ϕ is an expression of category ME_t, then $[\![\neg\phi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = 0$; otherwise, $[\![\neg\phi]\!]^{M,g} = 0$. Similarly for $[\![\phi \wedge \psi]\!]$, $[\![\phi \vee \psi]\!]$, $[\![\phi \rightarrow \psi]\!]$, and $[\![\phi \leftrightarrow \psi]\!]$.

(8) If $\phi \in ME_t$ and u is in Var_a, then $[\![\forall u\phi]\!]^{M,g} = 1$ iff for all e in D_a $[\![\phi]\!]^{M,g}u = 1$.

(9) If $\phi \in ME_t$ and u is in Var_a, then $[\![\exists u\phi]\!]^{M,g} = 1$ iff for some e in D_a $[\![\phi]\!]^{M,g}u = 1$.

The semantic value of an expression does not depend on variables that are not free in the expression. So we add the following definition.

The *denotation of an expression* of L relative to a model M is defined as follows :

(1) For any expression ϕ in ME_t, $[\![\phi]\!]^M = 1$ iff $[\![\phi]\!]^{M,g} = 1$ for every value assignment g.

(2) For any expression ϕ in ME_t, $[\![\phi]\!]^M = 0$ iff $[\![\phi]\!]^{M,g} = 0$ for every value assignment g.

4. A type model D of a language L

We define the completeness of binary relations on a type model D. Reynolds have introduced ω-complete relations ([5]).

DEFINITION 4.1. (1) A binary relation $R \subseteq D \times D$ is ω-complete if and only if

$$(U < d^{(i)} >_{i \in \omega},\ U < f^{(i)} >_{i \in \omega}) \in R$$

whenever for all $i \in \omega$, $(d^{(i)},\ f^{(i)}) \in R$ where $< d^{(i)} >_{i \in \omega}$, $< f^{(i)} >_{i \in \omega}$ are increasing chains in D.
(2) $R \subseteq D \times D$ is complete if and only if R is ω-complete and $(t,\ t) \in R$.

Let us construct a type model D of a higher-order type-theoretic language L. Let E be a singleton of type e. Starting from $D_0 = \{t\}$ a chain of approximations of a type model D is built by defining $D_{n+1} = E + < D_n, D_n >$ where $+$ represents disjoint sum and $< D_n, D_n >$ is the space of all continuous mappings from D_n to D_n, and embedding each D_n in D_{n+1} by a suitable projection pair (i_n, p_n) of D_n on D_{n+1} where $i_n : D_n \to D_{n+1}$, $p_n : D_{n+1} \to D_n$ with the properties $p_n \circ i_n = id_{D_n}$, $i_n \circ p_n \subseteq id_{D_{n+1}}$. A standard way of building D is by using Scott's inverse limit construction([4] [6]). The inverse limit of this chain can be defined as a set

$$D = \{< d^{(n)} >_{n \in \omega} \mid d^{(n)} = p_n(d^{(n+1)})\}.$$

Each D_n can be embedded in D by a projection pair $(i_n,\ p_n)$. If $d \in D_n$, we identify d with $i_n(d) \in D$. There we can assume $D_0 \subseteq D_1 \subseteq \cdots \subseteq D_n \subseteq \cdots \subseteq D$. Let d_n stand for $i_n \circ p_n(d)$. It holds $d_n = i_n \circ p_n(d) \subseteq d$. Also if $d \in D_n$, then we have $d_n = d$. Now we may take the type model D of L into account of the equational form $D = E + < D, D >$.

Defining a partial ordering $\leqq$ on D_n by $d \leqq f$ if and only if $d(a) \leqq f(a)$ for all $a \in D_n$, the set of all continuous functions from D_n to

D_n is a complete partial ordered set (c.p.o.s) and the disjoint sum of $E + < D_n, D_n >$ is a complete one, too.

Scott([8]) obtained D by other construction as, for example, the one based on his information systems. The existence of continuous projections $(-)_n : D \to D$ needs us some suitable properties of mappings $(-)_n$ as Scott's approach did. Moreover notice that the inverse limit construction can be carried on is the category c.p.o.. Especially we do not need to assume that D is a domain in the usual sense.

5. A topos generated by types $\mathbb{D}(L)$

The complete partial ordered set D of recursive and polymorphic types for the language L gives rise to a category $\mathbb{D}(L)$. The objects are the partial equivalence relations (p.e.r.) $[\![\alpha]\!]$ of $\alpha \in T^0$. A partial equivalence relation (p.e.r.) is a symmetric and transitive binary relation over D. The idea of using partial equivalence relations to interpret types was introduced in Scott([7]). An arrow $[\![\alpha]\!] \to [\![\beta]\!]$ is a transformation system from *a* p.e.r. $[\![\alpha]\!]$ to *a* p.e.r. $[\![\beta]\!]$. We may think of the objects of $\mathbb{D}(L)$ as type structures of sentences or knowledges and of the arrows as new representations of types or linguistic transformations. We may regard an object $[\![\alpha]\!]$ in $\mathbb{D}(L)$ as a representative tree structure of types based on type t. The arrow $f : [\![\alpha]\!] \to [\![\beta]\!]$ of $\mathbb{D}(L)$ are triples $([\![\alpha]\!], |f|, [\![\beta]\!])$, where $|f|$ is an element of product $[\![\alpha]\!] \times [\![\beta]\!]$. We may think of f as denoting a relation between the sets $[\![\alpha]\!]$ and $[\![\beta]\!]$. Equality between relations $f, g : [\![\alpha]\!] \rightrightarrows [\![\beta]\!]$ is defined thus :

$$f \cdot = \cdot g \text{ means } |f| = |g|.$$

The identity $1_{[\![\alpha]\!]} : [\![\alpha]\!] \to [\![\alpha]\!]$ is defined by

$$1_{[\![\alpha]\!]} = \{ < a, a' > \in [\![\alpha]\!] \times [\![\alpha]\!] | a = a' \}.$$

Composition of relations $f : [\![\alpha]\!] \times [\![\beta]\!]$ and $g : [\![\beta]\!] \times [\![\gamma]\!]$ is defined by

$|gf| = \{< a, c > \in [\![\alpha]\!] \times [\![\gamma]\!] | \exists_{b \in [\![\beta]\!]} (< a, b > \in |f| \wedge < b, c > \in |g|)\}.$

It is easily seen that $\mathbb{D}(L)$ is a category.

A cartesian closed category is a category $\mathbb{D}$ with finite products (hence having a terminal object) such that, for each object A of $\mathbb{D}$, the functor $(-) \times A : \mathbb{D} \to \mathbb{D}$ has a right adjoint, denoted by $(-)^A : \mathbb{D} \to \mathbb{D}$. This means that, for all objects A, B and C of $\mathbb{D}$, there is an isomorphism

$$Hom_{\mathbb{D}}(A \times B, C) \overset{\sim}{\to} Hom_{\mathbb{D}}(A, C^B)$$

and moreover, this isomorphism is natural in A, B and C.

THEOREM 5.1. *$\mathbb{D}(L)$ forms a cartesian closed category.*

PROOF. The terminal object 1 of $\mathbb{D}(L)$ is defined by $1 = \{*\}$, while products are defined by

$$[\![\alpha]\!] \times [\![\beta]\!] \equiv \{< a, b > | a \in [\![\alpha]\!] \wedge b \in [\![\beta]\!]\}.$$

The arrows $0_{[\![\alpha]\!]} : [\![\alpha]\!] \to 1, \Pi_{[\![\alpha]\!],[\![\beta]\!]} : [\![\alpha]\!] \times [\![\beta]\!] \to [\![\alpha]\!]$ and

$\Pi'_{[\![\alpha]\!],[\![\beta]\!]} : [\![\alpha]\!] \times [\![\beta]\!] \to [\![\beta]\!]$ are defined thus :

$|0_{[\![\alpha]\!]}| \equiv [\![\alpha]\!] \times \{*\} \equiv \{< a, * > \in [\![\alpha]\!] \times 1 | a \in [\![\alpha]\!]\},$

$|\Pi_{[\![\alpha]\!],[\![\beta]\!]}| \equiv \{<< a, b >, a > \in ([\![\alpha]\!] \times [\![\beta]\!]) \times [\![\alpha]\!] | \ a \in [\![\alpha]\!] \ \wedge b \in [\![\beta]\!]\},$

$|\Pi'_{[\![\alpha]\!],[\![\beta]\!]}| \equiv \{<< a, b >, a > \in ([\![\alpha]\!] \times [\![\beta]\!]) \times [\![\beta]\!] | \ a \in [\![\alpha]\!] \ \wedge b \in [\![\beta]\!]\}.$

Moreover, if $f : [\![r]\!] \to [\![\alpha]\!]$ and $g : [\![r]\!] \to [\![\beta]\!]$, we define $< f, g >: [\![\gamma]\!] \to [\![\alpha]\!] \times [\![\beta]\!]$ by $| < f, g > | \equiv \{< c, < a, b >> \in [\![\gamma]\!] \times ([\![\alpha]\!] \times [\![\beta]\!]) | < c, a > \in |f| \wedge < c, b > \in |g|\}$.

Now we define

$$[\![\beta]\!]^{[\![\alpha]\!]} \equiv \{\rho \in [\![\alpha]\!] \times [\![\beta]\!] | \rho \ : \ [\![\alpha]\!] \to [\![\beta]\!]\}.$$

We also define $\varepsilon_{[\beta],[\alpha]} : [\![\beta]\!]^{[\alpha]} \times [\![\alpha]\!] \to [\![\beta]\!]$ by

$|\varepsilon_{[\beta],[\alpha]}| \equiv \{<< \rho, a >, b >\in (([\![\alpha]\!] \times [\![\beta]\!]) \times [\![\alpha]\!]) \times [\![\beta]\!] | \rho : [\![\alpha]\!] \to [\![\beta]\!] \wedge < a, b >\in \rho\}$.

Moreover, if $h : [\![\alpha]\!] \times [\![\beta]\!] \to [\![\gamma]\!]$ then $h^* : [\![\alpha]\!] \times [\![\gamma]\!]^{[\beta]}$ is obtained. Thus

$$|h^*| \equiv \{< a, \rho >\in [\![\alpha]\!] \times ([\![\beta]\!] \times [\![\gamma]\!]) | a \in [\![\alpha]\!] \wedge \rho : [\![\beta]\!] \to [\![\gamma]\!] \wedge \forall b \in [\![\beta]\!] \; \exists c \in [\![\gamma]\!](<< a, b >, c >\in |h| \wedge < b, c >\in \rho)\}.$$

Let us consider a valuation on an designated object Ω to objects of $\mathbb{D}(L)$. A valuation is a function $V : a \to \Omega$ where $a \in [\![\alpha]\!]$. Let us take $\Omega = \{1/(n+1) | n = 0, 1, 2, \cdots\}$. We say $[\![\alpha]\!]$ is *well-typed* whenever, for every $a \in [\![\alpha]\!]$, a is degenerated to type t under the operations of iterative formulations $\beta(\alpha)$. This means that, for every $a \in [\![\alpha]\!]$, there exists a number n such that $\beta^n(a) \in D_0$. On the other case we have an irreducible type $[\![\alpha]\!]'$ for which the formulation operation can not be applicable no longer, i.e., $[\![\alpha]\!]' \in D_n$ for some $n \neq 0$. We call this number n the *irreducible degree* of $[\![\alpha]\!]$. Now let us assign the valuation as follows : for every $a \in [\![\alpha]\!]$,

(i) $V(a) = 1$ whenever $[\![\alpha]\!]$ is well-typed,

(ii) $V(a) = 1/(n+1)$ whenever the irreducible degree of $[\![\alpha]\!]$ is n.

The valuation is extended to a function $V : [\![\alpha]\!] \to \Omega$ by the rules for all $[\![\alpha]\!]$ in $\mathbb{D}(L)$:

(i) $V(\sim [\![\alpha]\!]) = 1 - V([\![\alpha]\!])$

(ii) $V([\![\alpha]\!] \wedge [\![\beta]\!]) = \min(V([\![\alpha]\!]), V([\![\beta]\!]))$

(iii) $V([\![\alpha]\!] \vee [\![\beta]\!]) = \max(V([\![\alpha]\!]), V([\![\beta]\!]))$

(iv) $V([\![\alpha \to \beta]\!]) = V(\sim [\![\alpha]\!] \vee [\![\beta]\!])$

(v) $V([\![\forall \varphi \alpha]\!]) = \operatorname{Inf}_i(V(d_i))$, where $d_i \in [\![\forall \varphi \alpha]\!]$

(vi) $V([\![\exists \varphi \alpha]\!]) = \operatorname{Sup}_i(V(d_i))$, where $d_i \in [\![\exists \varphi \alpha]\!]$

DEFINITION 5.2. An (elementary) *Topos* is a cartesian closed category in which the subobject functor is representable. What this means is that there is given and an object Ω, called the *subobject classifier* and a natural isomorphism

$$\text{Sub} \cong \text{Hom}\ (-, \Omega).$$

More precisely, it means that there is given an arrow $T : 1 \to \Omega$ such that

(i) for every arrow $h : \alpha \to \Omega$, an equalizer of h and $TO_\Omega : \alpha \to 1 \to \Omega$ exists, call it a *kernel* of h and write

$$ker\ \ h : \text{Ker}\ \ h \to \alpha;$$

(ii) for every monomorphism $m : \beta \to \alpha$, there is a unique arrow $char\ \ m : \alpha \to \Omega$, called its *characteristic* morphism, such that m is a kernel of $char\ \ m$. The following square is a pullback :

$$\begin{array}{ccc} \beta & \xrightarrow{\ m\ } & \alpha \\ {\scriptstyle O_\beta}\big\downarrow & & \big\downarrow{\scriptstyle \text{char}\ m} \\ 1 & \xrightarrow{\ T\ } & \Omega \end{array}$$

Now we shall show that the category $\mathbb{D}(L)$ associated with language L is a topos. In view of Theorem 5.1, it remains only to produce a subobject classifier. The subobject classifier and the canonical arrow $T : 1 \to \Omega$ are defined as follows :

$$\Omega \equiv \{1/(n+1) \mid n = 0, 1, 2, \cdots\} \equiv \{i_n \mid n = 0, 1, 2, \cdots\}$$
$$|T| \equiv \{< *, i_n > \ \mid i_n \in \Omega\}.$$

Next lemma is obtained immediately and will be used to prove the existence of a topos of $\mathbb{D}(L)$.

LEMMA 5.3. *A function* $f : [\![\alpha]\!] \times [\![\beta]\!]$ *is monic if and only if* $f^{-1}f\cdot = \cdot 1_{[\![\alpha]\!]}\cdot$

THEOREM 5.4. (Ω, T) *is a subobject classifier of* $\mathbb{D}(L)$.

PROOF. If $m : [\![\beta]\!] \to [\![\alpha]\!]$ is monic in $\mathbb{D}(L)$, we define its characteristic morphism $char\ \ m : [\![\alpha]\!] \to \Omega$ by

$$|char\ \ m| = \{< a, i_n > \in [\![\alpha]\!] \times \Omega | i_n = (\exists_{b \in [\![\beta]\!]} < b, a > \in |m|)\}.$$

Then we obtain two properties of topos as follows :
(i) $char(ker\ \ h)\cdot = \cdot h$,
(ii) $ker(char\ \ m) \cong m$,
where $h : [\![\alpha]\!] \to \Omega$ and $m : [\![\beta]\!] \to [\![\alpha]\!]$ is monic. In fact, every arrow $h : [\![\alpha]\!] \to \Omega$ in $\mathbb{D}(L)$ has a kernel $ker\ \ h$: Ker $h \to [\![\alpha]\!]$ that is, an equalizer of h and $TO_{[\![\alpha]\!]}$ such that

$$\text{Ker } h \equiv \{a \in [\![\alpha]\!] | < a, i_n > \in |h| \text{ for some } n's\},$$

$$|ker\ \ h| \equiv \{< a, a > \in [\![\alpha]\!] \times [\![\alpha]\!] | < a, i_n > \in |h| \text{ for some } n's\}.$$

So that the following square is a pullback :

$$\begin{array}{ccc} \text{Ker } h & \longrightarrow & 1 \\ {\scriptstyle \text{ker } h}\downarrow & & \downarrow{\scriptstyle T} \\ [\![\alpha]\!] & \xrightarrow{h} & \Omega \end{array}$$

Then, we get by the concept of well- typed objects and irreducible degrees,

$|char(ker\ \ h)|$

$$\cdot = \cdot \ \{< a, i_n > \in [\![\alpha]\!] \times \Omega | i_n = (\exists_{a' \in [\![\alpha]\!]} < a', a > \in |ker\ \ h|) \text{ for some } n's\}$$

$$\cdot = \cdot \ \{< a, i_n > \in [\![\alpha]\!] \times \Omega | i_n = (\exists_{a' \in [\![\alpha]\!]} < a, i_n > \in |h| \wedge a' = a) \text{ for some } n's\}$$

$$\cdot = \cdot \ \{< a, i_n > \in [\![\alpha]\!] \times \Omega | i_n = (< a, i_n > \in |h|) \text{ for some } n's\}$$

$$\cdot = \cdot \ \{< a, i_n > \in [\![\alpha]\!] \times \Omega | < a, i_n > \in |h| \text{ for some } n's\}$$
$$\cdot = \cdot \ |h|.$$

Next, suppose $m : [\![\beta]\!] \to [\![\alpha]\!]$ is monic. Define $g : [\![\beta]\!] \to \ker\ (char\ \ m)$ by $|g| \equiv |m|$, then we get, by lemma 5.3.,

$$|gg^{-1}|\cdot = \cdot|m^{-1}m|\cdot = \cdot 1_{[\beta]}.$$

On the other hand

$$\begin{aligned} |gg^{-1}|\cdot &= \cdot \ |mm^{-1}| \\ &\cdot = \cdot \ \{< a, a > \in [\![\alpha]\!] \times [\![\alpha]\!] |_{\exists a' \in [\![\alpha]\!]} < a', a > \in |m|\} \\ &\cdot = \cdot \ \{< a, a > \in [\![\alpha]\!] \times [\![\alpha]\!] | < a, i_n > \in |char\ \ m| \text{ for some } n's\} \\ &\cdot = \cdot \ \{< a, a > \in [\![\alpha]\!] \times [\![\alpha]\!] | a \in \ \text{Ker}(char\ \ m)\} \\ &\cdot = \cdot \ |1_{\text{ker}(char\ \ m)}|\cdot \end{aligned}$$

Therefore g is an isomorphism, hence $m \cong ker(char\ \ m)$.

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