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Relation between Clifford Semigroups and Abelian Regular Rings

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ABSTRACT. The theory of inverse semigroups has many features in common with the theory of groups. Many different properties of semigroup become the same condition on ring. In this paper, we want to find the properties of semigroups which is preserved by the properties of ring. Also we find that many different properties become the equivalent conditions.

1. Introduction

Boolean ring is very important because every complemented distributive lattice, Boolean algebras and Boolean rings are all the same objects. What are the common properties of division ring and Boolean ring? If we see division ring from another point of view, we can find that division ring is a union of multiplicative groups. In fact, if D is a division ring, then $D = D^* \cup \{0\}$, where D^* is the set of all nonzero elements, and D^*,$\{0\}$ itself form a multiplicative group. Also in Boolean ring, every element of it is one-element group. Since ring is a semigroup under multiplicative operation, many properties of ring determined by properties of semigroup multiplication. In this paper, we find that if a ring becomes a disjont union of multiplicative groups, this ring has many common properties in multiplicaive operation. Also we find that many sufficient and necessary conditions of rings which become disjoint union of multiplicative group.

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2. Preliminaries

We collect the needed definitions and results for the sake of reference . Throughout this paper, let R be a ring with or without 1 and let S be a semigroup. A semigroup S is called a *union of groups* if each of it's elements is contained in some subgroup of S. If S is a union of groups, then a has an inverse in some subgroup containing a. So $a = aa^{-1}a$ for some a^{-1} in S. An element a of a semigroups S is *regular* if $a = axa$ for some $x \in S$, S is *regular* if all its elements are regular. We note that if $axa = a$ then $e = ax$ is an idempotent element of S such that $ea = a$. For $e^2 = (ax)(ax) = ax = e$ and $ea = axa = a$. Similary, $f = xa$ is an idempotent such that $af = a$. We also note that if a is a regular element of R, then the principal right ideal $aS^1 = a \cup aS$ generated by a is just aS, for $a = af$ implies $a \in aS$. The notion of regularity was introduced by J.Von Neumann, and the following lemma is straightforward.

LEMMA 2.1[4, Lemma 1.13]. *An element a of a semigroup S(ring R) is regular if and only if the principal right and left ideal of S(ring R) generated by a has an idempotent generator.*

An element a of a semigroup S is called *completely regular* if $a = axa$ and $xa = ax$ for some $x \in S$. S is *completely regular* if all its elements are completely regular.

LEMMA 2.2[12, Lemma 1.7.5]. *The followings are equivalent for any $\mathcal{H}$-class H of a semigroup S:*

(1) *H contains an idempotent,*
(2) *H is a maximal subgroup of S,*
(3) *Every element of H is completely regular,*
(4) *H contains a completely regular element.*

From the above lemma, if G is a maximal subgroup of S, then G

is a $\mathcal{H}$-class of S. Hence a semigroup S is a union of its subgroup if and only if S is completely regular. An element $e_x \in S$ is called a *partial identity* for $x \in S$ if $e_x x = x e_x = x$, and if $ex = xe = x$ for some $e \in S$, then $e_x e = e e_x = e_x$. If $x \in S$ has a partial identity, the element $y \in S$ is called a *partial inverse* of x if $xy = yx = e_x$ and $e_x y = y e_x = y$. If x has a partial identity it must be unique. Also the partial inverse of x , if exists, is unique and usually denoted by x^{-1}.

LEMMA 2.3[1, Proposition 2.2]. *A semigroup S is union of groups if and only if every element in S have a partial identity and a partial inverse.*

One of the main themes of this paper concerns the influence of the properties of the idempotents of a ring on the structure of the ring as a whole. In pursuance of this theme, let us define a *Clifford semigroup* to be a regular semigroup S in which the idempotent are central, i.e., in which $ex = xe$ for every idempotent e and every x in S.

LEMMA 2.4 [8, THEOREM 2.1]. *The following statements are equivalent:*

(1) *S is a Clifford semigroup,*
(2) *S is a semilattice of groups,*
(3) *S is a strong semilattice of groups,*
(4) *S is regullar and each $\mathcal{H}$-class has a unique idempotent.*

3. Rings which are union of multiplicative groups

A ring R is called a *union of groups* if a semigroup $(R, .)$ is a union of groups.

EXAMPLE 3.1.

(1) Division ring is a union of groups.
(2) Boolean ring is a union of groups. In fact, Boolean ring is a ring in which each element is a multiplicative group.

(3) Direct product of division rings is not a division ring in general, but it is a union of groups.

A regular ring R is called *abelian* provided all idempotents in R are central. Obviously, any commutative regular ring is abelian, as is any direct product of division ring.

THEOREM 3.2. *For a ring R the following conditions are equivalent:*

(1) *Idempotents commute with each other,*
(2) *Idempotents are contained in center.*

PROOF. (2) implies (1): it is trivial. (1) implies (2): If $e \in R$ is an an idempotent, then $e + xe - exe$ is idempotent for any $x \in R$. Hence $e = e(e + xe - exe) = (e + xe - exe)e = e + xe - exe$, so $exe = xe$. Similarly, we have $exe = ex$. Thus e is central.

LEMMA 3.3 [6, Theorem 3.2]. *For a regular ring R, the following conditions are equivalent:*

(1) *R is abelian,*
(2) *R/P is a division ring for all prime ideal P of R,*
(3) *R has no nonzero nilpotent elements,*
(4) *All right (left) ideals of R are two-sided.*

A semigroup S is called *left (right) regular* if for each element a of S, there exists an element x in S such that $a = xa^2 (a = a^2x)$.

For characterizations of such a semigroup, see [4, Theorem 4.2]. As the following theorem shows that the condition is left - right symmetric for ring R, and are in fact regular.

THEOREM 3.4. *A ring R is right regular if and only if it is abelian regular.*

An idal I of R is called *completely semiprime* if $a^2 \in I, a \in R$ implies $a \in I$.

LEMMA 3.5[4, Theorem 1.27]. *The following assertions concerning semigroups are equivalent:*

(1) *S is right simple and contains an idempotent,*
(2) *S is the direct product $G \times E$ of a group G and a right zero semigroup E.*

THEOREM 3.6. *The followings are equivalent conditions on ring R:*

(1) *R is a union of groups,*
(2) *R is right regular,*
(3) *Every right ideal of R is completely semiprime,*
(4) *R is a union of disjoint groups under multiplication.*

PROOF. If(1) holds, then Lemma 2.3 means every element in R has a partial inverse ; for any element a of R, $a = aa^{-1}a, aa^{-1} = e_a, a = ae_a = aaa^{-1}$. So $a \in aRa, a \in a^2R$. Thus (1) implies (2). If some subgroups G_e and G_f of $(R, .)$ are not disjoint, there is an element a in R such that $ae_a = e_a a = a, aa' = a'a = e_a \ af_a = f_a a = a, aa'' = a''a = f_a$ where e_a, f_a are multiplicative identities of G_e, G_f respectively and a', a'' are inverses of a in G_e, G_f. Now $f_a = a'' = (e_a a)a'' = e_a(aa'') = e_a f_a = (a'a)f_a = a'(af_a) = a'a = e_a$. $a' = a/e_a = a'f_a = a.(aa'') = (a'a)a'' = e_a a'' = f_a a'' = a''$. Hence $G_e \cup G_f$ is also a group with identity $e_a = f_a$, and we have shown that (1) implies(4). Let R be right regular, and let I be a right ideal of R. Let $aRa \subset I$. Then $a \in a^2R = aaR \subset a^2R(aR) \subset aRaR \subset I$. Thus I is a semiprime ideal, and so (2) implies(3). Now assume (2). Since R is right regular, the principal right ideal generated by a is aR for every $a \in R$. Let $R_a = \{x \in R | xR = aR\}$. If $x, y \in R_a$, then $xyR = yR$. Since R is right regular,$aR = a^2R$ for every $a \in R$. So $xyR = x^2R = xR$. Thus the set R_a is a subsemigroup of $(R,.)$. To show that the set R_a is a right simple, let I be any right ideal of $R_a, b \in I$. If $x \in R_a$, then we must show that $x = bt$ for some $t \in R_a$. Since R_a is subsemigroup of R, $xR = bR = bxR$. Now $bx \in R_a$, as we have shown, $bxR = aR = xR$. So $x = bxy$ for some $y \in R$. Since R is right regular , we have $y = y^2z$ for some z in R . Hence $x = bxy = (bxy)yz = (xy)z$. Thus $xR = xyR$, and so $xy \in R_a$. This shows that R_a is right simple. Since R is regular , $a = axa$ for some x in R. Now ax is an idempotent belong to R_a and R_a is a right simple semigroup containing an idempotent. By Lemma 3.5, R_a is a direct product of a group and a band, and so is a union of groups. Hence (2) implies (1).

REMARK. Of couse "right" may be replaced by "left" in Theorem 3.4, since union of groups is left - right symmetric property.

If a ring R is a union of groups , we have $R = \cup R_e$, where R_e is a group, but we had no ideal at all where to look for the product of an element x in R_e and an element y in R_f , or even whether the product of R_e and R_f was contained in a single R_a . We have an attempt to improve above theorem by giving semilattice of rings. We shall say that R is *intra - regular* if, for any element a of R there exist x and y in R such that $a = xa^2y$.

THEOREM 3.7. *The followings are equivalent:*

(1) *R is intra - regular,*
(2) *Every ideal of R is completely semiprime,*
(3) *The principal ideals $J(a)$ of R constitute of a semilattice Y under intersection.*

PROOF. (1) implies (2) Let T be an ideal of R. If $aRa \subset T$, then $a \in Ra(Ra^2R)R \subset T$. (2) implies (1) Since $a^4 \in Ra^2R$ and

Ra^2R is ideal, $a^2 \in Ra^2R$ and so $a \in Ra^2R$. (2) implies (3): i) Since $a \in Ra^2R \subset RaR$, $J(a) = RaR$ is the principal ideal generated by a. ii) $J(ab) = J(ba)$ for every a, b in R. For $(ab)^2 = abab \in RbaR = J(ba)$, $J(ab) \subset J(ba)$ and equality follows by symmetry. iii) $J(ab) = J(a) \wedge J(b)$ for every a, b in R. Clearly $J(ab) \subset (J(a) \cap J(b))$. Conversely, let $c \in J(a) \cap J(b)$, say $c = uav = xby$ with $u, v, x, y \in R$. Then $c^2 = x(byua)v \in J(byua) = J(abyu) \subset J(ab)$. Since $J(ab)$ is completely prime and $c \in J(ab)$, $J(a) \cap J(b) \subset J(ab)$ and the equality follows. Thus the set $Y = \{J(a) | a \in R\}$ is a semilattice under intersection, and the mapping a onto $J(a)$ is an isomorphism of $(R, .)$ upon $(Y, \cap)$. The inverse image of the element $J(a)$ of Y is the set $S_a = \{x \in R | J(x) = J(a)\}$. In particular, S_a is a subsemigroup of $(R,.)$. For if $x, y \in S - a$, then $J(x) = J(y) = J(a)$ and so $J(xy) = J(x) \wedge J(y) = J(a), xy \in S_a$. Thus R is the semilattice Y of the mutually disjoint semigroup S_a. (3) implies (1): $J(a) = J(a) \wedge J(a) = J(a^2)$ whence $a \in J(a^2) \subset Ra^2R$.

In the proof of the above theorem we have that R is the semilattice of semigroup S_a. But we have not any informations of S_a. In fact S_a is a simple semigroup.

THEOREM 3.8. *R is intra-regular if and only if R is the semilattice of simple semigroups.*

PROOF. Let I be any ideal of S_a. If $b \in S_a$, then $SaS = SbS$. Let $x \in S_a$. Then $SxS = SaS$, and so $SbS = SxS \subset I$. Since $x \in Sx^2S \subset SxS, x \in I$. This shows that S_a is a simple semigroup. Since right(left) regular ring is intra regular, we have the following corollary.

COROLLARY 3.9. *If a ring is a union of groups, then it is a semilattice of simple semigroups.*

At first glance this does not look like progress at all, since simple semigroups are more complicated than groups. The progress is , however, real and lies in the gross multiplication fomular $J(a) * J(b) \subset J(ab)$. We call a simple semigroup S *completely simple* if it satisfies the minimal conditions on principal left ideals or principal right ideals.

THEOREM 3.10 [Ree's Theorem]. *Let G be a group, I, J be nonempty set and $P = (p_{j,i})$ be a $J \times I$ matrix with entries in G. Let $S = G \times I \times J$ and define a binary operation on S by the rull that $(a, i, j)(b, m, n) = (aP_{jm}b, i, n)$. Then S is a completely simple semigroup. Conversely, any completely simple semigroup is isomorphic to one constructed in this manner.*

PROOF. [8, Theorem 2.11].

THEOREM 3.11. *The followings are equivalent:*

(1) *R is a union of groups,*
(2) *R is a union of completely simple semigroups,*
(3) *R is a semilattice Y of completely simple semigroup $S_t(t \in Y)$, where Y is the semilattice of principle ideals of R.*

PROOF. (3) implies (2): It is trivial. (2) implies (1). Let S be a completely simple semigroup. If $a \in S$, then $a \in L$ for some minimal left ideal of S. Also $a^2 \in L$ and hence $S^1a^2 = S^1a = L$ By the dual argument $a^2S^1 = aS^1$. Let H_a be the any H -class of S. Then $a^2 \in H_a$. Hence $H_{a^2} = H_a$. By Green's theorem[8, Theorem 2.5] H_a is a subgroup of S and so S is a union of groups. (1) implies (3) is trivial.

If a is an element of a ring R, we say that a' is a *g-inverse* of a if $aa'a = a, a'aa' = a'$. Notice that an element with a g-inverse is necessarily regular. Less obviously, every regular element has a g-inverse, for if $axa = a$ we need only to define $a' = xax$.

THEOREM 3.12. *If a has a unique g-inverse a', then a' is partial identity for a.*

PROOF. Since idempotent aa' is central, by Theorem 3.2, $(aa')a = a(aa') = a$. If for some $e \in R, ea = ae = a$, then $(aa')e = e(aa') = (ea)a' = aa'$. Thus aa' is a partial identity for a. Also $aa' = a(a'aa') = (a'a)(aa') = a'(aa')a = a'a$ and $(aa')a' = a'(aa') = a'$. Hence a' is a partial inverse of a.

THEOREM 3.13. *The followings are equivalent:*

(1) *R is a regular ring and idempotent elements commute,*
(2) *Every element of R has a unique g-inverse,*
(3) *Every finitely generated right(left) ideal of R contains a unique idempotent generator.*

PROOF. (2) implies (1): Let e, f be idempotents and $x = (ef)'$. Then $(fxe)^2 = f(xefx)e = fxe$. Also we can have ef is a g-inverse of fxe. Since fxe is idempotent, it is its own unique inverse. So $fxe = ef$. Thus ef is idempotent. Similarly fe is also idempotent. Hence, $(ef)fe(ef) = ef, (fe)ef(fe) = fe$ and so fe is a g-inverse of ef. We finally obtain that $ef = fe$. (1) implies (3): In regular ring, every finitely generated right ideal is principal[11,p.68] we need only to show that it is generated by a unique idempotent. Let I be a finitely generated right ideal of R. Then $I = aR$ for some $a \in R$. $e = aa'$ is an idempotent element of R such that $ea = a$. Clearly $aR = eR$. Suppose e and f are idempotents generating I. Then $eR = fR = I$. Thus $ef = f$ and $fe = e$ and so $e = f$. (3) implies (1): From Theorem 1.6 in [6], R is regular and so we need only to show that g-inverse are unique. Let b and c be g-inverse of a. So $abR = acR, Rba = Rca$. Then $ab = ac$ and $ba = ca$. Hence $b = bab = bac = cac = c$.

THEOREM 3.14. *For a ring R, the followings are equivalent:*

(1) *R is abelian,*
(2) *R is an union of groups,*
(3) *R is right regular,*
(4) *R is a strong semilattice of groups,*
(5) *R is regular and idempotents are commute,*
(6) *Every right ideal of R is completely semiprime.*

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