The Output SINR of the Linearly Constrained Broadband Beamformer

선형 제한 조건을 갖는 광대역 빔 형성기의 출력 SINR

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ABSTRACT

In this paper, we derive expressions for the output signal-to-interference plus noise ratio(SINR) of the linearly constrained broadband beamformer in noncoherent situations using a vector approach. The incoming broadband signals are assumed to have flat spectra.

요 약

본 논문에서는 벡터적 접근 방법을 이용하여 noncoherent 상황에서 선형 제한 조건을 갖는 광대역 빔 형성기의 출력 SINR식을 도출하였다. 여기서 입사되는 광대역 신호들은 평탄한 스펙트럼을 갖는다고 가정하였다.

Ⅰ. Introduction

Adaptive beamformers have been receiveing much interest in the area of sonar, radar, communication and seismic systems. The adaptive array system involves the weighting of received signals at a sensor array optimally so that the output closely approximates a desired signal from a look direction while minimizing the contributions from interference directions. The adaptive arrays with directional constraints have been discussed in the literature[1, 2]. The array output power is minimized subject to certain directional constraints by finding an optimal set of weights using some recursive algorithm. In [1], the optimum weights were found by the method of Lagrange multipliers. In [3], expressions for the output SINR of the linearly constrained narrowband beamformer were presented. We derive the expression for the output SINR of the linearly constrained beamformer for broadband signals using orthogonality in vector space. This result is useful in studying pattern effects, because it allows one to calculate the output SINR directly without finding the weight vector[4]. Although only a single interference is considered, the results presented may be extended to the case of multiple interferences. Numerical results are included.

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Ⅱ. The output SINR of the linearly constrained beamformer for broadband signals

Assume that the incident signals are broadband in nature. One definition of broadband is in the context of frational bandwidth, defined as the signal bandwidth as a percentage of the center frequency. It is reasonable to defined signals with fractional bandwidths much greater than 1% as broadband[5]. The linearly constrained beamformer in the broadband case is shown schematically in Fig. 1, consisting of K omnidirectional equispaced elements, and L weights per channel. Suppose two signals are incident on the array, one desired and one interference. Also, suppose thermal noise is present on each element signal. The desired signal is assumed to be broadside along the array, i.e. $\theta_s = 0°$. If we set $s(k)$ as the desired signal, $n(k)$ as the interference, and $u(k)$ as zero mean uncorrelated white Gaussian noise with variance σ_u^2, then the output vector $\underline{X}(k)$ of the elements at the k-th observation period(snapshot) can be expressed as

$$\underline{X}(k) = [\,\underline{x}^T(k)\,\underline{x}^T(k-1)\,\cdots\,\underline{x}^T(k-L+1)\,]^T \qquad (1)$$

where $\underline{x}(k-m) = [\,x_1(k-m)\,x_2(k-m)\cdots x_K(k-m)\,]^T$

$$x_i(k-m) = s(k-m) + n(k-m)\cos(2\pi f_o(i-1)\tau) + u_i(k-m)$$

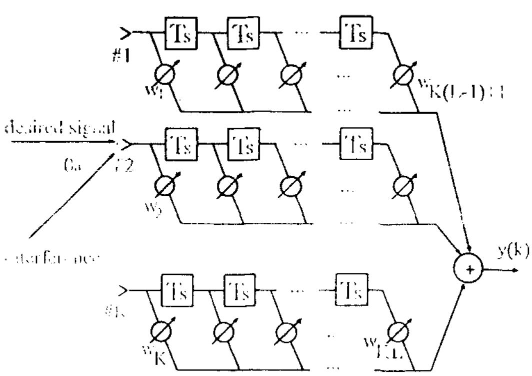

Fig 1. Schematic diagram of a linearly constrained beam former for broadband signals.

그림 1. 선형 제한 조건을 갖는 광대역 빔 형성기의 구조도.

$$\tau = \frac{d}{c}\sin\theta_n \qquad 1 \le i \le K, \quad 0 \le m \le L-1$$

Here, f_o denotes the center frequency, d is the distance between neighboring elements, c is the propagation velocity, and τ is a presteering delay. θ_n are the incident angles of the interference signal. The superscript T denotes the transpose. If there are M snapshots, a KL by M matrix $\underline{X}_M$ can be defined as follows.

$$\underline{X}_M = [\,\underline{X}(1)\,\underline{X}(2)\,\cdots\,\underline{X}(M)\,] \qquad (2)$$

The beamformer output $y(k)$ is given by

$$y(k) = W(k)^T\,X(k) \qquad (3)$$

where $\underline{W}(k) = [\,w_1(k)\,w_2(k)\,\cdots\,w_{KL}(k)\,]^T$

The output vector $\underline{Y}_M$ may be defined as :

$$\underline{Y}_M = [\,y(1)\,y(2)\,\cdots\,y(M)\,]^T \qquad (4)$$

The weights are chosen to minimize the total mean output power as a way of rejecting interferences incident on the array, subject to the set of linear constraints which maintain the frequency response in the look-direction as follows :

$$\underline{C}^T\,\underline{W}(k) = \underline{F} \qquad (5)$$

where $\underline{C} = \begin{bmatrix} \underline{1}_K & \underline{0}_K & \cdots\cdots & & \\ & & \cdots\cdots & \underline{0}_K & \underline{1}_K \end{bmatrix}$

$\underline{F} = [\,f_1 \quad f_2 \quad \cdots\cdots \quad f_L\,]^T$

$\underline{1}_K$ is an all 1's vector of length K, and $\underline{0}_K$ is a zero vector of length K. The implication of equation (3) above is shown in Fig.2. In Fig.2. the receiver noise component is omitted, but the analyses all include the effects of receiver noise. The beamformer output $y(k)$ is a point on the surface π determined by the points $\{x_1(k-m)\,x_2(k-m)\,\cdots\,x_K(k-m),\ m=0,\ 1,\ \cdots,\ L-1\}$. Optimum weights for the beamformer are determined so that the output power is minimized. To minimize the output power, output vector $\underline{Y}_M$ must be perpendicular to the surface π. As shown in Fig.2, since $x_i(k-m)-y(k)$ is a point on the surface π, if $\underline{Y}_M$ is perpendicular to π the following equation is satisfied.

Fig 2. The signals in vector space.
그림 2. 벡터 공간에서 신호들.

$$\langle (\underline{X}(k)-\underline{1}_{KL}\, y(k))^T,\ y(k)\rangle = \underline{0}_{KL} \tag{6}$$

$$\begin{aligned}\langle \underline{X}(k)^T,\ y(k)\rangle &= \langle y(k)\,\underline{1}_{KL}^T,\ y(k)\rangle \\ &= \underline{1}_{KL}\langle y(k),\ y(k)\rangle\end{aligned} \tag{7}$$

$\underline{0}_{KL}$ is a zero vector of length KL, and $\underline{1}_{KL}$ is an all 1's vector of length KL. $\langle\ \rangle$ is defined as

$$\langle \underline{f},\ \underline{g}\rangle = \underline{f}^T\,\underline{g} \tag{8}$$

From equation (3), (7) and (8),

$$\underline{X}(k)\,\underline{X}(k)^T\,\underline{W}(k) = \underline{1}_{KL}\,\underline{W}(k)^T\,\underline{X}(k)\,\underline{X}(k)^T\underline{W}(k) \tag{9}$$

$$\begin{aligned}\underline{W}(k) &= \underline{R}_{xx}(k)^{-1}\,\underline{1}_{KL}\,\underline{W}(k)^T\,\underline{R}_{xx}(k)\,\underline{W}(k) \\ &= \underline{R}_{xx}(k)^{-1}\,\underline{C}\,\underline{1}_L\,\underline{W}(k)^T\,\underline{R}_{xx}(k)\,\underline{W}(k)\end{aligned} \tag{10}$$

$\underline{X}(k)\,\underline{X}(k)^T$ in equation (9) is an autocorrelation matrix $\underline{R}_{xx}(k)$ of the elements output at the k-th time sample. Premultiplying $\underline{C}^T$ to both sides of equation (10), and from equation (5)

$$\underline{F} = \underline{C}^T\,\underline{R}_{xx}(k)^{-1}\,\underline{C}\,\underline{1}_L\,\underline{W}(k)^T\,\underline{R}_{xx}(k)\,\underline{W}(k) \tag{11}$$

$$\underline{1}_L\,\underline{W}(k)^T\,\underline{R}_{xx}(k)\,\underline{W}(k) = (\underline{C}^T\,\underline{R}_{xx}(k)^{-1}\,\underline{C})^{-1}\,\underline{F} \tag{12}$$

From equation (12) and (10)

$$\underline{W}(k) = \underline{R}_{xx}(k)^{-1}\,\underline{C}\,(\underline{C}^T\,\underline{R}_{xx}(k)^{-1}\,\underline{C})^{-1}\,\underline{F} \tag{13}$$

Assuming that the input is a wide sense stationary process

$$\underline{W}_{opt} = \underline{R}_{xx}^{-1}\,\underline{C}\,(\underline{C}^T\,\underline{R}_{xx}^{-1}\,\underline{C})^{-1}\,\underline{F} \tag{14}$$

Equation (14) is equal to that of Frost[1]. Therefore, the beamformer output becomes

$$\begin{aligned}y(k) &= \underline{X}(k)^T\,\underline{W}_{opt} \\ &= \underline{X}(k)^T\,\underline{R}_{xx}^{-1}\,\underline{C}\,(\underline{C}^T\,\underline{R}_{xx}^{-1}\,\underline{C})^{-1}\,\underline{F}\end{aligned} \tag{15}$$

The beamformer output can be divided into the desired signal component $s_0(k)$ and interference plus additive noise signal component $j_o(k)$.

$$\begin{aligned}s_o(k) &= a_1 s(k) + a_2 s(k-1) + \cdots + a_L s(k-L+1) \\ &= \underline{S}^T\,\underline{A}\end{aligned} \tag{16}$$

$$j_o(k) = y(k) - s_o(k) \tag{17}$$

where $\underline{S} = [\,s(k)\ s(k-1)\ \cdots\ s(k-L+1)\,]^T$

$\underline{A} = [\,a_1\ \ a_2\ \ \cdots\ \ a_L\,]^T$

Here, a_i is a scalar. Using the property that the space spanned by column vector $\underline{S}$ and $j_o(k)$ are orthogonal, $\underline{A}$ is found as

$$\underline{A} = \underline{R}_{ss}^{-1}\,\underline{R}_{sx}^{-1}\,\underline{C}(\underline{C}^T\,\underline{R}_{xx}^{-1}\,\underline{C})^{-1}\,\underline{F} \tag{18}$$

Here, $\underline{R}_{ss}$ is the L by L autocorrelation matrix of the desired signal, $\underline{R}_{sx}$ is the L by KL cross-correlation matrix of the desired signal and the sensor output signal. We assume here that the desired signal has a flat spectrum over its specified frequency band, as shown by Fig.3, where δf_0 denotes the bandwidth of the desired signal. The incoming interference is also flat spectra similar to the desired signal. Then, the Wiener-Khinchin theorem[6] allows us to find the elements of $\underline{R}_{xx}$, $\underline{R}_{ss}$ and $\underline{R}_{sx}$. The elements of R_{xx} may be obtained as follows :

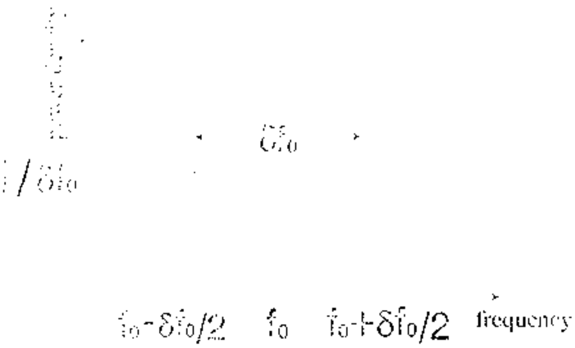

Fig 3. Modelled frequency spectrum of broadband signal.
그림 3. 모델링된 광대역 신호의 주파수 스펙트럼.

$$\underline{R}_{xx}(m, n) = \mathrm{Re}\Big[\frac{1}{\delta f_0}\int_{f_0-\frac{\delta f_0}{2}}^{f_0+\frac{\delta f_0}{2}} \exp(j2\pi f\tau_{mn})\,df\Big]$$

$$+\mathrm{Re}\Big[\frac{1}{\delta f_0}\int_{f_0-\frac{\delta f_0}{2}}^{f_0+\frac{\delta f_0}{2}} \exp\{j2\pi f(\varepsilon_{mn}\sin\theta_n/c$$

$$+\tau_{mn})\}\,df\Big] + \sigma_u^2\,\delta(m-n)$$

$$= \cos(2\pi f_0\tau_{mn})\,Sinc(\pi\delta f_0\tau_{mn})$$

$$+\cos(2\pi f_0(\varepsilon_{mn}\sin\theta_n/c+\tau_{mn}))$$

$$Sinc(\pi\delta f_0(\varepsilon_{mn}\sin\theta_n/c+\tau_{mn})) + \sigma_u^2\,\delta(m-n) \qquad (19)$$

where $\varepsilon_{mn} = (p-p')d$, $\tau_{mn} = (q-q')T_s$

Considering that the m-th and n-th matrix element correspond to the q-th tap of the p-th element and the q'-th tap of the p'-th element, respectively, we have the following relations :

$$m = p + K(q-1) \qquad n = p' + K(q'-1) \qquad (20)$$

where $m, n = 1, 2, \cdots, KL$ $\quad p, p' = 1, 2, \cdots, K$
$q, q' = 1, 2, \cdots, L$

The elements of $\underline{R}_{ss}$ may be obtained as :

$$\underline{R}_{ss}(m, n) = \mathrm{Re}\Big[\frac{1}{\delta f_0}\int_{f_0-\frac{\delta f_0}{2}}^{f_0+\frac{\delta f_0}{2}} \exp(j2\pi f\tau_{mn})\,df\Big]$$

$$= \cos(2\pi f_0\tau_{mn})\,Sinc(\pi\delta f_0\tau_{mn}) \qquad (21)$$

where $\tau_{mn} = (m-n)T_s$
$m, n = 1, 2, \cdots, L$

Finally, the elements of $\underline{R}_{sx}$ may be obtained by :

$$\underline{R}_{sx}(m, n) = \mathrm{Re}\Big[\frac{1}{\delta f_0}\int_{f_0-\frac{\delta f_0}{2}}^{f_0+\frac{\delta f_0}{2}} \exp(j2\pi f\tau_{mn})\,df\Big]$$

$$= \cos(2\pi f_0\tau_{mn})\,Sinc(\pi\delta f_0\tau_{mn}) \qquad (22)$$

where $\tau_{mn} = (q-q')T_s$
$m = 1, 2, \cdots, L$ $\qquad n = 1, 2, \cdots, KL$
$q, q' = 1, 2, \cdots, L$ $\qquad m = q$
$n = p' + K(q'-1)$ $\qquad p' = 1, 2, \cdots, K$

T_s is the time delay between the adjacent taps, $Sinc(x)$ denotes the $\sin(x)/x$, and $\delta(\cdot)$ is the Kronecker delta function. Therefore, from equations above the output SINR is obtained as follows :

$$SINR_o = \frac{E[s_0^2(k)]}{E[j_0^2(k)]}$$

$$= \frac{\underline{A}^T\,\underline{R}_{ss}\,\underline{A}}{\underline{F}^T(\underline{C}^T\,\underline{R}_{xx}^{-1}\,\underline{C})^{-1}\underline{F} - \underline{A}^T\,\underline{R}_{ss}\,\underline{A}} \qquad (23)$$

$E[\cdot]$ denotes the expectation operator. In section Ⅲ we show curves of the output SINR computed from (18) to (23) and discuss how the signal bandwidth δf_0 affects the array performance.

Ⅲ. Numerical results

For the results presented, a linear array of one-half wavelength spacing is used. The desired signal is assumed to be broadside along the array. The interference signal has an identical center frequency and bandwidth as the desired signal and is assumed to be incident at an angle of 30°. Also, it is assumed that there is no correlation between the desired and interference signals. The center frequency f_o of the signal is $0.25f_s$ where f_s is the sampling frequency. The linearly constrained beamformer is designed to pass signals with fractional bandwidth($\delta f_o/f_o$) of 0.4.

Fig.4 shows the output SINR as a function of the bandwidth when the input SNR and input INR (Interference-to-Noise Ratio) are 20 dB and 40 dB, respectively. The figure shows that as long as the input signal bandwidth and the designed beamformer bandwidth are identical, the output SINR remains constant, and as the input signal bandwidth increases further, the output SINR decreases. For example, Fig.4 shows that when the array has 3 weights per channel, the output SINR drops from 21 dB for $\delta f_o/f_o=0.4$ to 15 dB for $\delta f_o/f_o=1.0$. Also, the figure shows that for the same bandwidth, output SINR increases as the number of weights per channel increases. Fig.5 shows SINR as the incident angle of the interference varies, when the number of elements is 4, and the fractional bandwidth of the input signal is 0.4. We can show that the number of weights per channel increases, output SINR increases.

Ⅳ. Conclusions

In this paper, we have derived an expression for the output SINR of the linearly constrained beamformer for broadband signals in a noncoherent situation using the concept of vector space. If

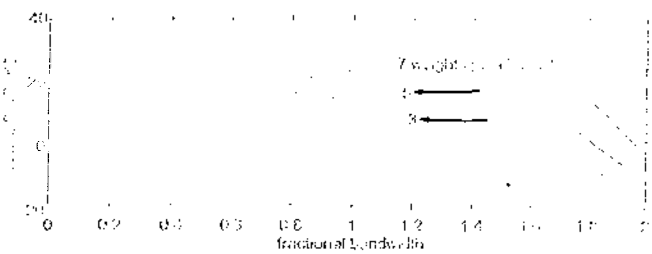

Fig 4. Output SINR versus fractional bandwidth. $\theta_n=30°$, SNR = 20 dB, INR = 40 dB.
그림 4. 출력 SINR 대 fractional bandwidth.

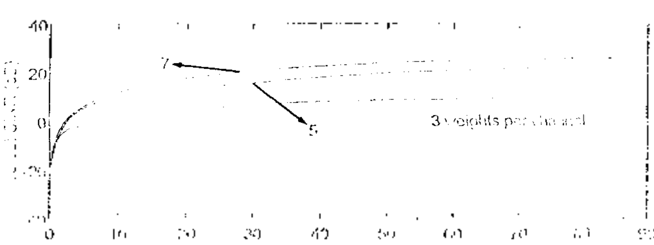

Fig 5. Output SINR versus interference incident angle. $\delta f_o/f_o=0.4$, SNR = 20 dB, INR = 40 dB.
그림 5. 출력 SINR 대 interference incident angle.

the bandwidth of the input signal remains identical to the designed beamformer, a constant output SINR is maintained. As the bandwidth increases further, output SINR is decreases. Also, the result shows that for the same bandwidth, output SINR increases as the number of weights per channel increases. If the correlation coefficients are included, the results could be extended to the case of coherent situations such as multiple propagation paths (multipaths) or smart jamming[6].

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