ON SOME PROPERTIES OF BOUNDED X^* – VALUED FUNCTIONS

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1. Introduction

Suppose that X is a Banach space with continuous dual X^{**}, (Ω, Σ, μ) is a finite measure space. $f : \Omega \to X^*$ is a weakly measurable function such that $x^{**}f \in L_1(\mu)$ for each $x^{**} \in X^{**}$ and $T_f : X^{**} \to L_1(\mu)$ is the operator defined by $T_f(x^{**}) = x^{**}f$.

In this paper we study the properties of bounded X^*- valued weakly measurable functions and bounded X^*- valued *weak** measurable functions.

Throughout the paper X will denote the unit ball of X by B_X. An operator $T_f : X^{**} \to L_1(\mu)$ is said to be (w^*, norm) - continuous provided that net $T_f(x_j{**})$ converges to $T_f(x^{**})$ in the norm topology of $L_1(\mu)$ whenever (x_α^{**}) is a net which converges to x^{**} in the *weak** topology of X^{**}.

A function : $(\Omega, \Sigma, \mu) \to X^*$ is weakly measurable if $x^{**}f$ is measurable for every $x^{**} \in X^{**}$. A function : $(\Omega, \Sigma, \mu) \to X^*$ is *weak** measurable if $x\ f$ is measurable for every $x \in X$.

An operator $T_f : X^{**} \to L_1(\mu)$ which is defined by $T_f(x^{**}) = x^{**}f$ is weakly compact if the norm closure of $T_f(B_X^{**})$ is weakly compact. A subset K of $L_1(\mu)$ is called uniformly integrable if $\lim_{\mu(E)\to 0} \int_E |f|d\mu = 0$ uniformly in $f \in K$.

2. Main Theorems

Theorem 1. *If $f : \Omega \to X^*$ is bounded weakly measurable function, then f is (w^*, norm) - sequentially continuous.*

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Proof. *If x_n^{**} converges to x^{**} in the weak* topology of X^{**}, then $x_n^{**}f$ converges to $x^{**}f$ pointwise. Since x_n^{**} converges to x^* in the weak* topology of X^{**}, by the principle of uniform boundedness $Sup_{n\to\infty} \| x_n{}^{**} \| < \infty$ and by hypothesis there exists $M > 0$ such that $Sup_{x\in\Omega} \| f \| < M$.*

Since $|x_i^{**}| \leq sup \| x_{n'}^{**} \| \leq M$, by Lebesque's bounded convergence theorem

$$\lim_{n\to\infty} \| x_n^{**}f - x^{**}f \| = \lim_{n\to\infty} \int_\Omega |x_n^{**}f - x^{**}f| d\mu = 0$$

Thus T_f is (w*, norm) - sequentially continuous.

Lemma. *A subset of $L_1(\mu)$ is relatively weakly compact if and only if it is bounded and uniformly integrable.*

Proof. *Let $K \subset L_1(\mu)$ be relatively weakly compact. Then K is bounded and if (f_n) is a sequence in K, then (f_n) has a weakly convergent subsequence.*

Hence there is a subsequence (f_{nj}) such that $\lim_j \int_E f_{nj} d\mu$ exists for all $E \in \Sigma$.

It follows immediately that K is uniformly integrable.

Conversely, suppose K is bounded and uniformly integrable. Let (f_n) be a sequence in K. Then there is a countable field $\mathcal{F}$ such that f_n is measurable relative to the σ- field Σ_1, generated by $\mathcal{F}$.

By diagonal procedure, select a subsequence (f_{nj}) such that $\lim_j \int_E f_{nj} d\mu = F(E)$ exists for all $E \in \mathcal{F}$.

Since K is uniformly integrable, there exists $f \in L_1(\Sigma_1,\ \mu)$ such that

$$\lim_j \int_\Omega f_{nj}\ g\ d\mu = \int_\Omega f\ g\ d\mu$$

for each $g \in L_\infty(\Sigma_1,\ \mu)$. Hence $f_{nj} \to f$ is weakly in $L_1(\Sigma_1,\ \mu)$, But $f_{nj} \to f$ is weakly $L_1(\mu)$, Hence K is relatively compact.

Theorem 2. *If $f : \Omega \to X^*$ is bounded weakly measure function, then $T_f : X^{**} \to L_1(\mu)$ is locally compact operator.*

Proof. *Since $f : \Omega \to X^*$ is bounded there exists a number M such that $sup\{\| f(x) \|; x \in \Omega\} \leq M$. If x^{**} belongs to B_X^{**},*

$$\| T_f(x^{**}) \| = \int_\Omega |x^{**}f| d\mu = \int_\Omega \| f(x) \| \ d\mu = M\mu(\Omega).$$

Hence $T_f(B_X{}^{**})$ *is norm bounded. If* $\varepsilon > 0$, $\mu(B) < \frac{\varepsilon}{M}$ *then*

$$\int_E \| f \| \, d\mu \leq M\mu(E) < \varepsilon \text{ and if } \mu(E) < \frac{\varepsilon}{M} \text{ and } x^{**} \in B_X{}^{**},$$

$$\int_E |T_f(x^{**})| d\mu = \int_E |x^{**} f| d\mu \leq \int_E \| f \| \, d\mu < \varepsilon$$

Hence $T_f(B_X{}^{**})$ *is uniformly integrable. By Dunford theorem,* $T_f(B_X{}^{**})$ *is relatively weakly compact. Therefore* $T_f : X^{**} \to L(\mu)$ *is weakly compact operator.*

Theorem 3. *Suppose that* (Ω, Σ, μ) *is a measure space,* $f_n : \Omega \to X^*$ *is bounded weak***-measurable for each* $n \in N$, $\{f_n : n \in N\}$ *is uniformly bounded and there is a real valued function* gx *on* Ω *such that* $xf_n \to gx$ a.e.$[\mu]$. *Then there is an* $f : \Omega \to X^*$ *such that* $xf = gx$ a.e.$[\mu]$ *for each* $x \in X$.

Proof. *Suppose the hypothesis are satisfied. Let* M_n *be a* $sup\{\| f_n(x^*) \| : x^* \in X^*\}$, *since* $\{f_n : n \in N\}$ *is uniformly bounded,* $M = \sup M_{n<\infty}$.

Let $K_M(0)$ *denote the closed ball of radius* M *with center at the origin of* X^*, *then* $K_M(0)$ *is weak** *compact and* $(K_M(0), w^*)^\Omega$ *is compact in pointwise topology.*

Since (f_n) *is a net in the compact space* $(K_M(0), w^*)^\Omega$, *there are a subnet* (f_{n_k}) *of* (f_n) *and a function* $f : \Omega \to K_M(0)$ *such that* (f_{n_k}) *converges to* f *pointwise in the* w^**- topology. But then* $xf = g_X$ a.e. $[\mu]$ *for each* $x \in X$

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