A NOTE ON THE VALUATION

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1. Introduction

Classically, valuation theory is closely related to the theory of divisors and conversely. If D is a Dedekined ring and K is its quotient field, then we can clearly construct the theory of divisors on D (or K), and then we can induce all the valuations on K ([3]). In particular, if K is a number field and A is the ring of algebraic integers, then since Z is Dedekind, A is a Dedekind rign and K is the field of fractions of A. Thus we can construct the theory of divisors in K and equivalently we can induce the set of all valuations in K. Since a number field K is a subset of C, if we embed K into R (or C), then we get trivial absolute values in R or C, which is called archimedean absolute value. Otherwise, we say non-archimedean absolute value or valuation in K.

In this note, we let K be an algebraic number field, and v be a valuation in K. Let E be a finite extension of K and w be a valuation which extends v into E (we write $w|v$). We consider the completion of K and E with respect to the corresponding valuations v, w respectively. By using the property of local degree which is denoted by $[E_w : K_v] = n_w$, we show that the norm of E over K is the product of local norms and trace is the sum of local traces which is usual in norm and trace.

2. Basic facts

Let K be a field. An absolute value on K is a real valued function $|\ |_v : K \to R$ (reals) defined as follows.

D1. $\forall x \in K,\ |x|_v \geq 0,\ |x|_v = 0 \Leftrightarrow x = 0.$
D2. $\forall x, y \in K,\ |xy|_v = |x|_v |y|_v.$
D3. $\forall x, y \in K,\ |x + y|_v \leq |x|_v + |y|_v.$

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If an absolute value satisfies D4 instead of D3:

D4. $\forall x, y \in K,\ |x+y|_v \leq max\{|x|_v, |y|_v\}$,

then its absolute value $|\ |_v$ is called a valuation or a non-archimedean absolute value ([2],[4],[6]).

Let K be a number field, and let $|\ |_p$ be the p-adic valuation on Q (rationals). For the integral closure A of Z in K which is a Dedekind ring, let $\mathcal{M}$ be a maximal ideal of A such that $\mathcal{M} \cap Z = (p)$. Then for $\pi A = \mathcal{M} \cdots$, and $p = \pi^r u$, and thus we define $|\pi|_{\mathcal{M}} = (\frac{1}{p})^{1/r}$, which is a $\mathcal{M}$-adic valuation on K. Thus, from our point of view of Dedekind rings and integral closure, we can recover all the valuations on K which induce p-adic valuations on Q. The set of absolute values on K consisting of the $\mathcal{P}$-adic absolute values $|\ |_{\mathcal{P}}$ ($\mathcal{P}$ is a prime ideal of A) and of the absolute values induced by extending K into R(reals) or C(complexes) will be called the canonical set denoted by M_K. The real and complex absolute values in M_K are called archimedean [5] (otherwise, non-archimedean).

For $|\ |_v \in M_K$ we can form the completion K_v of K as follows. Since $|\ |_v$ induces a topology on K, we consider Cauchy sequences in K. It is clear that all Cauchy sequences form a ring such that the set of null sequences forms a maximal ideal. We put

$$K_v = \{Cauchy\ sequences\}/\{null\ sequences\}.$$

Then K_v is a field which is called the completion of K.

Then we have the following ([2],[6]):

(i) K can be regarded as a subfield of K_v,

(ii) The absolute value $|\ |_v$ in K can be extended to K_v by continuity.

3. Norms and Traces

As before, let K be a number field with an absolute value $|\ |_v \in M_K$, and let E be a finite extension of K. Let $\overline{K}_v$ be the algebraic closure of K_v. Then we have embeddings $\sigma, \tau : E \to \overline{K_v}$ over K. If there exists an isomorphism $\lambda : \tau E \cdot K_v \to \sigma E \cdot K_v$ which is the identity on K_v, then we say that σ and τ are conjugate over K_v. We have the following ([6]):

"Two embeddings $\sigma, \tau : E \to \overline{K_v}$ over K give rise to the same absolute value on E if and only if σ and τ are conjugate over K_v."

Lemma. *In the above situation, we assume that $[E : K] = n$. For each $v \in M_K$, let w be an absolute value on E extending $|\ |_v = v$ and let n_w be the local degree $[E_w : K_v](= n_w)$. Then*

$$n = \sum_{w|v} n_w.$$

Proof. *We may identify E_w as a composite extension EK_v of E and K_v. Thus $n_w = [E_w : K_v] =$ the number of embeddings $E \to \overline{K_v}$, which are congugate each other. Since we can regard that E is a finite separable extension over K,*

$$n = [E : K] = \sum_{w|v} [E_w : K_v] = \sum_{w|v} n_w. \quad ///$$

Proposition. *Let K be a number field.*

Take an absolute value $v_0 = |\ |_{v_0} \in M_Q$.

(a) For every $\alpha \in K$

$$\prod_{v|v_0} |\alpha|_v{}^{n_v} = |N_{K/Q}(\alpha)|_{v_0}.$$

(b)

$$K \otimes_Q Q_{v_0} \cong \sum_{v|v_0} K_v.$$

Proof. *(a): We assume that $[K : Q] = n$. Then*

$$N_{K/Q}(\alpha) = \prod_{i=1}^{n} \sigma_i(\alpha) \quad (\alpha \in K),$$

where $\sigma_i : K \to \overline{Q}_{v_0}$ are embeddings over Q for $i = 1, \cdots, n$. By Lemma,

$$|N_{K/Q}(\alpha)|_{v_0} = \left| \prod_{i=1}^{n} \sigma_i(\alpha) \right|_{v_0} = \prod_{v|v_0} |\alpha|_v{}^{n_v},$$

where $n_v = [K_v : Q_{v_0}]$.

(b): *We assume that* $[K : Q] = n = \sum_{v|v_0} n_v$, *where* $n_v = [K_v : Q_{v_0}]$. *Thus*

$$K \otimes_Q Q_{v_0} \cong Q_{v_0} \oplus \cdots \oplus Q_{v_0}(n - times),$$

and

$$K_v \cong Q_{v_0} \oplus \cdots \oplus Q_{v_0}(n_v - times, v|v_0).$$

Therefore, we have

$$K \otimes_Q Q_{v_0} \cong \sum_{v|v_0} K_v.$$

Theorem. *Let* E *be a finite extension over* K *and for* $|\ |_v \in M_K$, *let* $|\ |_w$ *be an extension of* $|\ |_v$ *to* E. *We assume that* N_w *is the local norm from* E_w *to* K_v *and that* T_w *is the local trace. Then*

$$N_{E/K}(\alpha) = \prod_{w|v} N_w(\alpha), \quad T_{E/K}(\alpha) = \sum_{w|v} T_w(\alpha)$$

for each $\alpha \in E$.

Proof. *We assume that* $[E : K] = n$. *Then*

$$N_{E/K}(\alpha) = \prod_{i=1}^{n} \sigma_i(\alpha)$$

where $\sigma_i : E \to \overline{K}_v$ *are embeddings over* K *for* $i = 1, 2, \cdots, n$. *Then by Lemma,*

$$N_{E/K}(\alpha) = \prod_{i=1}^{r} \sigma_{i_1}(\alpha) \cdots \sigma_{i_{n_w}}(\alpha), \quad n = \sum_{w|v} n_w$$
$$= \prod_{w|v} N_w(\alpha).$$

Similarly,

$$T_{E/K}(\alpha) = \sum_{i=1}^{r} \sigma_{i_1}(\alpha) \cdots \sigma_{i_{n_w}}(\alpha), \quad n = \sum_{w|v} n_w$$
$$= \sum_{w|v} T_w(\alpha). \quad ///$$

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