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$H^1(\Omega)$-NORM ERROR ANALYSIS UNDER NUMERICAL QUADRATURE RULES BY THE P-VERSION OF THE FINITE ELEMENT METHOD

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1. Introduction

Let Ω be a closed and bounded polygonal domain in R^2, or a closed line segment in R^1 with boundary Γ, such that there exists an invertible mapping $T : \widehat{\Omega} \to \Omega$ with the following correspondence:

$$\widehat{x} \in \widehat{\Omega} \longleftrightarrow x = T(\widehat{x}) \in \Omega, \tag{1.1}$$

and

$$\widehat{t} \in U_p(\widehat{\Omega}) \longleftrightarrow t = \widehat{t} \circ T^{-1} \in U_p(\Omega), \tag{1.2}$$

where $\widehat{\Omega}$ denotes the corresponding reference elements $\widehat{I} = [-1, 1]$ and $\widehat{I} \times \widehat{I}$ in R^1 and R^2 respectively,

$$\begin{aligned} &U_p(\widehat{\Omega}) \\ &= \{\, \widehat{t} : \widehat{t} \text{ is a polynomial of degree} \le p \text{ in each variable on } \widehat{\Omega}\, \}, \end{aligned} \tag{1.3}$$

and

$$U_p(\Omega) = \{\, t : \widehat{t} = t \circ T \in U_p(\widehat{\Omega})\, \}. \tag{1.4}$$

We consider the following model problem of elliptic equations: Find $u \in H_0^1(\Omega)$, such that

$$-\operatorname{div}(a\nabla u) + bu = f \quad \text{in} \quad \Omega \subset R^2, \tag{1.5}$$

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right) + bu = f \quad \text{in} \quad \Omega \subset R^1, \tag{1.6}$$

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where the functions a, b and f satisfy a compatibility condition to ensure a solution exists, and

$$H_0^m(\Omega) = \{u \in H^m(\Omega) \,:\, u \text{ vanishes on } \Gamma\}. \tag{1.7}$$

For sake of simplicity, we assume that

$$0 < A_1 \le a(x) \le A_2 \quad \text{for all} \quad x \in \Omega, \tag{1.8}$$

$$0 \le b(x) \le B_2 \quad \text{for all} \quad x \in \Omega, \tag{1.9}$$

and

$$f \in L_2(\Omega). \tag{1.10}$$

In addition, we also assume that there exists a constant $M \ge 1$ such that

$$\|T\|_{j,\infty,\widehat{\Omega}},\ \|T^{-1}\|_{j,\infty,\Omega} \le A \quad \text{for} \quad 0 \le j \le M, \tag{1.11}$$

$$\|\widehat{J}\|_{j,\infty,\widehat{\Omega}},\ \|\widehat{J}^{-1}\|_{j,\infty,\Omega} \le A \quad \text{for} \quad 0 \le j \le M-1, \tag{1.12}$$

where $\widehat{J}$ and $\widehat{J}^{-1}$ denote the Jacobians of T and T^{-1} respectively. Then, as seen in theorem 4.3.2 of [1], we obtain the following correspondence: For any $\alpha \in [1, \infty]$, $0 \le \beta \le M$,

$$\widehat{t} \in W^{\beta,\alpha}(\widehat{\Omega}) \longleftrightarrow t = \widehat{t} \circ T^{-1} \in W^{\beta,\alpha}(\Omega) \tag{1.13}$$

with norm equivalence

$$C_1 \|t\|_{\beta,\alpha,\Omega} \le \|\widehat{t}\|_{\beta,\alpha,\widehat{\Omega}} \le C_2 \|t\|_{\beta,\alpha,\Omega}. \tag{1.14}$$

Here, using the p-version of the finite element method without subdividing Ω we derive the following discrete variational form of (1.5)–(1.6): Find $u_p \in S_{p,0}(\Omega)$ satisfying

$$B(u_p, v_p) = (f, v_p)_\Omega \quad \text{for all} \quad v_p \in S_{p,0}(\Omega), \tag{1.15}$$

where

$$(1.16) \qquad B(u,v) = \int_\Omega a\nabla u \cdot \nabla v\,dx + \int_\Omega buv\,dx,$$

$$(1.17) \qquad (f,v)_\Omega = \int_\Omega fv\,dx,$$

and

$$(1.18) \qquad S_{p,0}(\Omega) = U_p(\Omega) \cap H_0^1(\Omega).$$

In [4] and [5], M. Suri obtained an estimate

$$(1.19) \quad \|u-u_p\|_{1,\Omega} \le C\,p^{-(r-1)}\|u\|_{r,\Omega} \quad \text{for all} \quad u \in H_0^r(\Omega),\ r \ge 1.$$

But, its optimal result follows under the assumption that T is a sufficiently smooth mapping and all integrations in (1.15) are performed exactly. In practice, the integrals in (1.15) are seldom computed exactly. In this paper,when some numerical quadrature rules are used for calculating the integrations in the stiffness matrix and the load vector of (1.15) we give its variational form and derive an estimate of $\|u-\widetilde{u}_p\|_{1,\Omega}$ where $\widetilde{u}_p$ is an approximation satisfying (2.5). This paper also treats the case when $T : \widehat{\Omega} \to \Omega$ may not be smooth enough. In section 2, we consider a scheme of numerical quadrature rules and give some materials to be used later. In section 3, we obtain the main results under the influence of numerical quadrature rules and mappings. Some numerical experiments are contained in section 4.

2. Numerical quadrature rules and some materials

We consider numerical quadrature rules I_k defined on the reference element $\widehat{\Omega}$ by

$$(2.1) \qquad I_k(\widehat{f}) = \sum_{i=1}^{n(k)} \widehat{w}_i^k\,\widehat{f}(\widehat{x}_i^k) \sim \int_{\widehat{\Omega}} \widehat{f}(\widehat{x})\,d\widehat{x},$$

where k is a positive integer. Let $G_p = \{I_k\}$ be a family of quadrature rules I_k with respect to $U_p(\widehat{\Omega})$, $p = 1,2,3,\ldots$, satisfying the following properties : For each $I_k \in G_p$,

(K1) $\widehat{w}_i^k > 0$ and $\widehat{x}_i^k \in \widehat{\Omega}$ for $i = 1,\ldots,n(k)$.

(K2) $I_k(\widehat{f}^2) \le C_1 \|\widehat{f}\|^2_{0,\widehat{\Omega}}$ for all $\widehat{f} \in U_p(\widehat{\Omega})$.

(K3) $C_2 \|\widetilde{f}\|^2_{0,\widehat{\Omega}} \le I_k(\widetilde{f}^2)$ for all $\widetilde{f} \in \widetilde{U}_p(\widehat{\Omega})$,

where $\widetilde{U}_p(\widehat{\Omega}) = \{ \dfrac{\partial \widehat{f}}{\partial \widehat{x}_i} : \widehat{f} \in U_p(\widehat{\Omega}) \} \subset U_p(\widehat{\Omega})$.

(K4) $I_k(\widehat{f}) = \int_{\widehat{\Omega}} \widehat{f}(\widehat{x})\, d\widehat{x}$ for all $\widehat{f} \in U_{d(k)}(\widehat{\Omega})$,

where $d(k) \ge \widetilde{d}(p) > 0$.

We also get a family $G_{p,\Omega} = \{I_{k,\Omega}\}$ of numerical quadrature rules with respect to $U_p(\Omega)$, which are defined on Ω by

$$(2.2)\ I_{k,\Omega}(f) = \sum_{i=1}^{n(k)} w_i^k f(x_i^k) = \sum_{i=1}^{n(k)} \widehat{w}_i^k \widehat{J}(\widehat{x}_i^k)(f \circ T)(\widehat{x}_i^k) = I_k(\widehat{J}\widehat{f}).$$

Now, we denote by DF the $n \times n$ Jacobian matrix of $F : R^n \to R^n$, and define two discrete inner products

$$(2.3)\qquad (u,v)_{l,\Omega} = I_{l,\Omega}(uv) \quad \text{on} \quad \Omega,$$

$$(2.4)\qquad (\widehat{u},\widehat{v})_{l,\widehat{\Omega}} = I_l(\widehat{u}\,\widehat{v}) \quad \text{on} \quad \widehat{\Omega}.$$

Then, using quadrature rules I_m and I_l in G_p we obtain the following actual problem of (1.15): Find $\widetilde{u}_p \in S_{p,0}(\Omega)$, such that

$$(2.5)\qquad B_{m,\Omega}(\widetilde{u}_p, v_p) = (f, v_p)_{l,\Omega} \quad \text{for all} \quad v_p \in S_{p,0}(\Omega),$$

where

$$(2.6)\ B_{m,\Omega}(\widetilde{u}_p, v_p) = \sum_{i,j=1}^{n} \left(\widehat{a}\,\widehat{a}_{ij} \frac{\partial \widehat{\widetilde{u}}_p}{\partial \widehat{x}_i}, \frac{\partial \widehat{v}_p}{\partial \widehat{x}_j} \right)_{m,\widehat{\Omega}} + (\widehat{J}\,\widehat{b}\,\widehat{\widetilde{u}}_p, \widehat{v}_p)_{m,\widehat{\Omega}},$$

$$(2.7)\qquad (f, v_p)_{l,\Omega} = (\widehat{J}\,\widehat{f}, \widehat{v}_p)_{l,\widehat{\Omega}},$$

and $\widehat{a}_{ij}$ denote the entries of the matrix

$$\widehat{J}(\widehat{DT^{-1}})(\widehat{DT^{-1}})^t.$$

The following Lemma gives the ellipticity of $B_{m,\Omega}(\cdot\,,\cdot)$ in our approximate problem (2.5).

LEMMA 2.1. *There exists a constant $C > 0$ such that*

$$(2.8)\qquad C\|v\|^2_{1,\Omega} \le B_{m,\Omega}(v,v) \quad \textit{for all} \quad v \in S_{p,0}(\Omega).$$

Proof. We easily see that $aAA^ta^t \geq (1/\|A^{-1}\|^2)aa^t$ for any invertible matrix A and row-vector a. It follows from (1.11) and (1.12) that

$$\begin{aligned} I_{m,\Omega}(\nabla v \cdot \nabla v) &= I_m(\widehat{J}(\nabla\widehat{v}\widehat{DT^{-1}})(\nabla\widehat{v}\widehat{DT^{-1}})^t) \\ &\geq C\, I_m(\widehat{J}\nabla\widehat{v} \cdot \nabla\widehat{v})\, 1/\|DT\|^2_{0,\infty,\Omega} \\ &\geq C\, I_m(\nabla\widehat{v} \cdot \nabla\widehat{v})\, 1/\|\widehat{J}^{-1}\|_{0,\infty,\Omega}\, 1/\|T\|^2_{1,\infty,\widehat{\Omega}}\,. \end{aligned}$$

Hence, we have from (1.8), (1.9) and (K3) that

$$\begin{aligned} B_{m,\Omega}(v,v) &\geq C\, I_{m,\Omega}(\nabla v \cdot \nabla v) \\ &\geq C\, I_m(\nabla\widehat{v} \cdot \nabla\widehat{v}) \\ &\geq C\, |v|^2_{1,\widehat{\Omega}}\,, \end{aligned}$$

which completes the proof by the Friedrichs' inequality.

The following results shall be used for deriving the estimate $\|u - \widetilde{u}_p\|_{1,\Omega}$.

LEMMA 2.2. *For each integer $l \geq 0$, there exists a sequence of projections*

$$\Pi^l_p : H^l(\widehat{\Omega}) \to U_p(\widehat{\Omega}), \quad p = 1, 2, 3, \ldots, \quad \textit{such that}$$

$$\text{(2.9)} \qquad \Pi^l_p \widehat{v}_p = \widehat{v}_p \quad \textit{for all} \quad \widehat{v}_p \in U_p(\widehat{\Omega}),$$

(2.10)
$$\|\widehat{u} - \Pi^l_p\widehat{u}\|_{s,\widehat{\Omega}} \leq C\, p^{-(r-s)}\|\widehat{u}\|_{r,\widehat{\Omega}} \;\textit{for all}\quad \widehat{u} \in H^r(\widehat{\Omega})$$
$$\textit{with} \quad 0 \leq s \leq l \leq r.$$

Proof. See [5, Lemma 3.1].

LEMMA 2.3. *There exists a sequence of projections*

$$P_p^1 : H_0^1(\Omega) \to S_{p,0}(\Omega), \quad p = 1,2,3,\ldots, \textit{such that}$$

(2.11)
$$\|u - P_p^1 u\|_{s,\Omega} \leq C\, p^{-(r-s)} \|u\|_{r,\Omega} \quad \textit{for all} \quad u \in H_0^r(\Omega)$$
$$\textit{with} \quad 0 \leq s \leq 1 < r.$$

Proof. See [5, Theorem 4.2].

Let $u \in H_0^1(\Omega)$ be the solution of (1.5)–(1.6). Then, we easily see from (1.13) that $\widehat{u} = u \circ T \in H_0^1(\widehat{\Omega})$. But, under the case where T is a non-smooth mapping, $u \in H^r(\Omega)$ does not always guarantee $\widehat{u} \in H^r(\widehat{\Omega})$, $r > 1$. It may be possible to be $\widehat{u} \in H^k(\widehat{\Omega})$ for $k < r$. The following Lemma indicates an estimate for $\|u - u_p\|_{1,\widehat{\Omega}}$, which is slightly different from that of (1.19).

LEMMA 2.4. *Let u_p be an approximation of u which satisfies* (1.15). *We assume that $\widehat{u} = u \circ T \in H^k(\widehat{\Omega})$ under the mapping $T : \widehat{\Omega} \to \Omega$. Then, we have*

$$\|u - u_p\|_{1,\Omega} \leq C\, p^{-(k-1)} \|\widehat{u}\|_{k,\widehat{\Omega}} \,. \tag{2.12}$$

Proof. We easily see from Lemma 2.3 that there exists a sequence of projections

$$\widehat{P}_p^1 : H_0^1(\widehat{\Omega}) \to U_p(\widehat{\Omega}) \cap H_0^1(\widehat{\Omega}), \quad p = 1,2,\ldots, \tag{2.13}$$

such that

$$\|\widehat{w} - \widehat{P}_p^1 \widehat{w}\|_{1,\widehat{\Omega}} \leq C\, p^{-(r-1)} \|\widehat{w}\|_{r,\widehat{\Omega}} \quad \text{for all} \quad \widehat{w} \in H_0^r(\widehat{\Omega}). \tag{2.14}$$

Since $\widehat{u} = u \circ T \in H_0^1(\widehat{\Omega})$, clearly $U_p(\widehat{\Omega}) \cap H_0^1(\widehat{\Omega})$ contains $\widehat{P}_p^1 \widehat{u}$. Let $w_p = (\widehat{P}_p^1 \widehat{u}) \circ T^{-1}$. Then, from (1.13) we see that

$$w_p \in U_p(\Omega) \cap H_0^1(\Omega) = S_{p,0}(\Omega).$$

Hence, it follows from (1.14) and (2.14) that

$$
\begin{aligned}
(2.15) \qquad \|u - w_p\|_{1,\Omega} &\le C\,\|u \circ T - w_p \circ T\|_{1,\widehat{\Omega}} \\
&\le C\,\|\widehat{u} - \widehat{P}_p^1 \widehat{u}\|_{1,\widehat{\Omega}} \\
&\le C\,p^{-(k-1)} \|\widehat{u}\|_{k,\widehat{\Omega}}\,.
\end{aligned}
$$

The Lemma follows from

$$
(2.16) \qquad \|u - u_p\|_{1,\Omega} \le C \inf_{w_p \in S_{p,0}(\Omega)} \|u - w_p\|_{1,\Omega}\,.
$$

Let us now give the following technical Lemma which follows from Lemma 2.2 .

LEMMA 2.5. *For* $\widehat{\Omega} \subset R^n$, *let* $\widehat{u} \in H^s(\widehat{\Omega})$ *with* $s \ge n$. *Then the projection* Π_p^n *from Lemma 2.2 satisfies*

$$
(2.17) \qquad \|\widehat{u} - \Pi_p^n \widehat{u}\|_{0,\infty,\widehat{\Omega}} \le C\,p^{-(s-\frac{n}{2})} \|\widehat{u}\|_{s,\widehat{\Omega}}.
$$

Proof. By interpolation results (see [9, Theorem 3.2] and [11, Theorem 6.2.4]) we have that

$$
(2.18) \qquad \|\widehat{u} - \Pi_p^n \widehat{u}\|_{0,\infty,\widehat{\Omega}} \le C\,\|\widehat{u} - \Pi_p^n \widehat{u}\|^{\frac{1}{2}}_{\frac{n}{2}+\varepsilon,\widehat{\Omega}} \|\widehat{u} - \Pi_p^n \widehat{u}\|^{\frac{1}{2}}_{\frac{n}{2}-\varepsilon,\widehat{\Omega}} \quad \text{for} \quad 0 < \varepsilon \le \tfrac{1}{2}.
$$

We also have from Lemma 2.2 that

$$
(2.19) \qquad \|\widehat{u} - \Pi_p^n \widehat{u}\|_{r,\widehat{\Omega}} \le C\,p^{-(s-r)} \|\widehat{u}\|_{s,\widehat{\Omega}} \quad \text{for} \quad 0 \le r \le n \le s.
$$

Hence, taking with $r = \frac{n}{2} + \varepsilon$ and $r = \frac{n}{2} - \varepsilon$ in (2.19) we obtain $\|\widehat{u} - \Pi_p^n \widehat{u}\|^{\frac{1}{2}}_{\frac{n}{2}+\varepsilon,\widehat{\Omega}} \|\widehat{u} - \Pi_p^n \widehat{u}\|^{\frac{1}{2}}_{\frac{n}{2}-\varepsilon,\widehat{\Omega}} \le C\,p^{-(s-\frac{n}{2})} \|\widehat{u}\|_{s,\widehat{\Omega}}$, which completes the proof from (2.18).

3. $H^1(\Omega)$-norm error estimate under numerical quadrature rules and mappings

$\|u-\widetilde{u}_p\|_{1,\Omega}$ depends on several separate terms. The first dependence is on the error $\|u-u_p\|_{1,\Omega}$ with respect to the mapping T discussed in previous section 2. Next, the error will depend upon the smoothness of $\widehat{a}$, $\widehat{a}_{ij}$, $\widehat{b}$ and $\widehat{f}$ with the Jacobian $\widehat{J}$ of T.

LEMMA 3.1. *Let u be the exact solution of* (1.5)–(1.6) *and $\widetilde{u}_p$ an approximation of u which satisfies* (2.5). *Then there exists a constant C independent of m, l such that*

$$\begin{aligned}\|u-\widetilde{u}_p\|_{1,\Omega} \leq C\Big[&\inf_{u_p\in S_{p,0}(\Omega)}\Big\{ \|u-u_p\|_{1,\Omega} \\ &+ \sup_{w_p\in S_{p,0}(\Omega)} \frac{|B(u_p,w_p)-B_{m,\Omega}(u_p,w_p)|}{\|w_p\|_{1,\Omega}}\Big\} \\ &+ \sup_{w_p\in S_{p,0}(\Omega)} \frac{|(f,w_p)_\Omega-(f,w_p)_{l,\Omega}|}{\|w_p\|_{1,\Omega}}\Big].\end{aligned}$$

Proof. It is similar to the technique in [1, Theorem 4.1.1].

In Lemma 3.1, the third factor that $\|u-\widetilde{u}_p\|_{1,\Omega}$ depends upon is the smoothness of $\widehat{f}$ and $\widehat{J}$ with the mapping T. In this connection, we shall use the following Lemma.

LEMMA 3.2. *Let $I_l \in G_p$ be a quadrature rule on $\widehat{\Omega}\subset R^n$ which satisfies $d(l)-p-1>0$, and let $\widehat{f}\in H^\gamma(\widehat{\Omega})$ and $\widehat{J}\in H^\delta(\widehat{\Omega})$ with $min(\gamma,\delta)\geq n$. Then, for any $w_p\in S_{p,0}(\Omega)$ we have the following estimate*

$$\begin{aligned}(3.2)\qquad &\frac{|(f,w_p)_\Omega-(f,w_p)_{l,\Omega}|}{\|w_p\|_{1,\Omega}} \\ &\leq C\{\, q^{-(\gamma-\frac{n}{2})}\|\widehat{f}\|_{\gamma,\widehat{\Omega}}(\,\|\widehat{J}\|_{0,\infty,\widehat{\Omega}}+\|\widehat{J}\|_{\delta,\widehat{\Omega}}\,) \\ &\quad +(d(l)-p-q)^{-(\delta-\frac{n}{2})}\|\widehat{J}\|_{\delta,\widehat{\Omega}}\,(\,\|\widehat{f}\|_{0,\infty,\widehat{\Omega}}+\|\widehat{f}\|_{\gamma,\widehat{\Omega}}\,)\,\},\end{aligned}$$

where q is a positive integer with $d(l)-p-q>0$ and C is independent of l, p and q.

Proof. Since $d(l) - p - 1 > 0$ there exists a positive integer q such that $d(l) - p - q > 0$. For arbitrary $\widehat{w}_1 \in U_{d(l)-p-q}(\widehat{\Omega})$ and $\widehat{w}_2 \in U_q(\widehat{\Omega})$ we let $\widehat{w} = \widehat{w}_1 \widehat{w}_2 \in U_{d(l)-p}(\widehat{\Omega})$. Then, due to (K4) it follows that

$$(\widehat{w}, \widehat{w}_p)_{l,\widehat{\Omega}} - (\widehat{w}, \widehat{w}_p)_{\widehat{\Omega}} = 0. \tag{3.3}$$

Since $(f, w_p)_\Omega = (\widehat{J}\widehat{f}, \widehat{w}_p)_{\widehat{\Omega}}$ and $(f, w_p)_{l,\Omega} = (\widehat{J}\widehat{f}, \widehat{w}_p)_{l,\widehat{\Omega}}$

$$\begin{aligned} &|(f, w_p)_\Omega - (f, w_p)_{l,\Omega}| \\ &\quad \le |(\widehat{J}\widehat{f}, \widehat{w}_p)_{\widehat{\Omega}} - (\widehat{w}, \widehat{w}_p)_{\widehat{\Omega}}| + |(\widehat{w}, \widehat{w}_p)_{l,\widehat{\Omega}} - (\widehat{J}\widehat{f}, \widehat{w}_p)_{l,\widehat{\Omega}}|. \end{aligned} \tag{3.4}$$

By the Schwarz inequality we obtain

$$\begin{aligned} &|(\widehat{J}\widehat{f}, \widehat{w}_p)_{\widehat{\Omega}} - (\widehat{w}, \widehat{w}_p)_{\widehat{\Omega}}| \\ &\le |(\widehat{J}\widehat{f} - \widehat{J}\widehat{w}_2, \widehat{w}_p)_{\widehat{\Omega}}| + |(\widehat{J}\widehat{w}_2 - \widehat{w}_1\widehat{w}_2, \widehat{w}_p)_{\widehat{\Omega}}| \\ &\le \|\widehat{J}(\widehat{f} - \widehat{w}_2)\|_{0,\widehat{\Omega}} \|\widehat{w}_p\|_{0,\widehat{\Omega}} + \|(\widehat{J} - \widehat{w}_1)\widehat{w}_2\|_{0,\widehat{\Omega}} \|\widehat{w}_p\|_{0,\widehat{\Omega}} \\ &\le (\|\widehat{J}\|_{0,\widehat{\Omega}} \|\widehat{f} - \widehat{w}_2\|_{0,\infty,\widehat{\Omega}} + \|\widehat{J} - \widehat{w}_1\|_{0,\infty,\widehat{\Omega}} \|\widehat{w}_2\|_{0,\widehat{\Omega}}) \|\widehat{w}_p\|_{0,\widehat{\Omega}} . \end{aligned} \tag{3.5}$$

Taking $\widehat{w}_1 = \Pi^n_{d(l)-p-q}(\widehat{J})$ and $\widehat{w}_2 = \Pi^n_q(\widehat{f})$ in Lemma 2.5 we have

$$\|\widehat{f} - \widehat{w}_2\|_{0,\infty,\widehat{\Omega}} \le C\, q^{-(\gamma - \frac{n}{2})} \|\widehat{f}\|_{\gamma,\widehat{\Omega}} , \tag{3.6}$$

and

$$\|\widehat{J} - \widehat{w}_1\|_{0,\infty,\widehat{\Omega}} \le C\, (d(l) - p - q)^{-(\delta - \frac{n}{2})} \|\widehat{J}\|_{\delta,\widehat{\Omega}} . \tag{3.7}$$

Moreover, by the triangle inequality and from Lemma 2.2

$$\begin{aligned} \|\widehat{w}_2\|_{0,\widehat{\Omega}} &\le \|\widehat{f}\|_{0,\widehat{\Omega}} + \|\widehat{f} - \widehat{w}_2\|_{0,\widehat{\Omega}} \\ &\le C\{ \|\widehat{f}\|_{\gamma,\widehat{\Omega}} + q^{-\gamma} \|\widehat{f}\|_{\gamma,\widehat{\Omega}} \} \\ &\le C \|\widehat{f}\|_{\gamma,\widehat{\Omega}} , \end{aligned} \tag{3.8}$$

and obviously

$$\|\widehat{J}\|_{0,\widehat{\Omega}} \le C \|\widehat{J}\|_{\delta,\widehat{\Omega}} . \tag{3.9}$$

Hence, by substituting the above results in (3.5) we have

$$(3.10)\quad |(\widehat{J}\widehat{f},\widehat{w}_p)_{\widehat{\Omega}} - (\widehat{w},\widehat{w}_p)_{\widehat{\Omega}}|$$
$$\le C\,\{q^{-(\gamma-\frac{n}{2})} + (d(l)-p-q)^{-(\delta-\frac{n}{2})}\}\,\|\widehat{f}\|_{\gamma,\widehat{\Omega}}\,\|\widehat{J}\|_{\delta,\widehat{\Omega}}\,\|\widehat{w}_p\|_{0,\widehat{\Omega}}\,.$$

Similarly, we can estimate the last term of the right side in (3.4), which can be rewritten as

$$(3.11)\quad |(\widehat{J}\widehat{f},\widehat{w}_p)_{l,\widehat{\Omega}} - (\widehat{w},\widehat{w}_p)_{l,\widehat{\Omega}}|$$
$$\le |(\widehat{J}\widehat{f},\widehat{w}_p)_{l,\widehat{\Omega}} - (\widehat{J}\,\widehat{w}_2,\widehat{w}_p)_{l,\widehat{\Omega}}| + |(\widehat{J}\,\widehat{w}_2,\widehat{w}_p)_{l,\widehat{\Omega}} - (\widehat{w}_1\widehat{w}_2,\widehat{w}_p)_{l,\widehat{\Omega}}|$$
$$= |(\widehat{J}(\widehat{f}-\widehat{w}_2),\widehat{w}_p)_{l,\widehat{\Omega}}| + |(\widehat{w}_2(\widehat{J}-\widehat{w}_1),\widehat{w}_p)_{l,\widehat{\Omega}}|\,.$$

Using the Schwarz inequality, we have from (3.6) and (K2) that

$$(3.12)\ |(\widehat{J}(\widehat{f}-\widehat{w}_2),\widehat{w}_p)_{l,\widehat{\Omega}}| \le (\widehat{J}(\widehat{f}-\widehat{w}_2),\ \widehat{J}(\widehat{f}-\widehat{w}_2))^{\frac{1}{2}}_{l,\widehat{\Omega}}\,(\widehat{w}_p,\widehat{w}_p)^{\frac{1}{2}}_{l,\widehat{\Omega}}$$
$$\le C\,\|\widehat{J}\|_{0,\infty,\widehat{\Omega}}\,\|\widehat{f}-\widehat{w}_2\|_{0,\infty,\widehat{\Omega}}\,\|\widehat{w}_p\|_{0,\widehat{\Omega}}$$
$$\le C\,q^{-(\gamma-\frac{n}{2})}\|\widehat{f}\|_{\gamma,\widehat{\Omega}}\,\|\widehat{J}\|_{0,\infty,\widehat{\Omega}}\,\|\widehat{w}_p\|_{0,\widehat{\Omega}}\ .$$

Moreover, from (3.6) and (3.7) we also obtain

$$(3.13)\quad |(\widehat{w}_2(\widehat{J}-\widehat{w}_1),\widehat{w}_p)_{l,\widehat{\Omega}}|$$
$$\le (\widehat{w}_2(\widehat{J}-\widehat{w}_1),\widehat{w}_2(\widehat{J}-\widehat{w}_1))^{\frac{1}{2}}_{l,\widehat{\Omega}}\,(\widehat{w}_p,\widehat{w}_p)^{\frac{1}{2}}_{l,\widehat{\Omega}}$$
$$\le C\,\|\widehat{J}-\widehat{w}_1\|_{0,\infty,\widehat{\Omega}}\,\|\widehat{w}_2\|_{0,\infty,\widehat{\Omega}}\,\|\widehat{w}_p\|_{0,\widehat{\Omega}}$$
$$\le C\,\|\widehat{J}-\widehat{w}_1\|_{0,\infty,\widehat{\Omega}}\,(\|\widehat{f}\|_{0,\infty,\widehat{\Omega}} + \|\widehat{f}-\widehat{w}_2\|_{0,\infty,\widehat{\Omega}})\,\|\widehat{w}_p\|_{0,\widehat{\Omega}}$$
$$\le C\,\{\,(d(l)-p-q)^{-(\delta-\frac{n}{2})}\|\widehat{J}\|_{\delta,\widehat{\Omega}}\,\|\widehat{f}\|_{0,\infty,\widehat{\Omega}}\,\|\widehat{w}_p\|_{0,\widehat{\Omega}}$$
$$+(d(l)-p-q)^{-(\delta-\frac{n}{2})}q^{-(\gamma-\frac{n}{2})}\,\|\widehat{J}\|_{\delta,\widehat{\Omega}}\,\|\widehat{f}\|_{\gamma,\widehat{\Omega}}\,\|\widehat{w}_p\|_{0,\widehat{\Omega}}\,\}.$$

Hence, combining (3.12) and (3.13) we estimate

$$(3.14)\quad |(\widehat{J}\widehat{f},\widehat{w}_p)_{l,\widehat{\Omega}} - (\widehat{w},\widehat{w}_p)_{l,\widehat{\Omega}}|$$
$$\le C\,\{q^{-(\gamma-\frac{n}{2})}\|\widehat{J}\|_{0,\infty,\widehat{\Omega}}\,\|\widehat{f}\|_{\gamma,\widehat{\Omega}}$$
$$+(d(l)-p-q)^{-(\delta-\frac{n}{2})}\|\widehat{J}\|_{\delta,\widehat{\Omega}}\,\|\widehat{f}\|_{0,\infty,\widehat{\Omega}}$$
$$+q^{-(\gamma-\frac{n}{2})}(d(l)-p-q)^{-(\delta-\frac{n}{2})}\|\widehat{J}\|_{\delta,\widehat{\Omega}}\,\|\widehat{f}\|_{\gamma,\widehat{\Omega}}\,\}\,\|\widehat{w}_p\|_{0,\widehat{\Omega}}\,.$$

Since the last term of the right side in (3.14) is dominated by the terms in (3.10) we derive

$$
\begin{aligned}
(3.15)\qquad & |(f,w_p)_\Omega-(f,w_p)_{l,\Omega}| \\
& \le C\{q^{-(\gamma-\frac{n}{2})}\|\widehat{f}\|_{\gamma,\widehat{\Omega}}(\|\widehat{J}\|_{0,\infty,\widehat{\Omega}}+\|\widehat{J}\|_{\delta,\widehat{\Omega}}) \\
& \quad + (d(l)-p-q)^{-(\delta-\frac{n}{2})}\|\widehat{J}\|_{\delta,\widehat{\Omega}}(\|\widehat{f}\|_{0,\infty,\widehat{\Omega}}+\|\widehat{f}\|_{\gamma,\widehat{\Omega}}\}\,\|\widehat{w}_p\|_{0,\widehat{\Omega}}\,.
\end{aligned}
$$

It is obvious from (1.14) that

$$
(3.16)\qquad \|\widehat{w}_p\|_{0,\widehat{\Omega}} \le C\,\|\widehat{w}_p\|_{1,\widehat{\Omega}} \le C\,\|w_p\|_{1,\Omega}\,.
$$

The Lemma follows from dividing with $\|w_p\|_{1,\Omega}$.

Now, we give the following Lemma which can be used for estimating the middle term in (3.1).

LEMMA 3.3. *Let* $\widehat{u}_p$, $\widehat{w}_p \in U_p(\widehat{\Omega})$ *and* $\widehat{f} \in L_\infty(\widehat{\Omega})$. *Then, for all* $\widehat{v}_q \in U_q(\widehat{\Omega})$, $\widehat{f}_r \in U_r(\widehat{\Omega})$ *with* $0<q\le p$ *and* $r=d(m)-p-q>0$ *we have*

$$
\begin{aligned}
(3.17)\qquad & |(\widehat{f}\,\widehat{u}_p,\widehat{w}_p)_{\widehat{\Omega}}-(\widehat{f}\,\widehat{u}_p,\widehat{w}_p)_{m,\widehat{\Omega}}| \\
& \le C\{\|\widehat{f}_r\|_{0,\infty,\widehat{\Omega}}\|\widehat{u}_p-\widehat{v}_q\|_{0,\widehat{\Omega}}+\|\widehat{f}-\widehat{f}_r\|_{0,\infty,\widehat{\Omega}}\|\widehat{u}_p\|_{0,\widehat{\Omega}}\}\,\|\widehat{w}_p\|_{0,\widehat{\Omega}}\ ,
\end{aligned}
$$

where C *is independent of* p, q *and* m.

Proof. For any $\widehat{f}_r \in U_r(\widehat{\Omega})$ we have

$$
\begin{aligned}
(3.18)\qquad & |(\widehat{f}\,\widehat{u}_p,\widehat{w}_p)_{\widehat{\Omega}}-(\widehat{f}\,\widehat{u}_p,\widehat{w}_p)_{m,\widehat{\Omega}}| \\
& \le |(\widehat{f}\,\widehat{u}_p,\widehat{w}_p)_{\widehat{\Omega}}-(\widehat{f}_r\widehat{u}_p,\widehat{w}_p)_{\widehat{\Omega}}| + |(\widehat{f}_r\widehat{u}_p,\widehat{w}_p)_{\widehat{\Omega}}-(\widehat{f}_r\widehat{u}_p,\widehat{w}_p)_{m,\widehat{\Omega}}| \\
& \quad + |(\widehat{f}_r\widehat{u}_p,\widehat{w}_p)_{m,\widehat{\Omega}}-(\widehat{f}\,\widehat{u}_p,\widehat{w}_p)_{m,\widehat{\Omega}}|\,.
\end{aligned}
$$

Thank to (K4),

$$
(3.19)\qquad (\widehat{f}_r\widehat{v}_q,\widehat{w}_p)_{\widehat{\Omega}}-(\widehat{f}_r\widehat{v}_q,\widehat{w}_p)_{m,\widehat{\Omega}} = 0 \quad \text{for any} \quad \widehat{v}_q \in U_q(\widehat{\Omega}).
$$

Hence,

$$(3.20)\qquad \begin{aligned} & |(\widehat{f}_r\widehat{u}_p,\widehat{w}_p)_{\widehat{\Omega}}-(\widehat{f}_r\widehat{u}_p,\widehat{w}_p)_{m,\widehat{\Omega}}| \\ \le\ & |(\widehat{f}_r\widehat{u}_p,\widehat{w}_p)_{\widehat{\Omega}}-(\widehat{f}_r\widehat{v}_q,\widehat{w}_p)_{\widehat{\Omega}}| + |(\widehat{f}_r\widehat{v}_q,\widehat{w}_p)_{m,\widehat{\Omega}}-(\widehat{f}_r\widehat{u}_p,\widehat{w}_p)_{m,\widehat{\Omega}}|. \end{aligned}$$

By the Schwarz inequality we obtain

$$(3.21)\qquad \begin{aligned} & |(\widehat{f}_r\widehat{u}_p,\widehat{w}_p)_{\widehat{\Omega}}-(\widehat{f}_r\widehat{v}_q,\widehat{w}_p)_{\widehat{\Omega}}| \\ \le\ & (\widehat{f}_r(\widehat{u}_p-\widehat{v}_q),\widehat{f}_r(\widehat{u}_p-\widehat{v}_q))^{\frac{1}{2}}_{\widehat{\Omega}}\,(\widehat{w}_p,\widehat{w}_p)^{\frac{1}{2}}_{\widehat{\Omega}} \\ \le\ & C\,\|\widehat{f}_r\|_{0,\infty,\widehat{\Omega}}\|\widehat{u}_p-\widehat{v}_q\|_{0,\widehat{\Omega}}\,\|\widehat{w}_p\|_{0,\widehat{\Omega}}\ . \end{aligned}$$

Also, from (K2) we have

$$(3.22)\qquad \begin{aligned} & |(\widehat{f}_r\widehat{v}_q,\widehat{w}_p)_{m,\widehat{\Omega}}-(\widehat{f}_r\widehat{u}_p,\widehat{w}_p)_{m,\widehat{\Omega}}| \\ \le\ & (\widehat{f}_r(\widehat{u}_p-\widehat{v}_q),\widehat{f}_r(\widehat{u}_p-\widehat{v}_q))^{\frac{1}{2}}_{m,\widehat{\Omega}}(\widehat{w}_p,\widehat{w}_p)^{\frac{1}{2}}_{m,\widehat{\Omega}} \\ \le\ & C\|\widehat{f}_r\|_{0,\infty,\widehat{\Omega}}(\widehat{u}_p-\widehat{v}_q,\widehat{u}_p-\widehat{v}_q)^{\frac{1}{2}}_{m,\widehat{\Omega}}(\widehat{w}_p,\widehat{w}_p)^{\frac{1}{2}}_{m,\widehat{\Omega}} \\ \le\ & C\|\widehat{f}_r\|_{0,\infty,\widehat{\Omega}}\|\widehat{u}_p-\widehat{v}_q\|_{0,\widehat{\Omega}}\|\widehat{w}_p\|_{0,\widehat{\Omega}}\ . \end{aligned}$$

Hence, combining (3.21) and (3.22) we estimate

$$(3.23)\qquad \begin{aligned} & |(\widehat{f}_r\widehat{u}_p,\widehat{w}_p)_{\widehat{\Omega}}-(\widehat{f}_r\widehat{u}_p,\widehat{w}_p)_{m,\widehat{\Omega}}| \\ \le\ & C\,\|\widehat{f}_r\|_{0,\infty,\widehat{\Omega}}\|\widehat{u}_p-\widehat{v}_q\|_{0,\widehat{\Omega}}\|\widehat{w}_p\|_{0,\widehat{\Omega}}\ . \end{aligned}$$

Similarly, since $\widehat{f}\in L_\infty(\widehat{\Omega})$ we obtain

$$(3.24)\qquad \begin{aligned} & |(\widehat{f}\,\widehat{u}_p,\widehat{w}_p)_{\widehat{\Omega}}-(\widehat{f}_r\widehat{u}_p,\widehat{w}_p)_{\widehat{\Omega}}| \\ \le\ & ((\widehat{f}-\widehat{f}_r)\widehat{u}_p,(\widehat{f}-\widehat{f}_r)\widehat{u}_p)^{\frac{1}{2}}_{\widehat{\Omega}}\,(\widehat{w}_p,\widehat{w}_p)^{\frac{1}{2}}_{\widehat{\Omega}} \\ \le\ & C\,\|\widehat{f}-\widehat{f}_r\|_{0,\infty,\widehat{\Omega}}\|\widehat{u}_p\|_{0,\widehat{\Omega}}\|\widehat{w}_p\|_{0,\widehat{\Omega}}\ , \end{aligned}$$

and

$$(3.25)\qquad \begin{aligned} & |(\widehat{f}_r\widehat{u}_p,\widehat{w}_p)_{m,\widehat{\Omega}}-(\widehat{f}\,\widehat{u}_p,\widehat{w}_p)_{m,\widehat{\Omega}}| \\ \le\ & ((\widehat{f}_r-f)\widehat{u}_p,(\widehat{f}_r-\widehat{f})\widehat{u}_p)^{\frac{1}{2}}_{m,\widehat{\Omega}}(\widehat{w}_p,\widehat{w}_p)^{\frac{1}{2}}_{m,\widehat{\Omega}} \\ \le\ & C\,\|\widehat{f}_r-\widehat{f}\|_{0,\infty,\widehat{\Omega}}(\widehat{u}_p,\widehat{u}_p)^{\frac{1}{2}}_{m,\widehat{\Omega}}(\widehat{w}_p,\widehat{w}_p)^{\frac{1}{2}}_{m,\widehat{\Omega}} \\ \le\ & C\,\|\widehat{f}_r-\widehat{f}\|_{0,\infty,\widehat{\Omega}}\|\widehat{u}_p\|_{0,\widehat{\Omega}}\|\widehat{w}_p\|_{0,\widehat{\Omega}}\ . \end{aligned}$$

The Lemma follows from (3.23), (3.24), (3.25) and (3.18).

For any $\widehat{f} \in H^r(\widehat{\Omega})$ with $\widehat{\Omega} \subset R^n$ and $r \geq n$ we denote

$$K_s(\widehat{f}) = \|\Pi_s^n \widehat{f}\|_{0,\infty,\widehat{\Omega}} \,. \tag{3.26}$$

Then, we easily see from Lemma 2.2 that

$$\begin{aligned} K_s(\widehat{f}) &\leq C\{\|\widehat{f}\|_{0,\infty,\widehat{\Omega}} + s^{-(r-\frac{n}{2})}\|\widehat{f}\|_{r,\widehat{\Omega}}\} \\ &\leq C\{\|\widehat{f}\|_{0,\infty,\widehat{\Omega}} + \|\widehat{f}\|_{r,\widehat{\Omega}}\}. \end{aligned} \tag{3.27}$$

Let us define

$$M_{p,q} = \max_{i,j} \|\widehat{a}_{ij}\|_{p,q,\widehat{\Omega}} \,, \tag{3.28}$$

where the subscript q will be omitted when $q = 2$.

LEMMA 3.4. *Let $I_m \in G_p$ be a quadrature rule defined on $\widehat{\Omega} \subset R^n$, which satisfies $d(m) - p - 1 > 0$. Let $\widehat{u} \in H^\sigma(\widehat{\Omega})$, $\widehat{a} \in H^\alpha(\widehat{\Omega})$, $\widehat{b} \in H^\beta(\widehat{\Omega})$, $\widehat{J} \in H^\delta(\widehat{\Omega})$ and $\widehat{a}_{ij} \in H^\rho(\widehat{\Omega})$ for $i,j = 1,\ldots,n$, such that $k_1 = \min(\alpha,\rho) \geq n$ and $k_2 = \min(\beta,\delta) \geq n$. Then, for any $w_p \in S_{p,0}(\Omega)$ and an approximation u_p which satisfies (1.15) we have*

$$\begin{aligned} &\frac{|B(u_p,w_p) - B_m(u_p,w_p)|}{\|w_p\|_{1,\Omega}} \\ &\leq C\{\min(q_1,q_2)^{-(\sigma-1)}\|\widehat{u}\|_{\sigma,\widehat{\Omega}} \\ &\quad + (r_1^{-(k_1-\frac{n}{2})}\|\widehat{a}\|_{\alpha,\widehat{\Omega}} M_\rho + r_2^{-(k_2-\frac{n}{2})}\|\widehat{J}\|_{\delta,\widehat{\Omega}}\|\widehat{b}\|_{\beta,\widehat{\Omega}})\|\widehat{u}\|_{1,\widehat{\Omega}}\}, \end{aligned} \tag{3.29}$$

where q_1, q_2 are two positive integers such that $0 < q_i \leq p$ and $r_i = d(m) - p - q_i > 0$ for $i = 1,2$.

Proof. For arbitrary $w_p \in S_{p,0}(\Omega)$ we have

$$\begin{aligned} &|B(u_p,w_p) - B_m(u_p,w_p)| \\ &\leq C\{\max_{i,j}|\left(\widehat{a}\,\widehat{a}_{ij}\frac{\partial \widehat{u}_p}{\partial \widehat{x}_i}, \frac{\partial \widehat{w}_p}{\partial \widehat{x}_j}\right)_{\widehat{\Omega}} - \left(\widehat{a}\,\widehat{a}_{ij}\frac{\partial \widehat{u}_p}{\partial \widehat{x}_i}, \frac{\partial \widehat{w}_p}{\partial \widehat{x}_j}\right)_{m,\widehat{\Omega}}| \end{aligned} \tag{3.30}$$

$$+ |(\widehat{J}\,\widehat{b}\,\widehat{u}_p, \widehat{w}_p)_{\widehat{\Omega}} - (\widehat{J}\,\widehat{b}\,\widehat{u}_p, \widehat{w}_p)_{m,\widehat{\Omega}}\,|\,\}.$$

We begin by estimating the last term in the right side of (3.30). It is obvious from (1.9) and (1.12) that $\widehat{J}\,\widehat{b} \in L_\infty(\widehat{\Omega})$. Since $d(m)-p-1 > 0$ it allows us to take q_2 such that $r_2 = d(m) - p - q_2 > 0$ and $0 < q_2 \le p$. Hence, using Lemma 3.3 with $\widehat{v}_q = \Pi^1_{q_2}\widehat{u}_p \in U_{q_2}(\widehat{\Omega})$ and $\widehat{f}_r = \Pi^n_{r_2}(\widehat{J}\,\widehat{b}) \in U_{r_2}(\widehat{\Omega})$ we obtain

$$\begin{aligned}(3.31)\qquad & |(\widehat{J}\,\widehat{b}\,\widehat{u}_p, \widehat{w}_p)_{\widehat{\Omega}} - (\widehat{J}\,\widehat{b}\,\widehat{u}_p, \widehat{w}_p)_{m,\widehat{\Omega}}\,| \\ & \le C\,\{\,\|\Pi^n_{r_2}(\widehat{J}\,\widehat{b})\|_{0,\infty,\widehat{\Omega}}\|\widehat{u}_p - \Pi^1_{q_2}\widehat{u}_p\|_{0,\widehat{\Omega}} \\ & \quad + \|\widehat{J}\,\widehat{b} - \Pi^n_{r_2}(\widehat{J}\,\widehat{b})\|_{0,\infty,\widehat{\Omega}}\|\widehat{u}_p\|_{0,\widehat{\Omega}}\,\}\,\|\widehat{w}_p\|_{0,\widehat{\Omega}}.\end{aligned}$$

Since $\widehat{J}\,\widehat{b} \in H^n(\widehat{\Omega})$ and from (3.27), clearly $K_{r_2}(\widehat{J}\,\widehat{b})$ is bounded by a fixed constant for all $r_2 = d(m) - p - q_2 > 0$. We see from (1.14) that

$$(3.32)\qquad \|\widehat{u} - \widehat{u}_p\|_{1,\widehat{\Omega}} \le C\,\|u - u_p\|_{1,\Omega},$$

and from Lemma 2.2 with $r = s = 1$ the boundedness of the projection $\Pi^1_{q_2}$ follows:

$$(3.33)\qquad \|\Pi^1_{q_2}(\widehat{u} - \widehat{u}_p)\|_{1,\widehat{\Omega}} \le C\,\|\widehat{u} - \widehat{u}_p\|_{1,\widehat{\Omega}}.$$

Thus, using the triangle inequality and from Lemma 2.4 and Lemma 2.2 we have

$$\begin{aligned}(3.34)\qquad & \|\widehat{u}_p - \Pi^1_{q_2}\widehat{u}_p\|_{0,\widehat{\Omega}} \le C\,\|\widehat{u}_p - \Pi^1_{q_2}\widehat{u}_p\|_{1,\widehat{\Omega}} \\ & \le C\{\,\|\widehat{u} - \widehat{u}_p\|_{1,\widehat{\Omega}} + \|\widehat{u} - \Pi^1_{q_2}\widehat{u}\|_{1,\widehat{\Omega}} + \|\Pi^1_{q_2}(\widehat{u} - \widehat{u}_p)\|_{1,\widehat{\Omega}}\,\} \\ & \le C\,\{\,\|\widehat{u} - \widehat{u}_p\|_{1,\widehat{\Omega}} + \|\widehat{u} - \Pi^1_{q_2}\widehat{u}\|_{1,\widehat{\Omega}}\,\} \\ & \le C\,\{\,p^{-(\sigma-1)} + q_2^{-(\sigma-1)}\,\}\,\|\widehat{u}\|_{\sigma,\widehat{\Omega}} \\ & \le C\,q_2^{-(\sigma-1)}\|\widehat{u}\|_{\sigma,\widehat{\Omega}}\ .\end{aligned}$$

Moreover, since $\widehat{J}\,\widehat{b} \in H^{k_2}(\widehat{\Omega})$ with $k_2 = \min(\beta, \sigma) \ge n$ it follows from Lemma 2.5 that

$$(3.35)\qquad \|\widehat{J}\,\widehat{b} - \Pi^n_{r_2}(\widehat{J}\,\widehat{b})\|_{0,\infty,\widehat{\Omega}}$$

$$\leq C r_2^{-(k_2-\frac{n}{2})} \|\widehat{J}\,\widehat{b}\|_{k_2,\widehat{\Omega}} \leq C r_2^{(k_2-\frac{n}{2})} \|\widehat{J}\|_{\delta,\widehat{\Omega}} \|\widehat{b}\|_{\beta,\widehat{\Omega}} ,$$

and clearly

$$\|\widehat{u}_p\|_{0,\widehat{\Omega}} \leq \|\widehat{u}_p\|_{1,\widehat{\Omega}} \leq C\|u_p\|_{1,\Omega} \leq C\|\widehat{u}\|_{1,\widehat{\Omega}} . \tag{3.36}$$

Hence, the above results yields from (3.31) that

$$\begin{aligned} &|(\widehat{J}\,\widehat{b}\,\widehat{u}_p, \widehat{w}_p)_{\widehat{\Omega}} - (\widehat{J}\,\widehat{b}\,\widehat{u}_p, \widehat{w}_p)_{m,\widehat{\Omega}}| \\ &\quad \leq C\{ q_2^{-(\sigma-1)} \|\widehat{u}\|_{\sigma,\widehat{\Omega}} + r_2^{(k_2-\frac{n}{2})} \|\widehat{J}\|_{\delta,\widehat{\Omega}} \|\widehat{b}\|_{\beta,\widehat{\Omega}} \|\widehat{u}\|_{1,\widehat{\Omega}} \} \|\widehat{w}_p\|_{0,\widehat{\Omega}} . \end{aligned} \tag{3.37}$$

We finally estimate the first term in (3.30). For any $\widehat{a}_{ij}$ $i,j = 1,\ldots,n$ we let q_1 be any integer such that $0 < q_1 \leq p$ and $r_1 = d(m) - p - q_1 > 0$. Then, since $\widehat{a}\,\widehat{a}_{ij} \in L_\infty(\widehat{\Omega})$, due to Lemma 3.3 with $\widehat{v}_q = \frac{\partial}{\partial \widehat{x}_i}(\Pi^1_{q_1}\widehat{u}_p)$ and $\widehat{f}_r = \Pi^n_{r_1}(\widehat{a}\,\widehat{a}_{ij})$, we have

$$\begin{aligned} &\left| \left(\widehat{a}\,\widehat{a}_{ij} \frac{\partial \widehat{u}_p}{\partial \widehat{x}_i}, \frac{\partial \widehat{w}_p}{\partial \widehat{x}_j} \right)_{\widehat{\Omega}} - \left(\widehat{a}\,\widehat{a}_{ij} \frac{\partial \widehat{u}_p}{\partial \widehat{x}_i}, \frac{\partial \widehat{w}_p}{\partial \widehat{x}_j} \right)_{m,\widehat{\Omega}} \right| \\ &\qquad \leq C\{ \|\Pi^n_{r_1}(\widehat{a}\,\widehat{a}_{ij})\|_{0,\infty,\widehat{\Omega}} \|\frac{\partial \widehat{u}_p}{\partial \widehat{x}_i} - \frac{\partial}{\partial \widehat{x}_i}(\Pi^1_{q_1}\widehat{u}_p)\|_{0,\widehat{\Omega}} \\ &\qquad\quad + \|\widehat{a}\,\widehat{a}_{ij} - \Pi^n_{r_1}(\widehat{a}\,\widehat{a}_{ij})\|_{0,\infty,\widehat{\Omega}} \|\frac{\partial \widehat{u}_p}{\partial \widehat{x}_i}\|_{0,\widehat{\Omega}} \} \|\frac{\partial \widehat{w}_p}{\partial \widehat{x}_j}\|_{0,\widehat{\Omega}} . \end{aligned} \tag{3.38}$$

Using Lemma 2.2 and Lemma 2.4 we easily see from the boundedness of $\Pi^1_{q_1}$ that

$$\begin{aligned} &\|\frac{\partial \widehat{u}_p}{\partial \widehat{x}_i} - \frac{\partial}{\partial \widehat{x}_i}(\Pi^1_{q_1}\widehat{u}_p)\|_{0,\widehat{\Omega}} \\ &\qquad \leq C\,\|\widehat{u}_p - \Pi^1_{q_1}\widehat{u}_p\|_{1,\widehat{\Omega}} \leq C\,q_1^{-(\sigma-1)} \|\widehat{u}\|_{\sigma,\widehat{\Omega}} . \end{aligned} \tag{3.39}$$

Also, clearly

$$\|\frac{\partial \widehat{u}_p}{\partial \widehat{x}_i}\|_{0,\widehat{\Omega}} \leq C\,\|\widehat{u}_p\|_{1,\widehat{\Omega}} \leq C\,\|\widehat{u}\|_{1,\widehat{\Omega}} , \tag{3.40}$$

and

$$\|\frac{\partial \widehat{w}_p}{\partial \widehat{x}_j}\|_{0,\widehat{\Omega}} \leq C\,\|\widehat{w}_p\|_{1,\widehat{\Omega}} . \tag{3.41}$$

Moreover, since $\widehat{a}\,\widehat{a}_{ij} \in H^{k_1}(\widehat{\Omega})$ with $k_1 = \min(\alpha,\rho) \geq n$ we obtain from Lemma 2.5 that

$$(3.42) \qquad \|\widehat{a}\,\widehat{a}_{ij} - \Pi^n_{r_1}(\widehat{a}\,\widehat{a}_{ij})\|_{0,\infty,\widehat{\Omega}} \leq C\, r_1^{-(k_1-\frac{n}{2})}\|\widehat{a}\|_{\alpha,\widehat{\Omega}} M_\rho \;.$$

So, from (3.39)-(3.42) and since $\|\Pi^n_{r_1}(\widehat{a}\,\widehat{a}_{ij})\|_{0,\infty,\widehat{\Omega}}$ is bounded, we conclude that

$$(3.43) \qquad \max_{i,j}\left|\left(\widehat{a}\,\widehat{a}_{ij}\frac{\partial \widehat{u}_p}{\partial \widehat{x}_i}, \frac{\partial \widehat{w}_p}{\partial \widehat{x}_j}\right)_{\widehat{\Omega}} - \left(\widehat{a}\,\widehat{a}_{ij}\frac{\partial \widehat{u}_p}{\partial \widehat{x}_i}, \frac{\partial \widehat{w}_p}{\partial \widehat{x}_j}\right)_{m,\widehat{\Omega}}\right|$$
$$\leq C\,\{\, q_1^{-(\sigma-1)}\|\widehat{u}\|_{\sigma,\widehat{\Omega}} + r_1^{-(k_1-\frac{n}{2})}\|\widehat{a}\|_{\alpha,\widehat{\Omega}}\, M_\rho \|\widehat{u}\|_{1,\widehat{\Omega}}\,\}\, \|\widehat{w}_p\|_{1,\widehat{\Omega}} \;.$$

Since $\|\widehat{w}_p\|_{0,\widehat{\Omega}} \leq C\|\widehat{w}_p\|_{1,\widehat{\Omega}} \leq C\,\|w_p\|_{1,\Omega}$, the Lemma follows from substituting (3.37) and (3.43) in (3.30) and dividing by $\|w_p\|_{1,\Omega}$.

By a direct application of Lemma 2.4, 3.2 and 3.4 to Lemma 3.1 we obtain the following main Theorem which gives an asymptotic, $H^1(\Omega)$-norm estimate for the rate of convergence with using numerical quadrature rules and the mapping $T : \widehat{\Omega} \to \Omega \subset R^n$.

THEOREM 3.5. *For any numerical quadrature rules* $I_m, I_l \in G_p$ *and for any mapping* $T : \widehat{\Omega} \to \Omega \subset R^n$ *which satisfies* (1.11)-(1.12), *we assume that* $\widehat{u} \in H^\sigma(\widehat{\Omega}), \widehat{a} \in H^\alpha(\widehat{\Omega}), \widehat{b} \in H^\beta(\widehat{\Omega}), \widehat{J} \in H^\delta(\widehat{\Omega}), \widehat{f} \in H^\gamma(\widehat{\Omega})$ *and* $\widehat{a}_{ij} \in H^\rho(\widehat{\Omega})$ *for each* $i,j = 1,\ldots,n$, *with* $\min(\alpha,\beta,\gamma,\delta,\rho) \geq n$. *Then, for any positive integers* q, q_1, q_2 *such that* $0 < q \leq d(l) - p - 1$ *and* $0 < q_i \leq \min(d(m) - p - 1,\, p)$ $i = 1,2$, *we have*

$$(3.44) \qquad \|u - \widetilde{u}_p\|_{1,\Omega} \leq C\,\{\min(q_1,q_2)^{-(\sigma-1)}\|\widehat{u}\|_{\sigma,\widehat{\Omega}}$$
$$+ (r_1^{-(k_1-\frac{n}{2})}\|\widehat{a}\|_{\alpha,\widehat{\Omega}}\, M_\rho + r_2^{-(k_2-\frac{n}{2})}\|\widehat{J}\|_{\delta,\widehat{\Omega}}\|\widehat{b}\|_{\beta,\widehat{\Omega}})\,\|\widehat{u}\|_{1,\widehat{\Omega}}$$
$$+ q^{-(\gamma-\frac{n}{2})}\|\widehat{f}\|_{\gamma,\widehat{\Omega}}\,(\|\widehat{J}\|_{0,\infty,\widehat{\Omega}} + \|\widehat{J}\|_{\delta,\widehat{\Omega}})$$
$$+ r^{-(\delta-\frac{n}{2})}\|\widehat{J}\|_{\delta,\widehat{\Omega}}\,(\|\widehat{f}\|_{0,\infty,\widehat{\Omega}} + \|\widehat{f}\|_{\gamma,\widehat{\Omega}})\,\}\,,$$

where $k_1 = \min(\alpha,\rho), k_2 = \min(\delta,\beta), r = d(l) - p - q$ *and* $r_i = d(m) - p - q_i$ *for* $i = 1,2$.

We see from Theorem 3.5 that the rate of convergence is essentially given by

$$
\begin{aligned}
(3.45) \qquad O(\ & \min(q_1, q_2)^{-(\sigma-1)} \\
& + \{ (d(m) - p - q_1)^{-(k_1 - \frac{n}{2})} + (d(m) - p - q_2)^{-(k_2 - \frac{n}{2})} \} \\
& + \{ q^{-(\gamma - \frac{n}{2})} + (d(l) - p - q)^{-(\delta - \frac{n}{2})} \} \ ).
\end{aligned}
$$

If m, l and q are large enough with $\min(q_1, q_2) = p$, then the rate of convergence is asymptotically $O(p^{-(\sigma-1)})$, which coincides with that of (2.12). Moreover, when the mapping T is sufficiently smooth the rate is in proportion to that of (1.19). In the case where $\widehat{a}, \widehat{a}_{ij}, \widehat{b}, \widehat{f}$ and $\widehat{J}$ are sufficiently smooth, i.e., k_1, k_2 and γ are large enough, even when $d(m) \approx 2p + 1$ with $q_1 = q_2 = p$ and $d(l) \approx p + 2$ with $q \approx 1$ the first term in (3.45) may dominate, so that the rate of convergence is asymptotically $O(p^{-(\sigma-1)})$ which is the same that of $\|u - u_p\|_{1,\Omega}$. More precisely, in G-L quadrature rules, using I_m and I_l with $(p+1)$-point and p-point G-L rules respectively we would obtain an asymptotic rate $O(p^{-(\sigma-1)})$.

When one of $\widehat{a}\,\widehat{a}_{ij}$, $\widehat{J}\,\widehat{b}$ and $\widehat{J}\,\widehat{f}$ is not smooth enough, either because one of them is not smooth in the original problem or because a non-smooth mapping T is used, the first term $\min(q_1, q_2)^{-(\sigma-1)}$ may be dominated by one of the other terms. In this situation, using an overintegration with a sufficient number of m or l we may reduce the error $\|u - \widetilde{u}_p\|_{1,\Omega}$ until the first term dominates again. In practice, when $\widehat{a}\,\widehat{a}_{ij}$ or $\widehat{J}\,\widehat{b}$ is not smooth we may increase the value of $d(m)$ with $q_1 = q_2 \approx p$. When $\widehat{J}\,\widehat{f}$ is not sufficiently smooth we also increase both of $d(l)$ and q. The next section contains some numerical experiments with using G-L quadrature rules.

4. Numerical experiments

We consider the following one-dimensional problem:

$$
-\frac{d}{dx}(a\frac{du}{dx}) = f \quad \text{on} \quad \Omega = [0, 1]
$$

with $u(0) = u(1) = 0$.

Here, a and f are chosen in such a way that the exact solution is $u(x) = e^x \sin(x) - e^1 \sin(1)\, x$. Of course, the simulations have no need for the knowledge of the exact solution u.

EXAMPLE 4.1. We choose $a(x) = \cos(x)$, then a, f and u are sufficiently smooth functions. Hence we expect that under the smooth mapping T, the rate of convergence nearly follows the best fit of u. Moreover, even when we use p-point G-L rules $L_m = L_l = L_p$ with no overintegrations we would obtain the optimal results. But, if we take the mapping $T(\widehat{x}) = ((2+\varepsilon)^\alpha - (1-\widehat{x}+\varepsilon)^\alpha)/((2+\varepsilon)^\alpha - \varepsilon^\alpha)$ with $\alpha = 2.5$ and $\varepsilon = 0.001$, then this mapping is not smooth enough. It causes the non-smoothness of $\widehat{a}_{ij}$. In fact, $\widehat{a}_{ij} = 1/\widehat{J}$ has a pole at $\widehat{x} = -1.001$ which is very close to $\widehat{x} = -1$. In order to recover optimal results, an overintegration $L_m (m > p)$ must be used for calculating all integrals in the stiffness matrix. Figure 4.1 represents the results when overintegrations L_m are used for computing integrals in the stiffness matrix and all integrations in the load vector are nearly exact $(L_l,\, l = 1000)$. On the other hand, $\widehat{J}\widehat{f}$ has no poles under the mapping T. Hence we expect that there is no use for an overintegration L_l to calculate all integrations in the load vector. Figure 4.2 shows this phenomenon in the case where overintegrations $L_l (l \geq p)$ are used and $L_m (m = 1000)$.

EXAMPLE 4.2. We choose $a(x) = 1/(x+w)^\alpha$ for $w > 0$ and $\alpha \geq 1$, and take the mapping $T(\widehat{x})$ as the same that of example 4.1. If w is near to zero, then $a(x)$ and $f(x)$ have poles near to $x = 0$ in the original problem. Hence we need the overintegrations L_m and L_l in both of the stiffness matrix and the load vector. Moreover, $\widehat{a}\,\widehat{a}_{ij}$ is more singular than that of $\widehat{a}$ under the influence of the mapping T. When $w = 0.001$ and $\alpha = 1$, Figure 4.3 shows the results in the case where we use an overintegration scheme of $L_m (m \geq p)$ and $L_l (l = 1000)$. When we choose $\alpha = 2$ and $w = 0.1$, the results in Figure 4.4 follows under the case where $L_m (m = 1000)$ and $L_l (l \geq p)$. We consider the following two-dimensional problem:

$$-\mathrm{div}(a \nabla u) = f \quad \text{on} \quad \Omega,$$

with $u(x) = 0$ on Γ.

EXAMPLE 4.3. In the case where the domain Ω is the trapezoid with vertices $A = (0,0), B = (2,0), C = (0,1), D = (1,1)$, we consider mapping $T : (\widehat{x}_1, \widehat{x}_2) \in \widehat{\Omega} \longrightarrow (x_1, x_2) \in \Omega$ given by

$$\begin{aligned} x_1 &= (\widehat{x}_1 + 1)(3 - \widehat{x}_2)/4, \\ x_2 &= (\widehat{x}_2 + 1)/2. \end{aligned}$$

We choose $a(x_1, x_2)$, $f(x_1, x_2)$ in such a way that

$$u(x_1, x_2) = x_1 x_2 (x_1 + x_2 - 2)(e^{(x_2 - 1)} - 1).$$

In particular, we take $a(x_1, x_2) = 1/(x_1 + w)$ with $w > 0$. If w is near to zero, then $a(x_1, x_2)$ has a singularity near to the x_2-axis, and also f is singular. Hence, even if the mapping T is smooth enough, $\widehat{a}\,\widehat{a}_{ij}$ and $\widehat{J}\,\widehat{f}$ are not sufficiently smooth, which is caused by the original problem. To obtain optimal results we may use overintegrations L_m and L_l. Figure 4.5 gives the results when overintegrations $L_m (m \geq p + 1)$ and $L_l (l = 100)$ are used for $w = 0.025$. When $w = 0.1$, Figure 4.6 shows the results in the case where $L_m (m = 50)$ and $L_l (l \geq p + 1)$ are used.

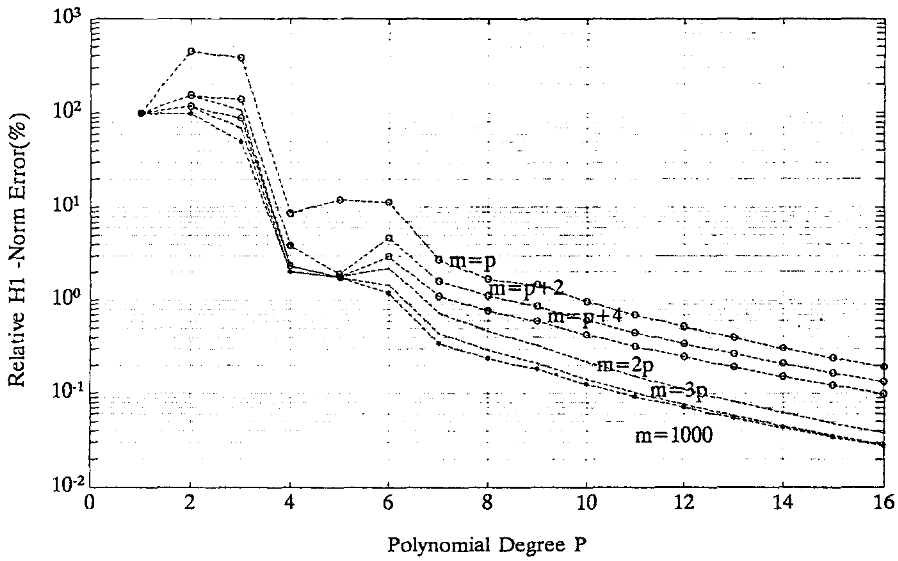

Figure 4.1

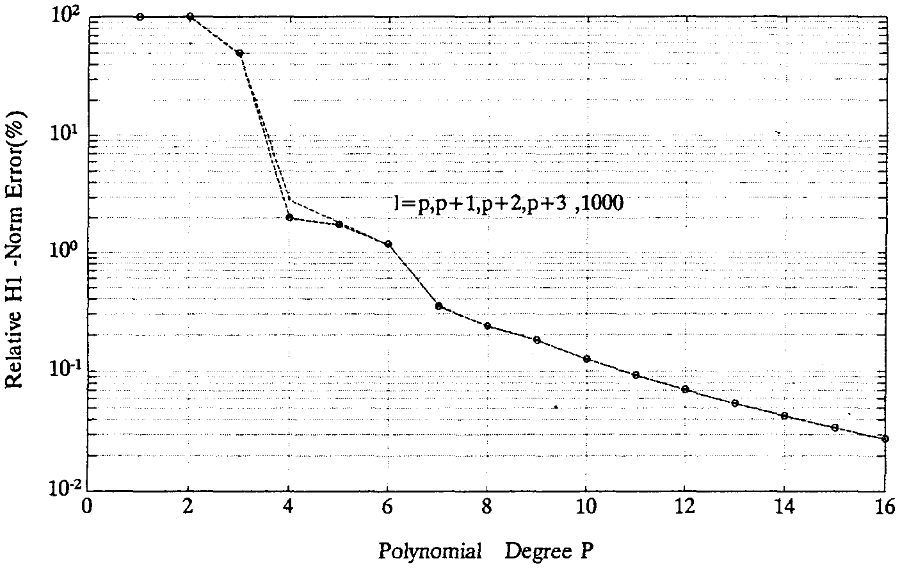

Figure 4.2

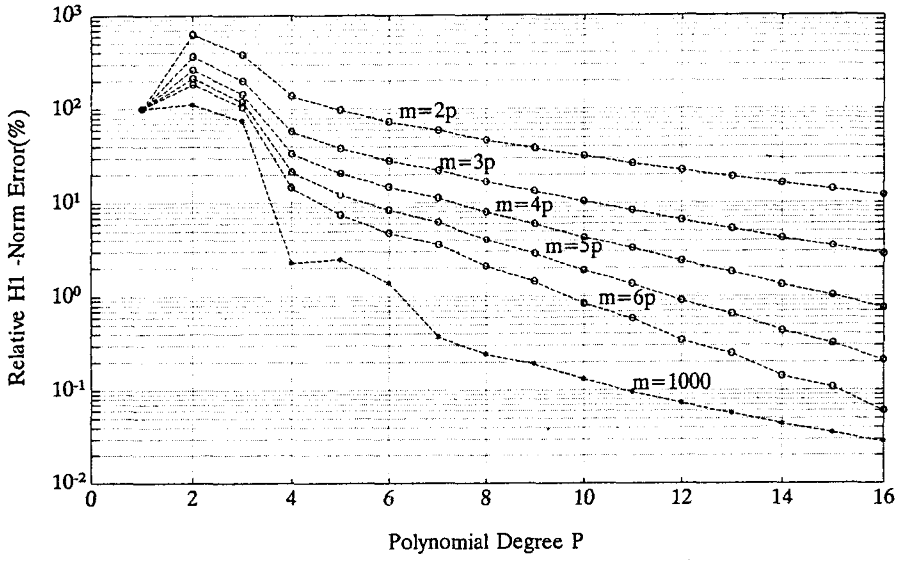

Figure 4.3

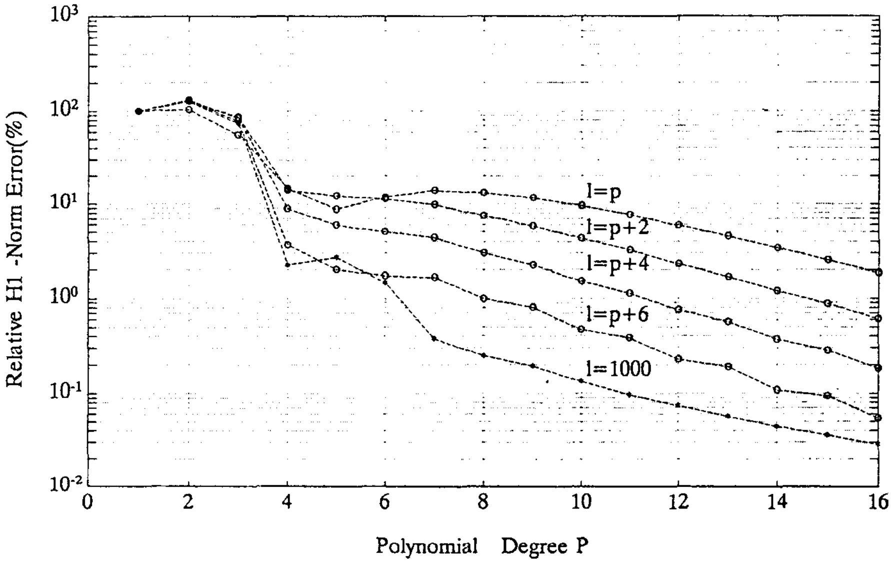

Figure 4.4

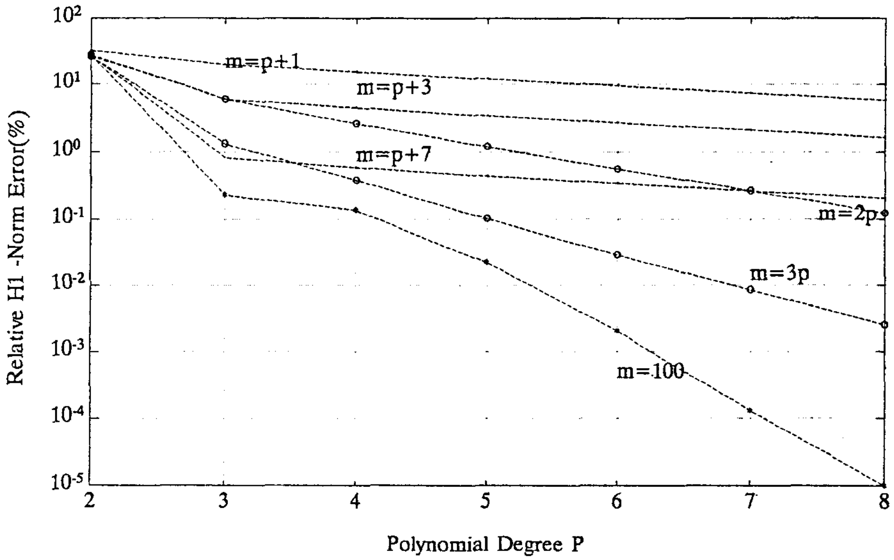

Figure 4.5

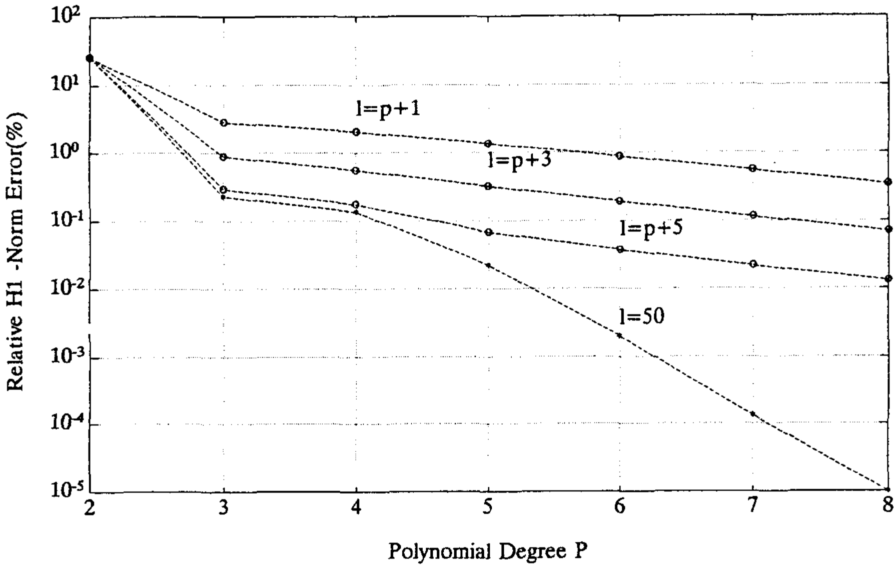

Figure 4.6

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