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UPPER BOUNDS FOR ASSIGNMENT FUNCTIONS

GWANG YEON LEE

Let $R = (r_1, r_2, \dots, r_m)$ and $S = (s_1, s_2, \dots, s_n)$ be positive integral vectors satisfying $r_1 + r_2 + \cdots + r_m = s_1 + s_2 + \cdots + s_n$, and let $\mathcal{U}(R, S)$ denote the class of all $m \times n$ matrices $A = [a_{ij}]$ where $a_{ij} = 0$ or 1 such that

$$(1.1) \qquad \sum_{k=1}^{n} a_{ik} = r_i, \ \sum_{k=1}^{m} a_{kj} = s_j, \ i = 1, \dots, m, \ j = 1, \dots, n.$$

Thus R is the *row sum vector* and S is the *column sum vector* of every matrix in $\mathcal{U}(R, S)$. We assume throughout that $\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} s_j$, i.e., $\mathcal{U}(R, S) \neq \emptyset$. Let $\overline{\mathcal{U}(R, S)}$ denote the *convex hull* of $\mathcal{U}(R, S)$. Let $X = [x_{ij}]$ be an $m \times n$ matrix. We define the *support* of X to be the set $\operatorname{supp}(X) = \{(i, j) : x_{ij} \neq 0\}$. The *assignment function* $\mathrm{P}_{R,S}(\cdot)$ is defined by

$$(1.2) \qquad \mathrm{P}_{R,S}(X) = \sum_{A \in \mathcal{U}(R,S)} \ \prod_{(i,j) \in \operatorname{supp}(A)} x_{ij}.$$

Brualdi, Hartfiel and Hwang [1] determined some bounds when $R = (1, \dots, 1)$ is the m-tuple of 1's and $S = (s_1, \dots, s_n)$, and the author determined the various bounds.

For integers $k, n, 1 \le k \le n$, let $V_{k,n}$ denote the set of all $n \times 1$ (0,1)-matrices whose entries have sum k. For real n-vectors, i.e., real $n \times 1$ matrices $\mathbf{x}$ and $\mathbf{y}$ we say that $\mathbf{x}$ *is majorized by* $\mathbf{y}$ (or $\mathbf{y}$ *majorizes* $\mathbf{x}$), written as $\mathbf{x} \prec \mathbf{y}$ if

$$(1.3) \qquad \max\{\mathbf{v}^t\mathbf{x} : \mathbf{v} \in V_{k,n}\} \le \max\{\mathbf{v}^t\mathbf{y} : \mathbf{v} \in V_{k,n}\}$$

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for all $k = 1, 2, \ldots, n$ and equality holds in (1.3) when $k = n$. $\mathbf{x}$ is said to be *submajorized by* $\mathbf{y}$, written as $\mathbf{x} \prec_w \mathbf{y}$, if (1.3) holds for all $k = 1, 2, \ldots, n$.

Let $U \subset \mathbf{R}^n$. A function $\varphi : U \to \mathbf{R}$ is called *Schur-convex* (*Schur-concave*) if for $\mathbf{x}$,$\mathbf{y} \in U$, $\mathbf{x} \prec \mathbf{y}$ implies that $\varphi(\mathbf{x}) \leq \varphi(\mathbf{y})$ (resp. $\varphi(\mathbf{x}) \geq \varphi(\mathbf{y})$). If, in addition, $\varphi(\mathbf{x}) < \varphi(\mathbf{y})$ whenever $\mathbf{x} \prec \mathbf{y}$ but $\mathbf{x}$ is not a rearrangement of $\mathbf{y}$, then φ is said to be *strictly Schur-convex* on U. Strictly Schur concavity on U is defined similarly. If $U = \mathbf{R}^n$, then φ is simply said to be Schur- convex or strictly Schur-convex omitting "on $\mathbf{R}^n$". Of course, φ is Schur-convex if and only if $-\varphi$ is Schur-concave.

Denote by $S_k(\mathbf{x})$ the kth *elementary symmetric function* of $\mathbf{x} = (x_1, x_2, \ldots, x_n)^t$. That is,

$$S_0(\mathbf{x}) \equiv 1, \quad S_1(\mathbf{x}) = \sum_{i=1}^{n} x_i, \quad S_2(\mathbf{x}) = \sum_{i<j} x_i x_j,$$

$$S_3(\mathbf{x}) = \sum_{i<j<k} x_i x_j x_k, \; \ldots, \quad S_n(\mathbf{x}) = \prod_{i=1}^{n} x_i.$$

Let $\mathbf{R}^n_+ = \{(x_1, \ldots, x_n) : 0 \leq x_i \text{ for all } i = 1, \ldots, n\}$ and let $\mathbf{R}^n_{++} = \{(x_1, \ldots, x_n) : 0 < x_i \text{ for all } i = 1, \ldots, n\}$.

LEMMA 1.[3]. *The function $S_k(\mathbf{x})$ is increasing and Schur-concave on $\mathbf{R}^n_+$. If $k \neq 1$, S_k is strictly Schur-concave on $\mathbf{R}^n_{++}$.*

Let $e_k = (1, \ldots, 1, 0, \ldots, 0)$ be the n-tuple vector such that the number of 1 is k.

THEOREM 2. *Let $\mathbf{x} \in \mathbf{R}^n_+$, $0 \leq x_i \leq 1$, $i = 1, 2, \ldots, n$. If $\mathbf{x} \prec e_k$, then $S_k(\mathbf{x}) \geq 1$ for all k, $2 \leq k \leq n$, with equality if and only if $\mathbf{x}$ is a rearrangement of e_k.*

Proof. If $\mathbf{x} \prec e_k$, then $S_k(\mathbf{x}) \geq 1$, by lemma 1. We will prove that the equality holds if and only if $\mathbf{x}$ is a rearrangement of e_k. If $\mathbf{x}$ is a rearrangement of e_k, then $S_k(\mathbf{x}) = 1$. Now, suppose that $\mathbf{x}$ is not rearrangement of e_k. Let t be the number of non-integers in $\mathbf{x}$. Then $2 \leq t \leq n$. Let $\mathcal{D} = \{\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{R}^n : x_1 \geq x_2 \geq \cdots \geq x_n\}$ and we may assume that $\mathbf{x} \in \mathcal{D}$, without loss of generality. By induction on

t, if $t = 2$, then we may assume that x_k, x_{k+1} are not integers. Since $\mathbf{x} \prec e_k$, $x_k + x_{k+1} = 1$, $x_1 = \cdots = x_{k-1} = 1$ and $x_{k+2} = \cdots = x_n = 0$.

$$\begin{aligned} S_k(\mathbf{x}) &= x_k + x_{k+1} + (k-1)x_k x_{k+1} \\ &= (k-1)x_k x_{k+1} + 1 > 1. \end{aligned}$$

Assume true for $n-1$, and consider n. Let $\mathbf{x}$ be the vector that have n nonintegers. We can choose the vector $\mathbf{y}$ such that $\mathbf{x} \prec \mathbf{y}$, $\mathbf{y} = (y_1, y_2, \ldots y_n) \in \mathcal{D}$ and the number of nonintegers is $n-1$. Then, by hypothesis, $S_k(\mathbf{y}) > 1$. Since S_k is Schur-concave, $S_k(\mathbf{x}) \geq S_k(\mathbf{y}) > 1$. Hence $S_k(\mathbf{x}) > 1$. Therefore, if $S_k(\mathbf{x}) = 1$, then $\mathbf{x}$ is a rearrangement of e_k. The proof is completed.

THEOREM 3. *Let $R = kE_m$ and $S = (s_1, \ldots, s_n)$ be positive integral vectors satisfying $\sum_{j=1}^n s_j = mk$. For $X \in \overline{\mathcal{U}(R,S)}$,*

$$P_{R,S}(X) \leq \prod_{i=1}^{m} S_k(x_{i1}, x_{i2}, \ldots, x_{in}) \tag{2.1}$$

with equality if and only if $X \in \mathcal{U}(R,S)$.

Proof. Since $R = kE_m = (k, k, \ldots, k)$, every term of the expansion

$$\mathrm{P}_{R,S}(X) = \sum_{A \in \mathcal{U}(R,S)} \prod_{a_{ij} \neq 0} x_{ij}$$

appear in the expansion of $\prod_{i=1}^m S_k(x_{i1}, \ldots, x_{in})$. And

$$(x_{1j_1}, x_{1j_2}, \ldots, x_{1j_k})(x_{2j_1}, x_{2j_2}, \ldots, x_{2j_k}) \cdots (x_{nj_1}, x_{nj_2}, \ldots, x_{nj_k})$$

appears in the expansion of $\prod_{i=1}^m S_k(x_{i1}, x_{i2}, \ldots, x_{in})$ but does not appear in the expansion of $\mathrm{P}_{R,S}(X)$. So $\mathrm{P}_{R,S}(X) \leq \prod_{i=1}^m S_k(x_{i1}, x_{i2}, \ldots, x_{in})$, for all $X \in \overline{\mathcal{U}(R,S)}$. If $X \in \mathcal{U}(R,S)$, then $\mathrm{P}_{R,S}(X) = 1$. Since each row vector of X is a rearrangement of e_k, $\prod_{i=1}^m S_k = 1$. Hence, for $X \in \mathcal{U}(R,S)$, $\mathrm{P}_{R,S}(X) = \prod_{i=1}^m S_k$. Now, we will prove that if $\mathrm{P}_{R,S}(X) = \prod_{i=1}^m S_k$, then $X \in \mathcal{U}(R,S)$. Suppose that $X \notin \mathcal{U}(R,S)$. Then there is a column with noninteger entries. Without loss of generality, we may assume that the first column have noninteger entries. Then the first column at least have s_1+1 many nonzero entries. So, without loss of generality, we may assume that $x_{i1}\neq 0$, $i=1,2,\dots,s_1+1$. Then $x_{11}x_{21}\cdots x_{s_1+11}>0$ and $x_{11}x_{21}\cdots x_{s_1+11}\varepsilon_1>0$ appear in the expansion of $\prod_{i=1}^m S_k$ but do not appear in the expansion of $\mathrm{P}_{R,S}(X)$. That is, $\mathrm{P}_{R,S}(X)<\prod_{i=1}^m S_k(x_{i1},\dots,x_{in})$. This is a contradiction. Therefore, $X\in\mathcal{U}(R,S)$.

COROLLARY 4. *Let $R=kE_m$ and $S=(s_1,\dots,s_n)$ be positive integral vectors satisfying $\sum_{j=1}^n s_j=km$. For all $X\in\overline{\mathcal{U}(R,S)}$,*

$$P_{R,S}(X)\leq\left[\binom{n}{k}\left(\frac{k}{n}\right)^k\right]^m. \tag{2.2}$$

Proof. Since $(\frac{k}{n},\dots,\frac{k}{n})\prec(x_{i1},\dots,x_{in})$ and S_k is a Schur-concave function,

$$S_k(x_{i1},\dots,x_{in})\leq S_k(\frac{k}{n},\dots,\frac{k}{n})=\binom{n}{k}\left(\frac{k}{n}\right)^k.$$

Therefore, by theorem 4,

$$\mathrm{P}_{R,S}(X)\leq\left[\binom{n}{k}\left(\frac{k}{n}\right)^k\right]^m.$$

COROLLARY 5. *Let $R=2E_m$ and $S=(s_1,\dots,s_n)$ be positive integral vectors satisfying $\sum_{j=1}^n s_j=2m$. For all $X\in\overline{\mathcal{U}(R,S)}$,*

$$P_{R,S}(X)\leq\left(2-\frac{2}{n}\right)^m. \tag{2.3}$$

If $n=2$, then the equality holds.

Proof. For all $i=1,2,\dots,m$, $(\frac{2}{n},\frac{2}{n},\dots,\frac{2}{n})\prec(x_{i1},x_{i2},\dots,x_{in})$. Since S_k is Schur-concave,

$$S_2(x_{i1},\dots,x_{in})\leq S_2(\frac{2}{n},\dots,\frac{2}{n})=\binom{n}{2}\left(\frac{2}{n}\right)^2.$$

Since $\mathrm{P}_{R,S}(X) \leq \prod_{i=1}^{m} S_2(x_{i1}, \ldots, x_{in})$,

$$\begin{aligned}\mathrm{P}_{R,S}(X) &\leq \left[\binom{n}{2}\left(\frac{2}{n}\right)^2\right]^m \\ &= \left[n(n-1)\left(\frac{2}{n^2}\right)\right]^m \\ &= \left(2 - \frac{2}{n}\right)^m .\end{aligned}$$

LEMMA 6. *For integer $n \geq 2$,*

$$\frac{n!}{n^n} \leq \left(\frac{1}{2}\right)^{\frac{n}{2}} \tag{2.4}$$

with equality holds for $n = 2$.

Proof. By induction on n, if $n = 2$ then equality holds. If $n = 3$, then $\frac{3!}{3^3} = \frac{2}{9} < \left(\frac{1}{2}\right)^{\frac{3}{2}}$. Assume true for $n-1$, and consider n. That is,

$$\begin{aligned}&\frac{(n-1)!}{(n-1)^{n-1}} \leq \left(\frac{1}{2}\right)^{\frac{n-1}{2}} \\ &\iff \frac{(n-1)!}{(n-1)^{n-1}} \frac{1}{n^{n-1}} \leq \left(\frac{1}{2}\right)^{\frac{n-1}{2}} \frac{1}{n^{n-1}} \\ &\iff \frac{n!}{n^n} \frac{1}{(n-1)^{n-1}} \leq \left(\frac{1}{2}\right)^{\frac{n-1}{2}} \frac{1}{n^{n-1}} \\ &\iff \frac{n!}{n^n} \leq \left(\frac{1}{2}\right)^{\frac{n-1}{2}} \frac{(n-1)^{n-1}}{n^{n-1}} \\ &= \left(\frac{1}{2}\right)^{\frac{n-1}{2}} \left(\frac{n-1}{n}\right)^{n-1} .\end{aligned}$$

$2 \leq (1 + \frac{1}{n-1})^{n-1}$ for $n \geq 2$, since $\left(1 + \frac{1}{n}\right)^n$ is an increasing function, $\left(1 + \frac{1}{n}\right)^n = 2$ for $n = 1$ and $2 < \left(1 + \frac{1}{n}\right)^n$ for $n \geq 2$. So, $(\frac{n}{n-1})^{n-1} \leq \frac{1}{2}$. Hence,

$$\frac{n!}{n^n} \leq \left(\frac{1}{2}\right)^{\frac{n-1}{2}} \frac{1}{2} < \left(\frac{1}{2}\right)^{\frac{n}{2}} .$$

Therefore, $\frac{n!}{n^n} \leq \left(\frac{1}{2}\right)^{\frac{n}{2}}$ for integer $n \geq 2$.

By the above lemma,

$$\left(\frac{n!}{n^n}\right)^2 \leq \left(\frac{1}{2}\right)^n \Longleftrightarrow (n!)^2 \leq \left(\frac{n^2}{2}\right)^n$$
$$\Longleftrightarrow (n!)^{\frac{2}{n}} \leq \frac{n^2}{2}$$
$$\Longleftrightarrow \frac{2}{n} \leq \frac{n}{(n!)^{\frac{2}{n}}}$$
$$\Longleftrightarrow \frac{2(n-1)}{n} \leq \frac{n(n-1)}{(n!)^{\frac{2}{n}}}$$
$$\Longleftrightarrow 2 - \frac{2}{n} \leq \frac{n(n-1)}{(n!)^{\frac{2}{n}}}$$
$$\Longleftrightarrow \left(2 - \frac{2}{n}\right)^m \leq \left(\frac{n(n-1)}{(n!)^{\frac{2}{n}}}\right)^m = \frac{(n!)^{m-\frac{2m}{n}}}{(n-2)!^m}. \tag{2.5}$$

Note that the right term in (2.5) is the upper bound when $R = 2E_m$ by the Theorem 2.1 [2]. Thus, in this case, the upper bound in (2.3) sharper than that of Theorem 2.1 [2].

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Department of Mathematics
Hanseo University
Seosan 352-820, Korea