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Analytical Comparison of Time-Dependent Mild-Slope Equations
時間依存 緩傾斜方程式의 理論的 比較

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Abstract: We analyze existing time-dependent mild-slope equations, which were developed by Smith and Sprinks (1975) (or, equivalently, Radder and Dingemans (1985)) and Kubo *et al.* (1992), in terms of the dispersion relation and energy transport. One-dimensionally in the horizontal direction, we compare the modulation of wave amplitudes for the time-dependent mild-slope equations against the linear Schrödinger equation. In view of the dispersion relation and modulation of wave amplitudes, Smith and Sprinks' model is more accurate in shallower water ($\bar{k}h \leq 0.2\pi$) and satisfies the linear Schrödinger equation in very shallow water ($\bar{k}h=0$), while Kubo *et al.*'s model is more accurate in deeper water ($\bar{k}h>0.2\pi$) and satisfies the linear Schrödinger equation at a point of intermediate water depth ($\bar{k}h=0.3\pi$). In view of the energy transport, Kubo *et al.*'s model is more accurate but yields singular solutions at some higher frequency range.

要　旨: 現存하는 時間依存 緩傾斜方程式으로 Smith와 Sprinks(1975)가 개발한 식(이와 대등한 정확도로 Radder와 Dingemans(1985)가 개발한 식)과 Kubo 등(1992)이 개발한 식이 있다. 分散關係式과 에너지 轉送의 관점에서 時間依存 緩傾斜方程式을 분석하였다. 수평방향으로 1차원적으로 時間依存 緩傾斜方程式의 振幅變調現象을 線形 Schrödinger式과 대비하여 비교하였다. 分散關係式과의 관점에서 보면, Smith와 Sprinks의 模型이 보다 얕은 水深($\bar{k}h \leq 0.2\pi$)에서 더 정확하고 아주 얕은 水深($\bar{k}h=0$)에서는 線形 Schrödinger式을 만족시키는 반면 Kubo 등의 模型은 보다 깊은 水深($\bar{k}h>0.2\pi$)에서 더 정확하고 遷移領域의 한 지점($\bar{k}h=0.3\pi$)에서 線形 Schrödinger式을 만족시킨다. 에너지 轉送의 관점에서 보면 Kubo 등의 模型이 더 正確하지만 높은 周波數 領域에서 解가 發散하는 短點이 있다.

1. INTRODUCTION

The time-dependent mild-slope equations are widely applicable from deep to shallow waters to both monochromatic and random waves. Also, the model equations treat the combined effects of refraction, diffraction, and reflection. So, the time-dependent mild-slope equations are quite useful tools for coastal engineers who want to predict the wave climate of the whole range of waters.

Combined refraction and diffraction was first studied by Ludwig (1966) in order to provide locally valid solutions for the Helmholtz equation near a caustic. The solution uses Airy functions, which are sinusoidal in the illuminated zone, damp exponentially in the shadow zone, and are transitional between sinusoidal and exponentially damping behavior in the vicinity of the caustic, with values of nearly double the incident wave on the caustic itself.

The combined refraction and diffraction of water waves on a slowly varying bottom was studied by Berkhoff (1972), who derived the mild-slope equation:

$$\nabla\cdot(CC_g\nabla\hat{\phi})+k^2CC_g\hat{\phi}=0 \tag{1}$$

where ∇ is the horizontal gradient operator, k is the local wavenumber, C and C_g are phase and group velocities, respectively, and the function $\hat{\phi}$ is related to the velocity potential ϕ by

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$$\phi=\frac{\cosh k(z+h)}{\cosh kh}\hat{\phi}e^{-i\omega t}. \tag{2}$$

Berkhoff's equation (1) is applicable only to monochromatic waves and is of elliptic type, and hence it requires the surrounding boundary conditions and significant work has been done on efficient solution methods.

A time-dependent mild-slope equation was first developed by Smith and Sprinks (1975) by means of Green's second identities applied to the velocity potential. The model equation is

$$\frac{\partial^2\tilde{\phi}}{\partial t^2}-\nabla\cdot(\overline{CC_g}\nabla\tilde{\phi})+(\bar{\omega}^2-\bar{k}^2\overline{CC_g})\tilde{\phi}=0 \tag{3}$$

where $\bar{C}$ and $\bar{C}_g$ are phase and group velocities, respectively, of a narrow-banded wave with carrier angular frequency $\bar{\omega}$ and wavenumber $\bar{k}$. The function $\tilde{\phi}$ is related to the velocity potential ϕ by

$$\phi=\frac{\cosh\bar{k}(z+h)}{\cosh\bar{k}h}\tilde{\phi}. \tag{4}$$

The time-dependent mild-slope equation (3) reproduces Berkhoff's equation (1) for monochromatic waves and reproduces the long wave equation in shallow water.

A system of the time-dependent mild-slope equations was derived based on the Hamiltonian theory of water waves by Radder and Dingemans (1985). The model equations are

$$\frac{\partial\eta}{\partial t}=-\nabla\cdot\left(\frac{\overline{CC_g}}{g}\nabla\tilde{\phi}\right)+\frac{(\bar{\omega}^2-\bar{k}^2\overline{CC_g})}{g}\tilde{\phi} \tag{5}$$

$$\frac{\partial\tilde{\phi}}{\partial t}=-g\eta \tag{6}$$

where g is the gravitational acceleration. The water surface elevation η may be eliminated from equations (5) and (6) in order to obtain Smith and Sprinks' equation (3).

The time-dependent mild-slope equation was extended by Booij (1981) to include the effects of ambient currents using a variational principle. Some errors were corrected by Kirby (1984) to obtain the model equation

$$\frac{D^2\tilde{\phi}}{Dt^2}+(\nabla\cdot\mathrm{U})\frac{D\tilde{\phi}}{Dt}-\nabla\cdot(\overline{CC_g}\nabla\tilde{\phi})+(\bar{\sigma}^2-\bar{k}^2\overline{CC_g})\tilde{\phi}=0 \tag{7}$$

where the total derivative D/Dt is

$$\frac{D}{Dt}=\left(\frac{\partial}{\partial t}+\mathrm{U}\cdot\nabla\right). \tag{8}$$

$\mathrm{U}(x, y)$ is the ambient current and $\bar{\sigma}$ is the wave intrinsic frequency which satisfies the dispersion relation

$$\bar{\sigma}^2=(\bar{\omega}-\bar{\mathbf{k}}\cdot\mathrm{U})^2=g\bar{k}\tanh\bar{k}h. \tag{9}$$

Kirby *et al.* (1992) presented a number of computations using the time-dependent mild-slope equations (5) and (6). They studied the propagation of wave groups in order to verify the linear dispersive properties of the model, and then tested the model against several existing data sets, including the wave focusing experiments by Berkhoff *et al.* (1982) (monochromatic waves) and Vincent and Briggs (1989) (monochromatic and random waves). They extended those time-dependent mild-slope equations to treat progressive nonlinear Stokes waves.

At the same time, Kubo *et al.* (1992) have developed a different type of time-dependent mild-slope equation

$$\nabla\cdot(\overline{CC_g}\nabla\hat{\phi})+\bar{k}^2\overline{CC_g}\hat{\phi}+i\nabla\cdot\left(\frac{\partial}{\partial\omega}(\overline{CC_g})\nabla\frac{\partial\hat{\phi}}{\partial t}\right)$$
$$+i\frac{\partial}{\partial\omega}(\bar{k}^2\overline{CC_g})\frac{\partial\hat{\phi}}{\partial t}=0 \tag{10}$$

where the function $\hat{\phi}$ is related to the velocity potential ϕ by

$$\phi=\frac{\cosh\bar{k}(z+h)}{\cosh\bar{k}h}\hat{\phi}e^{-i\bar{\omega}t}. \tag{11}$$

The time-dependent mild-slope equation (10) was derived by extending the terms CC_g and k^2CC_g in Berkhoff's equation (1) into Taylor series in $\Delta\omega$, and eliminating powers of $\Delta\omega$ using the relation $\partial\hat{\phi}/\partial t=-i\Delta\omega\hat{\phi}$ for narrow-banded spectra. They also showed the propagation of the wave groups to verify the linear dispersive properties of the model equation. The last two terms in equation (10) are added in order to correct Berkhoff's equation (1) for time-dependent problems. The values of the terms $\partial(\overline{CC_g})/\partial\omega$ and $\partial(\bar{k}^2\overline{CC_g})/\partial\omega$ are given by

$$\frac{\partial}{\partial\omega}(\overline{CC_g})=\frac{\bar{\omega}}{\bar{k}^2}\left(\frac{\bar{C}}{\bar{C}_g}-1+\frac{\bar{k}\,\bar{\omega}''}{\bar{C}_g}\right)=\frac{\bar{\omega}}{\bar{k}^2}\{2(\bar{n}-1)$$

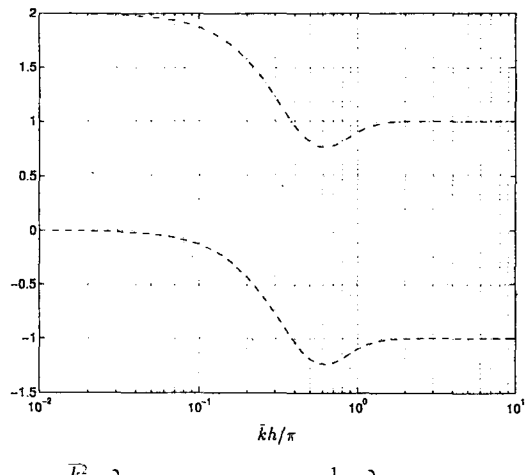

Fig. 1. $\frac{\bar{k}^2}{\bar{\omega}}\frac{\partial}{\partial\omega}(\overline{CC_g})$: dashed line, $\frac{1}{\bar{\omega}}\frac{\partial}{\partial\omega}(\bar{k}^2\overline{CC_g})$: dash-dotted line.

$$+\frac{2\bar{n}-1}{2\bar{n}}(1-(2\bar{n}-1)\cosh 2\bar{k}h)\} \tag{12}$$

$$\frac{\partial}{\partial\omega}(\bar{k}^2\overline{CC_g})=2\bar{\omega}+\bar{k}^2\frac{\partial}{\partial\omega}(\overline{CC_g})$$
$$=\bar{\omega}\{2\bar{n}+\frac{2\bar{n}-1}{2\bar{n}}(1-(2\bar{n}-1)\cosh 2\bar{k}h)\} \tag{13}$$

where $n=\bar{C}_g/\bar{C}=(1+2\bar{k}h/\sinh 2\bar{k}h)/2$.

The values of $\bar{k}^2/\bar{\omega}\ \partial(\overline{CC_g})/\partial\omega$ and $1/\bar{\omega}\ \partial(\bar{k}^2\overline{CC_g})/\partial\omega$ are shown in Figure 1, from which we see that the second to last term in equation (10) becomes asymptotically zero in very shallow water and the last term becomes asymptotically $+i\bar{\omega}(\partial\hat{\phi}/\partial t)$ and $+2i\bar{\omega}(\partial\hat{\phi}/\partial t)$ in very deep and shallow waters, respectively.

First, we compare the two models (3) (or, equivalently, (5) and (6)) and (10) from a geometric optics point of view, which yields the dispersion relation and the transport equation for wave energy. Second, we compare the two models one-dimensionally in the horizontal direction for constant water depth against the linear Schrödinger equation, which is the equation for modulation of wave amplitudes accurate to $O(\Delta k)^2$ and serves as a benchmark for other leading order envelope equations.

2. COMPARISON OF MODEL EQUATIONS BY GEOMETRIC OPTICS APPROACH

For the case where variations in the wave train and the domain topography are slow relative to the wavelength, the propagation of surface waves is often treated from the geometric optics point of view, which leads to the usual ray approximation.

For Smith and Sprinks' model, the geometric optics approximation is constructed by substituting the ansatz

$$\tilde{\phi}(x,y,t)=A(x,y,t)e^{i(\int k_x dx+\int k_y dy-\omega t)} \tag{14}$$

where $A(x,y,t)$ is a complex amplitude which modulates in space and time and k_x and k_y are the local wavenumbers in x and y directions, into equation (3), which yields

$$-(\omega^2-\bar{\omega}^2)A+(k^2-\bar{k}^2)\overline{CC_g}A-\nabla\cdot(\overline{CC_g}\nabla A)$$
$$-i\frac{1}{A}\nabla\cdot(A^2\mathbf{k}\overline{CC_g})=0. \tag{15}$$

Separation of real and imaginary parts of the resulting equation leads to an eikonal equation for the phase function

$$\frac{\omega^2-\bar{\omega}^2}{\overline{CC_g}}=k^2-\bar{k}^2-\frac{\nabla\cdot(\overline{CC_g}\nabla A)}{\overline{CC_g}A} \tag{16}$$

and a transport equation for wave energy

$$\nabla\cdot(A^2\mathbf{k}\overline{CC_g})=0. \tag{17}$$

It is usually assumed that for bottom of small slopes, the last term in equation (16) is second order in the small parameter and thus negligibly small (Keller, 1958). Retention of the small term allows for the inclusion of weak diffraction corrections in grid-based refraction schemes. Neglecting terms that are second-order small in equation (16) leads to the following dispersion relation

$$\frac{k}{\bar{k}}=\sqrt{1+\frac{\left(\frac{\omega}{\bar{\omega}}\right)^2-1}{\bar{n}}} \tag{18}$$

For Kubo *et al.*'s model, the geometric optics approximation is constructed by substituting the ansatz

$$\hat{\phi}(x,y,t)=\tilde{\phi}e^{i\bar{\omega}t}=A(x,y,t)e^{i(\int k_x dx+\int k_y dy-(\omega-\bar{\omega})t)} \tag{19}$$

into equation (10), which yields

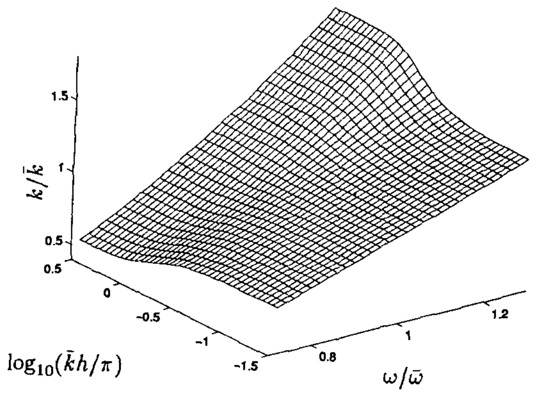

Fig. 2. Exact linear dispersion relations.

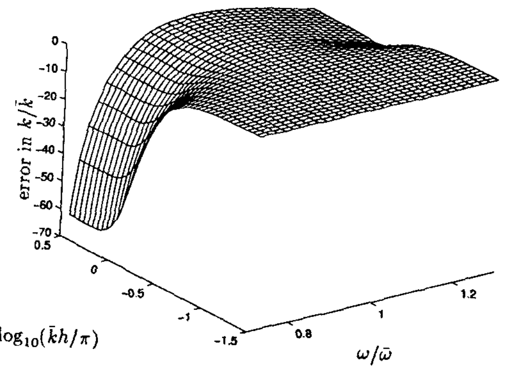

Fig. 3. Percent errors in $k/\bar{k}$ for Smith and Sprinks' model.

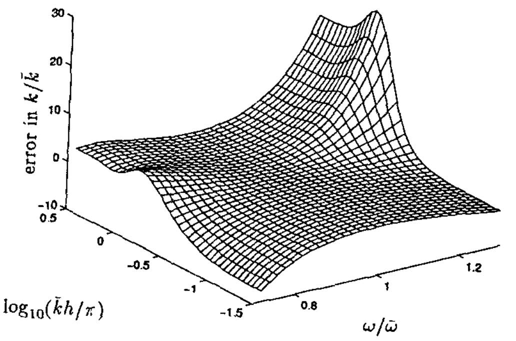

Fig. 4. Percent errors in $k/\bar{k}$ for Kubo *et al.*'s model.

$$(\omega-\bar{\omega})\left\{\frac{\partial}{\partial\omega}(\bar{k}^2\overline{CC_g})-k^2\frac{\partial}{\partial\omega}(\overline{CC_g})\right\}A-(k^2-\bar{k}^2)\overline{CC_g}A$$
$$+\nabla\cdot(\overline{CC_g}\nabla A)+(\omega-\bar{\omega})\nabla\cdot\left(\frac{\partial}{\partial\omega}(\overline{CC_g})\nabla A\right)$$
$$+i\frac{1}{A}\nabla\cdot\left[A^2\mathbf{k}\{\overline{CC_g}+(\omega-\bar{\omega})\frac{\partial}{\partial\omega}(\overline{CC_g})\}\right]=0. \qquad (20)$$

Separation of the real and imaginary parts of the resulting equation leads to an eikonal equation for the phase function

$$\frac{\omega-\bar{\omega}}{\overline{CC_g}}\left\{\frac{\partial}{\partial\omega}(\bar{k}^2\overline{CC_g})-k^2\frac{\partial}{\partial\omega}(\overline{CC_g})\right\}$$
$$=k^2-\bar{k}^2-\frac{\nabla\cdot(\overline{CC_g}\nabla A)}{\overline{CC_g}A}-$$
$$\frac{(\omega-\bar{\omega})}{\overline{CC_g}A}\nabla\cdot\left(\frac{\partial}{\partial\omega}(\overline{CC_g})\nabla A\right) \qquad (21)$$

and a transport equation for wave energy

$$\nabla\cdot[A^2\mathbf{k}\{\overline{CC_g}-(\omega-\bar{\omega})\frac{\partial}{\partial\omega}(\overline{CC_g})\}]=0. \qquad (22)$$

The last two terms in equation (21) represent weak diffraction with additional correcting term obtained by Taylor series expansion. Neglecting terms of diffraction in equation (21) leads to the following dispersion relation

$$\frac{k}{\bar{k}}=\sqrt{1+\frac{2\left(\frac{\omega}{\bar{\omega}}-1\right)}{\bar{n}+\left(\frac{\omega}{\bar{\omega}}-1\right)\frac{\bar{k}^2}{\bar{\omega}}\frac{\partial}{\partial\omega}(\overline{CC_g})}} \qquad (23)$$

...se dispersion relations, equations (18) and (23), can be compared against the exact dispersion relation for linear wave:

$$\left(\frac{\omega}{\bar{\omega}}\right)^2=\frac{k}{\bar{k}}\frac{\tanh kh}{\tanh \bar{k}h}. \qquad (24)$$

The exact dispersion relation is shown in Figure 2. The percent errors in $k/\bar{k}$ for Smith and Sprinks' model and Kubo *et al.*'s model are shown in Figures 3 and 4, which show that Kubo *et al.*'s model gives closer dispersion relation to the exact dispersion relation in deep and intermediate-depth waters, whereas Smith and Sprinks' model gives closer dispersion relation in shallow water. Figures 5-7 show the dispersion relations for exact solution, Smith and Sprinks' model, Kubo *et al.*'s model, and the linear Schrödinger equation in deep water ($\bar{k}h=2\pi$), intermediate-depth in water ($\bar{k}h=0.3\pi$), and in shallow water ($\bar{k}h=0.05\ \pi$). The dispersion relation for the linear Schrödinger equation is described by equation (34) which is accurate to $O(\Delta k)^2$. At deep

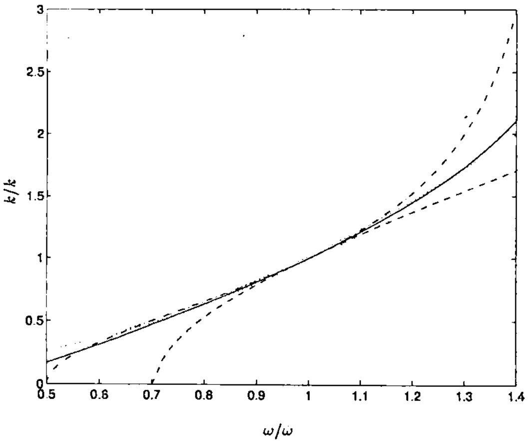

Fig. 5. Dispersion relations for $\bar{k}h=2\pi$ (dotted line: 2-percent confidence interval of exact solution, dashed line: Smith and Sprinks' model, dash-dotted line: Kubo *et al.*'s model, solid line: linear Schrödinger equation).

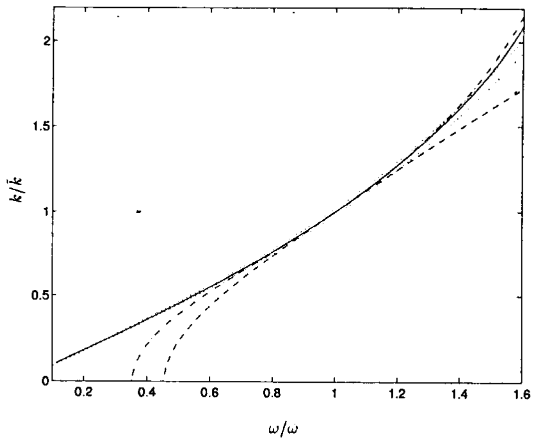

Fig. 6. Dispersion relations for $\bar{k}h=0.3\pi$ (dotted line: 2 percent confidence interval of exact solution, dashed line: Smith and Sprinks' model, dash-dotted line: Kubo *et al.*'s model, solid line: linear Schrödinger equation).

water with $\bar{k}h=2\pi$, Smith and Sprinks' model has lower valid boundary of $\omega/\bar{\omega}=0.7$ and Kubo *et al.*'s model has lower valid boundary of $\omega/\bar{\omega}=0.5$. At intermediate-depth water with $\bar{k}h=0.3\pi$, Smith and Sprinks' model has lower valid boundary of $\omega/\bar{\omega}=0.45$ and Kubo *et al.*'s model has lower valid boundary of $\omega/\bar{\omega}=0.35$. At shallow water with $\bar{k}h=0.05\pi$, Smith and Sprinks' mode has lower valid boundary of $\omega/\bar{\omega}=0.1$ and Kubo *et al.*'s model has lower valid boundary of $\omega/\bar{\omega}=0.5$. The higher ranges of $\omega/\bar{\omega}$ for valid solutions for both Smith and Sprinks' model and Kubo *et al.*'s model are much larger than the lower ranges of $\omega/\bar{\omega}$ for valid solutions.

The transport equations for wave energy can be compared against the transport equation for the linear wave energy:

$$\nabla\cdot(A^2\mathbf{k}CC_g)=0. \tag{25}$$

The exact linear shoaling coefficient can be obtained from equation (25) as

$$K_s=\frac{A}{A_0}=\sqrt{\frac{C_{g0}}{C_g}} \tag{26}$$

where the subscript 0 denotes the reference point. The linear shoaling coefficient for Smith and Sprinks' model can be obtained from equation (17) as

$$K_s=\frac{A}{A_0}=\sqrt{\frac{(k\overline{CC_g})_0}{k\overline{CC_g}}}. \tag{27}$$

The linear shoaling coefficient for Kubo *et al.*'s model can be obtained from equation (22) as

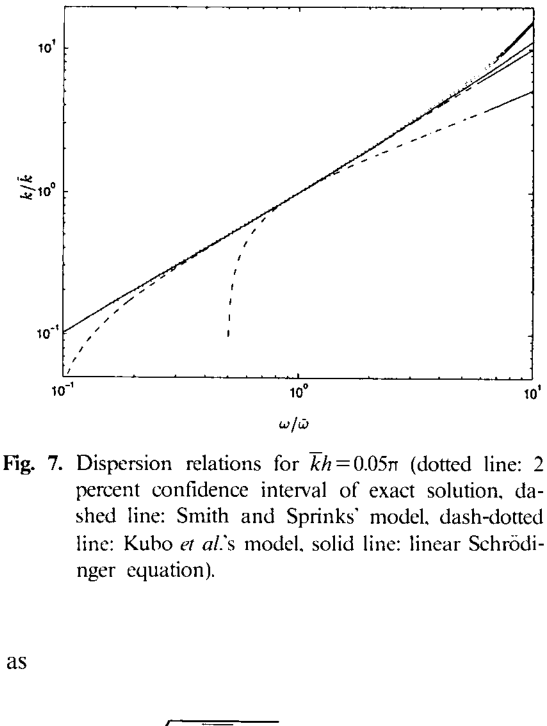

Fig. 7. Dispersion relations for $\bar{k}h=0.05\pi$ (dotted line: 2 percent confidence interval of exact solution, dashed line: Smith and Sprinks' model, dash-dotted line: Kubo *et al.*'s model, solid line: linear Schrödinger equation).

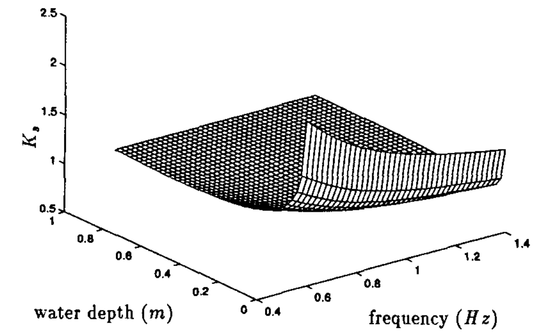

Fig. 8. Exact shoaling coefficients for $h_0=1$ m.

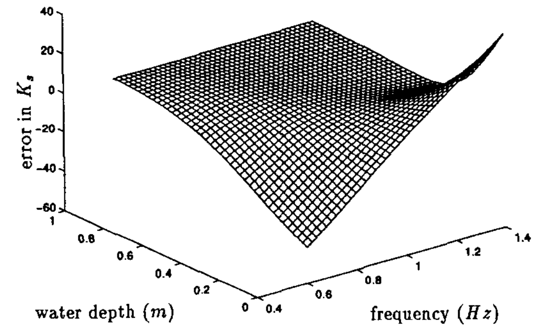

Fig. 9. Percent errors in shoaling coefficient for Smith and Sprinks' model ($h_0=1$ m, $\bar{f}=1$ Hz).

$$K_s=\frac{A}{A_0}=\sqrt{\frac{k_0\{(\overline{CC_g})_0+(\omega-\bar{\omega})\frac{\partial}{\partial\omega}(\overline{CC_g})_0\}}{k\{\overline{CC_g}+(\omega-\bar{\omega})\frac{\partial}{\partial\omega}(\overline{CC_g})\}}}\ . \quad (28)$$

The linear shoaling coefficient for Kubo *et al.*'s model is equal to the linear shoaling coefficient for Smith and Sprinks' model with additional correcting terms obtained by Taylor series expanding the terms CC_g and $(CC_g)_0$.

The exact linear shoaling coefficients for reference water depth $h_0=1$ m and frequencies from $f=0.6$ Hz to $f=1.4$ Hz are shown in Figure 8 where the shoaling coefficient decreases and then increases as water depth decreases. At lower frequencies, the turning starts deeper water than at higher frequencies, and the maximum shoaling coefficient at water depth $h=0.01$ m is 2.21 at the lowest frequency. Figures 9 and 10 show the percent errors for Smith and Sprinks' model and Kubo *et al.*'s model, respectively when the represer[illegible] quen[illegible]

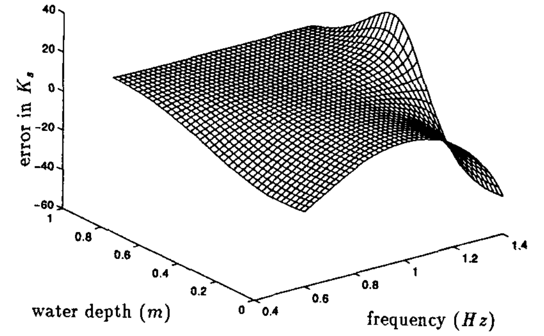

Fig. 10. Percent errors in shoaling coefficient for Kubo *et al.*'s model ($h_0=1$ m, $\bar{f}=1$ Hz).

Hz with $\bar{k}h_0=1.28\pi$ at $h_0=1$ m and $\bar{k}h=0.06\pi$ at $h=1$ cm. Smith and Sprinks' model gives smaller shoaling coefficients at lower frequencies and larger shoaling coefficients at higher frequencies in all water depths relative to the exact shoaling coefficients. Kubo *et al.*'s model gives smaller shoaling coefficients at both lower and higher frequencies in water depth shallower than 0.4 m and gives singular solutions of the shoaling coefficients at frequencies higher than $f=1.4$ Hz. The singularity happens when the value of $\overline{CC_g}+(\omega-\bar{\omega})\partial(\overline{CC_g})/\partial\omega$ becomes zero. The value of $\partial(\overline{CC_g})/\partial\omega$ is 0 in shallow water and $-\bar{\omega}/\bar{k}^2$ in deep water (see Figure 1), so the singularity of the shoaling coefficients happens always at higher frequencies. Overall, the shoaling coefficient for Kubo *et al.*'s model is more accurate than the shoaling coefficient for Smith and Sprinks' model, but yields singular solutions at some higher frequency range.

3. COMPARISON OF MODEL EQUATIONS IN VIEW OF MODULATION OF WAVE AMPLITUDES

For constant water depth, the two models can be compared against the linear Schrödinger equation, which is an equation for modulation of wave amplitudes accurate to $O(\Delta k)^2$ and serves as a benchmark for other leading order envelope equations. We analyze the problems in one dimension horizontally, *i.e.*, in x direction which is assumed to be the direction perpendicular to the wave crest. The velocity potential $\tilde{\phi}$ in Smith the Sprink[illegible] equation

(3) can be defined as

$$\tilde{\phi}(x,t)=Ae^{i(\int kdx-\omega t)}=\bar{A}(x,t)\,e^{i(\int \bar{k}dx-\bar{\omega}t)} \tag{29}$$

where $\bar{A}(x,t)$, which modulates in space and time, is the amplitude obtained by extracting the harmonic terms with carrier wavenumber $\bar{k}$ and angular frequency $\bar{\omega}$ from the velocity potential $\tilde{\phi}$. We have the following relations for the amplitude $\bar{A}$:

$$\bar{A}=Ae^{i(\int(k-\bar{k})dx-(\omega-\bar{\omega})t)} \tag{30}$$

$$\frac{\partial\bar{A}}{\partial t}=-i(\omega-\bar{\omega})\bar{A} \tag{31}$$

$$\frac{\partial\bar{A}}{\partial x}=i(k-\bar{k})\bar{A} \tag{32}$$

$$\frac{\partial^2\bar{A}}{\partial x^2}=-(k-\bar{k})^2\bar{A}. \tag{33}$$

The local angular frequency $\omega(k)$ can be approximated by a few terms in the Taylor series expansion to $O(\Delta k)^2$:

$$\omega=\bar{\omega}+\bar{\omega}'(k-\bar{k})+\bar{\omega}''\frac{(k-\bar{k})^2}{2} \tag{34}$$

where the superscript prime means the derivative with respect to the wavenumber. After multiplying equation (34) by $\bar{A}$ and rearranging, we have

$$(\omega-\bar{\omega})\bar{A}=\bar{C}_g(k-\bar{k})\bar{A}+\bar{\omega}''\frac{(k-\bar{k})^2}{2}\bar{A} \tag{35}$$

where $\bar{\omega}'$ is replaced by the group velocity $\bar{C}_g$. Then, the linear Schrödinger equation for modulation of wave amplitudes $\bar{A}$ can be obtained using the relations (30)-(33):

$$\frac{\partial\bar{A}}{\partial t}+\bar{C}_g\frac{\partial\bar{A}}{\partial x}-\frac{i}{2}\bar{\omega}''\frac{\partial^2\bar{A}}{\partial x^2}=0. \tag{36}$$

This linear Schrödinger equation (36) and the dispersion relation (34) for the linear Schrödinger equation will be used as a reference in comparing the accuracy of the two time-dependent mild-slope equations.

Substitution of equation (29) into Smith and Sprinks' equation (3) gives the equation for modulation of wave amplitudes $\bar{A}$:

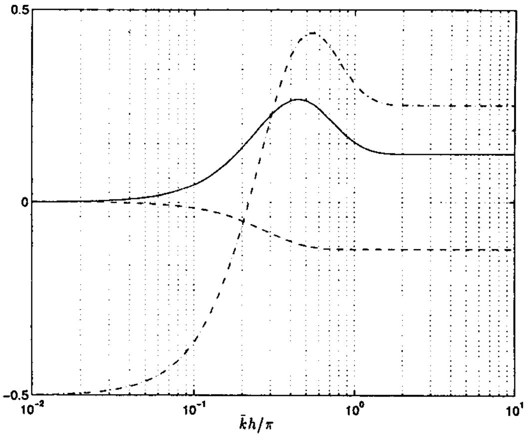

Fig. 11. $-\frac{\bar{k}^2}{\bar{\omega}}\frac{\bar{\omega}''}{2}$: solid line, $-\frac{\bar{k}^2}{\bar{\omega}}\frac{\bar{C}_g}{2\bar{\omega}}(\bar{C}-\bar{C}_g)$: dashed line, $-\frac{\bar{k}^2}{\bar{\omega}}\left\{\bar{\omega}''+\frac{\bar{C}_g}{2\bar{\omega}}(2\bar{C}_g-\bar{C})\right\}$: dash-dotted line.

$$-2i\bar{\omega}\left(\frac{\partial\bar{A}}{\partial t}+\bar{C}_g\frac{\partial\bar{A}}{\partial x}\right)+\frac{\partial^2\bar{A}}{\partial t^2}-\bar{C}\bar{C}_g\frac{\partial^2\bar{A}}{\partial x^2}=0 \tag{37}$$

or

$$\frac{\partial\bar{A}}{\partial t}+\bar{C}_g\frac{\partial\bar{A}}{\partial x}-\frac{i}{2}\frac{\bar{C}_g}{\bar{\omega}}(\bar{C}-\bar{C}_g)\frac{\partial^2\bar{A}}{\partial x^2}=0 \tag{38}$$

where we use the following relation accurate to $O(\Delta k)$:

$$\frac{\partial\bar{A}}{\partial t}+\bar{C}_g\frac{\partial\bar{A}}{\partial x}=0. \tag{39}$$

The velocity potential $\hat{\phi}$ in Kubo *et al.*'s equation (10) can be defined as

$$\hat{\phi}(x,t)=\tilde{\phi}(x,t)\,e^{i\bar{\omega}t}=Ae^{i(\int kdx-(\omega-\bar{\omega})t)}=\bar{A}(x,t)\,e^{i\int\bar{k}dx}. \tag{40}$$

Substitution of equation (40) into Kubo *et al.*'s equation (10) gives the equation for modulation of wave amplitudes $\bar{A}$:

$$2i\bar{\omega}\left(\frac{\partial\bar{A}}{\partial t}+\bar{C}_g\frac{\partial\bar{A}}{\partial x}\right)+\bar{C}\bar{C}_g\frac{\partial^2\bar{A}}{\partial x^2}$$
$$+i\frac{\partial}{\partial w}(\bar{C}\bar{C}_g)\left(\frac{\partial^3\bar{A}}{\partial x^2\partial t}+2i\bar{k}\frac{\partial^2\bar{A}}{\partial x\partial t}\right)=0 \tag{41}$$

or

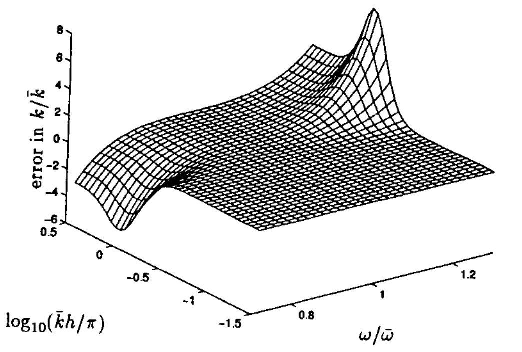

Fig. 12. Percent errors in $k/\bar{k}$ for linear Schrödinger equation.

$$\frac{\partial\bar{A}}{\partial t}+\bar{C}_g\frac{\partial\bar{A}}{\partial x}-i\left\{\bar{\omega}''+\frac{\bar{C}_g}{2\bar{\omega}}(2\bar{C}_g-\bar{C})\right\}\frac{\partial^2\bar{A}}{\partial x^2}=0 \qquad (42)$$

where we use relation (39) and neglect the term smaller than $O(\Delta k)^2$.

The comparison of equations (38) and (42) against the linear Schrödinger equation (36) shows that the wave envelope for both Smith and Sprinks' model and Kubo *et al.*'s model propagate correctly with the group velocity $\bar{C}_g$ in $O(\Delta k)$, but none of the models cannot predict the propagation of the wave envelope in $O(\Delta)^2$. Figure 11 shows the coefficients of $(\partial^2\bar{A}/\partial x^2)$ multiplied by $\bar{k}^2/\bar{\omega}$ in equations (36), (38), and (42). Smith and Sprinks' model satisfies the linear Schrödinger equation in very shallow water ($\bar{k}h=0$), and the error becomes larger at intermediate-depth water with largest error at $\bar{k}h=0.7\pi$ after which the error becomes smaller in deeper water and becomes constant in very deep water. Kubo *et al.*'s model satisfies the linear Schrödinger equation at a point of intermediate depth ($\bar{k}h=0.3\pi$) and, from the point, the error increases in both deeper water and shallower water with a positive error in deeper water and a negative error in shallower water. The maximum error occurs in very shallow water for Kubo *et al.*'s model. The coefficient of $(\partial^2\bar{A}/\partial x^2)$ for Smith and Sprinks' model is closer to the coefficient for the linear Schrödinger equation at $\bar{k}h\leq 0.2\pi$ than for Kubo *et al.*'s model. The coefficient of $(\partial^2\bar{A}/\partial x^2)$ for Kubo *et al.*'s model is closer to the coefficient for the linear Schrödinger equation at $\bar{k}h>0.2\pi$ than for Smith and Sprinks' model.

Figure 12 shows the percent errors in $k/\bar{k}$ for the linear Schrödinger equation, which can be compared with Figure 3 for Smith and Sprinks' model and Figure 4 for Kubo *et al.*'s model. These figures show that the percent errors in $k/\bar{k}$ for the linear Schrödinger equation are much smaller than those for Smith and Sprinks' model and Kubo *et al.*'s model in whole water depth.

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