A Constitutive Model for Plastic Damage of Ductile Materials

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Abstract

An elasto-plastic-damage constitutive model for ductile materials is proposed within the framework of a thermodynamic approach of continuum damage mechanics (CDM) in which internal irreversible thermodynamic changes of micro-structure of materials such as plastic deformation and damage evolution are considered as thermodynamic state variables. The new constitutive model can predict not only the elasto-plastic behaviors but also the sequential stiffness degradation process of ductile materials more rationally.

1. INTRODUCTION

When a material is subjected to external loading, cumulated internal damage results in fracturing. For ductile materials, plasticity or yielding is the first appearing material damage, followed by hardening and fracturing. In the classical continuum mechanics of solids, the elastic and plastic constitutive equations of a material are based on the general thermodynamic principles and on the assumption of a continuous material regardless of how large the deformation is. In the fracture mechanics, the energy dissipation during crack propagation and unstable sudden cracking can be estimated based on the assumption of predescribed characteristics of initial cracks.

However, in the theory of plasticity, a material undergoes infinite plastic deformation without cracking since no criteria on the crack initiation and propagation are defined. The fracture mechanics approach focuses on characteristic behaviors at the crack tip zone for given geometry of the crack. It cannot describe the global stiffness degradation due to micro-voids and micro-cracks in the material. At present, both fracture and plasticity theories as well as other phenomenological material modeling approaches cannot treat material constitutive relations differently for virgin and damaged parts of a material at the same time.

In circumstances where the material defects are distributed in a statistically homogeneous manner, it is advantageous to model the mechanism associated with material degradation within the continuum damage mechanics(CDM). In CDM, internal state variables are introduced, which may be regarded as a continuous measure of the material degradation. The

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CDM approach was introduced by Kachanov in 1958[1]. Within a thermodynamical framework by Truesdell as described in [2, 3], constitutive equations including continuum damage are formulated after introducing a number of internal state variables characterizing the irreversible microscopic changes of a material. Recently, various applications in engineering practices using CDM have been made, especially for creep behavior of materials under elevated temperature, low cycle fatigue fracture of ductile materials, and brittle fracture of concrete, rocks, and sea ices.

In this paper, an elasto-plastic damage model of ductile materials are proposed. Based on the thermodynamic framework of CDM approach, constitutive relations of internal state variables including plastic strains, hardening, and damage are obtained. An anisotropic damage tensor is used to describe the magnitude and direction of material damage. Numerical examples show the validity and applicability of the proposed constitutive model.

2. CONTINUUM DAMAGE MECHANICS

2.1 Isotropic Damage

Kachanov[1] considered that damage of metals in creep was caused by initiation of microcracks. He expressed the reduction ratio of a cross sectional area as damage parameter and expressed the relation between the effective stress and strain as follows(see Fig. 1).

$$\sigma^* = \frac{\sigma}{\Psi} \tag{1}$$

$$\varepsilon = \frac{\sigma*}{\mathrm{E}} = \frac{\sigma}{\mathrm{E}\Psi} \tag{2}$$

$$\dot{\Psi} = -\mathrm{A}\ (\ \sigma_{\max}\ /\ \Psi\ )^{\mathrm{n}} \tag{3}$$

Since this expression is for a one-dimensional state of stress, Ψ is a scalar quantity which cannot characterize the direction of internal damage. In this case, increasing rate of damage, i.e. damage evolution equation is assumed empirically in a form of simple differential equation and coefficients are determined from experiments. In the general case of a three-dimensional deformation and damage, the nature and thus definition of the effective stress become more complex.

2.2 Anisotropic Damage

A micro-crack or micro-void has a directional characteristic for a three-dimensional state of deformation, especially for brittle materials. Thus, an isotropic damage model has limitations for multi-directional states of stresses or brittle materials. Vakulenko and Kachanov[5] proposed a tensor damage concept to describe an anisotropic damage model. The tensor damage was obtained as an integral of a dyadic product of the unit normal vector of a crack surface and a displacement jump vector. Davison and Stevens[6] and Krajcinovic[16] studied spall damage of a brittle material by treating damage as a vector quantity with magnitude and direction. Murakami and Ohno[7] proposed a simple tensor damage model in which the Vakulenko-type damage tensor was modified by introducing that the direction of the normal vector of a crack was the same as that of a displacement jump vector. They showed that eigenvalues of the tensor damage meant physically the area reduction ratio in the direction

of eigenvectors. According to their attribution, a genetic anisotropic tensor damage can be written as follows:

$$\mathbf{\Omega} = \sum_{i=1}^{3} \omega_i (\nu_i \otimes \nu_i) \tag{4}$$

where ν_i is a principal directional cosine vector of tensor damage, ω_i is the reduction rate of effective cross sectional area in the direction of ν_i. Then, the effective stress tensor can be written as follows:

$$\sigma^* = \sigma \cdot \phi \tag{5}$$

$$\phi = (\mathbf{I} - \mathbf{\Omega})^{-1} \tag{6}$$

Here, $\mathbf{I}$ is a second-order identity tensor. Eq. (5) is a form of three-dimensional case expanded from a one-dimensional expression of Eq. (1).

3. FORMULATION OF GOVERNING EQUATIONS

Motions of a continuous body can be expressed with three basic equations: kinematics, balance laws, and constitutive equations. Strain components are defined from the kinematic relation of deformations. Equations of motion are derived from the balance laws of mass, linear momentum, and angular momentum. Finally, constitutive equations represent the relation between stresses and strains which are intrinsic characteristics of materials. Detailed definitions used in this paper are as follows.

3.1 Kinematics of a continuous medium

Deformation gradient tensor

We suppose that the position vector of a particle $\mathcal{P}$ in a continuous body $\mathcal{B}$ is denoted as $\mathbf{x}$ with respect to reference coordinate system. After deformation, considering the one-to-one correspondence between the undeformed and deformed configurations, the deformed spatial position vector of the particle $\mathcal{P}$ defined by $\mathbf{y}$ can be written as:

$$\mathbf{y} = \mathbf{y}(\mathbf{x}, \mathrm{t}) \tag{7}$$

The deformation gradient tensor $\mathbf{F}$ defines the relation between the vectors $\mathbf{dx}$ and $\mathbf{dy}$ representing the infinitesimal length of undeformed and deformed configuration respectively, and is written as:

$$\mathbf{dy} = \mathbf{F} \cdot \mathbf{dx} \tag{8}$$

$$\mathbf{F} = \mathrm{Grad}\ \mathbf{y} = \frac{\partial \mathbf{y}}{\partial \mathbf{x}} \tag{9}$$

And, the Jacobian J is defined by the determinant of $\mathbf{F}$ as:

$$\mathrm{J} = \det \mathbf{F} \tag{10}$$

Velocity gradient tensor

The velocity gradient tensor $\mathbf{L}$ can be expressed as:

$$\mathbf{L} = \text{grad}\mathbf{v} = \frac{\partial \mathbf{v}}{\partial \mathbf{y}} \tag{11}$$

where $\mathbf{v}$ is a velocity vector of the particle $\mathcal{P}$ described by an Eulerian concept.

$$\mathbf{v}(\mathbf{y}, \mathrm{t}) = \frac{\partial \mathbf{y}}{\partial \mathrm{t}} \tag{12}$$

The velocity gradient tensor $\mathbf{L}$ can be decomposed into two parts; the velocity strain (or stretching, or rate of deformation) tensor $\mathbf{D}$ which is the symmetric part of $\mathbf{L}$ and the spin tensor $\mathbf{W}$ which is the anti-symmetric part of $\mathbf{L}$:

$$\mathbf{L} = \mathbf{D} + \mathbf{W} \tag{13}$$

$$\mathbf{D} = \mathbf{L}_{\text{sym}} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T) \tag{14}$$

$$\mathbf{D} = \mathbf{L}_{\text{anti-sym}} = \frac{1}{2}(\mathbf{L} - \mathbf{L}^T) \tag{15}$$

Additive decomposition of the deformation gradient tensor

To deal with the elasto-plastic deformation of solid materials easily, it is convenient to decompose the deformation component into two parts; the plastic component representing permanent deformation of shape and the reversible elastic component. Nemat-Nasser[9] and Guo[10] suggested the additive decomposition method which expresses the deformation gradient tensor as an additive form of elastic and plastic components.

Assuming a homogeneous deformation condition where all material elements undergo the same deformation and the compatibility of deformation is satisfied automatically, a displacement vector $\mathbf{u}(= \mathbf{y} - \mathbf{x})$ at an arbitrary material point $\mathcal{P}$ can be decomposed into an elastic component ue and a plastic component $\mathbf{u}_p$:

$$\mathbf{u} = \mathbf{u}_\mathrm{e} + \mathbf{u}_\mathrm{p} \tag{16}$$

Then velocity $\dot{\mathbf{u}}(= \mathbf{v})$ can be expressed as:

$$\dot{\mathbf{u}} = \dot{\mathbf{u}}_\mathrm{e} + \dot{\mathbf{u}}_\mathrm{p} = \mathbf{v}_\mathrm{e} + \mathbf{v}_\mathrm{p} \tag{17}$$

and Eq. (16) is converted to the sum of position vectors as:

$$\mathbf{y} = \mathbf{y}_\mathrm{e} + \mathbf{y}_\mathrm{p} - \mathbf{x} \tag{18}$$

where $\mathbf{y}_e$ and $\mathbf{y}_\mathrm{p}$ are the position vectors of a material point $\mathcal{P}$ which would be occupied by an imagined configuration of pure elastic and plastic deformation respectively. Differentiating Eq. (18) partially with respect to $\mathbf{x}$ results in the deformation gradient $\mathbf{F}$ as an additive form of elastic and plastic components:

$$\mathbf{F} = \mathbf{F}_\mathrm{e} + \mathbf{F}_\mathrm{p} - \mathbf{I}. \tag{19}$$

Taking the material time derivative of Eq. (19), we can obtain:

$$\dot{\mathbf{F}} = \dot{\mathbf{F}}_e + \dot{\mathbf{F}}_p. \tag{20}$$

And from Eqns. (11) and (17), we can obtain the velocity gradient $\mathbf{L}$ as:

$$\begin{aligned} \mathbf{L} &= \frac{\partial \mathbf{v}}{\partial \mathbf{y}} = \frac{\partial \mathbf{v}_e}{\partial \mathbf{y}} + \frac{\partial \mathbf{v}_p}{\partial \mathbf{y}} \\ &= \mathbf{L}_e + \mathbf{L}_p. \end{aligned} \tag{21}$$

Inserting Eq. (21) into Eq. (14), the velocity strain $\mathbf{D}$ is also decomposed into elastic and plastic components:

$$\begin{aligned} \mathbf{D} &= \frac{1}{2}(\mathbf{L} + \mathbf{L}^T) = \frac{1}{2}(\mathbf{L}_e + \mathbf{L}_p + \mathbf{L}_e^T + \mathbf{L}_p^T) \\ &= \mathbf{D}_e + \mathbf{D}_p. \end{aligned} \tag{22}$$

Strain measures

When a material property depends on a deformation path such as in a large strain elasto-plastic deformation problem, constitutive equations should be described as a form of relationship between the rates of stress and strain. In that case, it is convenient to introduce the Eulerian measures of stress and strain which can show the realistic force and deformation state of each moment. Hence, a Eulerian strain ε is defined as a material time integration of the velocity strain $\mathbf{D}$:

$$\dot{\varepsilon} = \mathbf{D}\ , \quad \varepsilon = \int_{t_o}^{t} \mathbf{D}\ \mathrm{d}t. \tag{23}$$

The strain tensor ε can be comprehended as a three-dimensionally expanded form of the logarithmic (or true or natural) strain. Eqns. (22) and (23) lead to following additive decomposition of the strain as:

$$\dot{\varepsilon} = \mathbf{D}_e + \mathbf{D}_p = \dot{\varepsilon}_e + \dot{\varepsilon}_p. \tag{24}$$

3.2 Equations of Motion

Mass balance law

The mass conservation condition of a continuous body $\mathcal{B}$ gives the following equation:

$$\rho_o = \rho\ \mathrm{J} = \rho\ \det\ \mathbf{F}. \tag{25}$$

Balance of linear momentum

The balance law of linear momentum which is known as the Newton's 3rd law leads to the equation of motion as:

$$\mathrm{div}\ \sigma^T + \rho\mathbf{b} - \rho\frac{\partial \mathbf{v}}{\partial t} = 0 \tag{26}$$

where Cauchy stress tensor σ is an Eulerian stress measure implying the internal force per unit area at the current deformed configuration.

Balance of angular momentum

The balance law of angular momentum by the internal and external forces is applied to obtain the symmetry condition of the Cauchy stress tensor as:

$$\sigma = \sigma^{\mathrm{T}} \tag{27}$$

3.3 Energy Balance

The first law of thermodynamics

In a closed system, the first law of thermodynamics relates the work done and amount of heat input into the system to the change of energy stored in the system as:

$$\dot{\mathrm{E}}_{\mathrm{total}} = \mathrm{P}_{\mathrm{input}} + \mathrm{Q}_{\mathrm{input}} \tag{28}$$

where the power input $\dot{\mathrm{E}}_{\mathrm{total}}$ can be expressed as a sum of the rate of work done by internal and external forces and heat input rate $\mathrm{Q}_{\mathrm{input}}$ consists of the conduction through surface and the distributed internal heat source. As a consequence, using the energy conservation principle of Eq. (28), we can obtain the energy equation as:

$$\rho\dot{\mathrm{u}} = \sigma : \mathbf{D} + \rho\mathrm{r} - \mathrm{div}\ \mathbf{q} \tag{29}$$

where $\mathbf{q}$ is the heat flux vector and r is the strength of the distributed internal heat source per unit mass. The specific internal energy u can be expressed as a function of state variables characterizing the thermodynamic state of the system.

The second law of thermodynamics

The second law of thermodynamics states that in a closed continuous system, the rate of increase of entropy is always greater than the rate of entropy input by heat transfer. In other word, the causes of increasing the entropy of a system are not only the results of heat transfer but also the energy dissipation due to inelastic irreversible deformation processes in the material. The second law can be expressed as:

$$\dot{\mathrm{s}} \geq \frac{\mathrm{r}}{\theta} - \frac{1}{\rho}\,\mathrm{div}\frac{\mathbf{q}}{\theta} \tag{30}$$

where s is the specific entropy and θ is the absolute temperature. Eq. (30) is the well-known Clausius-Duhem inequality.

3.4 Constitutive Equation

The relations between state variables describing the internal thermodynamic states of continuum can be derived from the Clausius-Duhem inequality. It is called the constitutive equation expressing the mechanical characteristics of a material.

Table 1: Mechanical thermodynamic variables

State Variables				Associated Variables	
Observable		Internal			
strain	ε			stress	σ
temperature	θ			entropy	s
temp. gradient	$\mathbf{g}$			heat flux	$-\mathbf{q}$
		elastic strain	ε_e	stress	σ
		plastic strain	ε_p	stress	$-\sigma$
		other variables	$\mathbf{A}_k$		$\mathbf{V}_k$

Free energy potential

For the convenience of formulation, we introduce the Helmholz free energy density ψ that is the portion of the internal energy u available for doing work in the isothermal process as:

$$\psi = \mathrm{u} - \mathrm{s}\theta \tag{31}$$

Inserting Eq. (29) and (31) to Eq. (30), the Clausius-Duhem inequality yields:

$$\rho\mathcal{D} = \sigma : (\dot{\varepsilon}_e + \dot{\varepsilon}_p) - \rho(\dot{\psi} + \mathrm{s}\dot{\theta}) - \frac{1}{\theta}\mathbf{q}\cdot\mathbf{g} \geq 0 \tag{32}$$

where $\mathbf{g}$ is the gradient of temperature distribution and $\mathcal{D}$ is the energy dissipation power density representing the energy dissipation during the deformation process.

State variables

It can be postulated that the characteristics of macroscopic behavior of a material depends completely on the thermodynamic state of the material, which can be defined by a number of independent variables, i.e. so-called state variables[2, 3]. We will classify the state variables into two groups; one is the observable variables that can be observed externally and another is the internal variables related to the deformation history and dissipation process, thus can not be measured by the current deformed state of deformation. Lemaitre and Chaboche[11, 12, 13] defined the state variables together with the thermodynamically conjugated associated variables as shown in Table 1.

Thermodynamic restrictions

According to the principle of equipresence theorem, the form of constitutive equations which define the relations between state variables and associated variables can be written as:

$$\begin{aligned}
\psi &= \psi(\varepsilon_e,\ \theta,\ \mathbf{g},\ \mathbf{V}_k) \\
\sigma &= \sigma(\varepsilon_e,\ \theta,\ \mathbf{g},\ \mathbf{V}_k) \\
\mathrm{s} &= \mathrm{s}(\varepsilon_e,\ \theta,\ \mathbf{g},\ \mathbf{V}_k) \\
\mathbf{q} &= \mathbf{q}(\varepsilon_e,\ \theta,\ \mathbf{g},\ \mathbf{V}_k) \\
\mathbf{A}_k &= \mathbf{A}_k(\varepsilon_e,\ \theta,\ \mathbf{g},\ \mathbf{V}_k)
\end{aligned} \tag{33}$$

Hence, the material time derivative of the free energy $\dot{\psi}$ can be written as:

$$\dot{\psi} = \frac{\partial \psi}{\partial \varepsilon_e} : \dot{\varepsilon}_e + \frac{\partial \psi}{\partial \theta}\,\dot{\theta} + \frac{\partial \psi}{\partial \mathbf{g}} \cdot \dot{\mathbf{g}} + \frac{\partial \psi}{\partial \mathbf{V}_k} \cdot \dot{\mathbf{V}}_k \tag{34}$$

Inserting Eq. (34) into Eq. (32), the Clausius-Duhem inequality is finally converted to:

$$\rho \mathcal{D} = (\sigma - \rho \frac{\partial \psi}{\varepsilon_e}) : \dot{\varepsilon}_e + \sigma : \dot{\varepsilon}_p - \rho(s + \frac{\partial \psi}{\partial \theta})\dot{\theta} - \frac{\mathbf{q}}{\theta} \cdot \mathbf{g} - \rho \frac{\partial \psi}{\partial \mathbf{g}} \cdot \dot{\mathbf{g}} - \rho \frac{\partial \psi}{\partial \mathbf{V}_k} \cdot \dot{\mathbf{V}}_k \geq 0 \tag{35}$$

Under the assumption of the isothermal deformation process, the inequality (35) can yield one equation and one inequality as:

$$\sigma = \rho \frac{\partial \psi}{\varepsilon_e} \tag{36}$$

$$\begin{aligned} \rho \mathcal{D} &= \sigma : \dot{\varepsilon}_p - \rho \frac{\partial \psi}{\partial \mathbf{V}_k} \cdot \dot{\mathbf{V}}_k \geq 0 \\ &= \mathbf{X} \cdot \mathbf{J} \geq 0 \end{aligned} \tag{37}$$

where

$$\begin{aligned} \mathbf{X} &= \{\sigma, \mathbf{A}_k\}^T \\ \mathbf{J} &= \{\dot{\varepsilon}_p, \dot{\mathbf{V}}_k\} \\ \mathbf{A}_k &= -\rho \frac{\partial \psi}{\partial \mathbf{V}_k} \quad (k = 1, 2, \ , \ ) \end{aligned} \tag{38}$$

and $\mathbf{J}$ is the thermodynamic flux variable representing the irreversible energy dissipation process in materials and $\mathbf{X}$ stands for the conjugated thermodynamic force vector.

Dissipative potential and complementary law

To describe the mechanical energy dissipation process of Eq. (37), evolution equations of flux variables should be defined. First of all, we postulate the existence of a dissipation potential φ as:

$$\varphi(\mathbf{J}) = \frac{1}{2}\rho \mathcal{D} = \frac{1}{2}\,\mathbf{X}(\mathbf{J}) \cdot \mathbf{J} \tag{39}$$

Then, the thermodynamic conjugated force $\mathbf{X}$ can be derived from the normality rule to the φ surface:

$$\mathbf{X} = \frac{\partial \varphi}{\partial \mathbf{J}} \tag{40}$$

For convenience, however, the evolution of flux variables is expressed by functions of the conjugated force $\mathbf{X}$ in the form of complementary laws which are the normality rule to the complementary potential φ^*. The Legendre-Fenchel transformation enables us to define the complementary potential φ^* as:

$$\varphi^*(\mathbf{X}) = \mathrm{Sup}(\mathbf{X} \cdot \mathbf{J} - \varphi) \tag{41}$$

Table 2: State variables in plastic damage problem

Flux variables		Conjugated forces	
plastic strain	ε_p	stress	$-\sigma$
kinematic strain	$\mathbf{A}$	back stress	α
effective plastic strain	p	isotropic hard stress	R
damage	Ω	energy release rate	$\mathbf{Y}$

and evolution equations of flux variables can be derived from the normality rule to the φ^* surface as:

$$\mathbf{J} = \frac{\partial \varphi*(\mathbf{X})}{\partial \mathbf{X}} \tag{42}$$

4. FORMULATION OF ELASTO-PLASTIC DAMAGE CONSTITUTIVE EQUATIONS

4.1 General

The constitutive equation for the elasto-plastic deformation and damage evolution equation of ductile materials are derived within the framework of the thermodynamic formulation process.

Flux variables

To represent the internal changes of the microscopic state of materials and energy dissipation phenomena in the elasto-plastic deformation process, thermodynamic flux variables and their conjugated forces are listed in Table 2.

Effective stress tensor and damage tensor

From the Eqns. (4) (5), the stress tensor σ^* which is a more realistic than the conventional stress σ in a damaged material is defined as following:

$$\sigma^* = \sigma \cdot \phi, \quad \phi = (\mathbf{I} - \Omega)^{-1} \tag{43}$$

In Eq. (43), however, since σ^* is not a symmetric tensor, the effective stress tensor $\tilde{\sigma}$ is redefined by considering only the symmetric components of σ^*

$$\begin{aligned} \tilde{\sigma} &= {\sigma^*}_{\text{sym}} = \frac{1}{2}(\sigma^* + \sigma^{*\mathrm{T}}) = \frac{1}{2}(\phi \cdot \sigma + \sigma \cdot \phi) \\ &= \mathbf{\Phi} : \sigma \end{aligned} \tag{44}$$

The damage tensor $\mathbf{\Phi}$ is a symmetric fourth order tensor representing the relation between the effective stress tensor $\tilde{\sigma}$ and the conventional stress tensor.

Energetic identification hypothesis

In the stress-strain relations of a material deteriorated by the evolution of damage, damage tensor $\mathbf{\Phi}$ represents the diminishing rate of effective sectional area. $\mathbf{\Phi}$ can be defined by various manners according to the basic assumptions. We adopted the energetic identification hypothesis proposed by Sidoroff[18]. For the first step, the complementary energy in a virgin material free from damage can be expressed as:

$$\mathrm{W}^{\mathrm{e}}(\sigma, \mathbf{0}) = \frac{1}{2}\sigma^{\mathrm{T}} : \mathbf{C}_{\mathrm{e}}^{-1} : \sigma \tag{45}$$

where $\mathbf{C}_e$ is the elastic stiffness tensor satisfying the relation $\sigma = \mathbf{C}_{\mathrm{e}} : \varepsilon_{\mathrm{e}}$. We postulate that the complementary energy of Eq. (45) in a damaged material can be expressed by a function of the effective stress tensor $\tilde{\sigma}$ instead of σ:

$$\begin{aligned} \mathrm{W}^{\mathrm{e}}(\sigma,\ \mathbf{\Phi}) &= \mathrm{W}^{e}(\tilde{\sigma},\ \mathbf{0}) \\ &= \frac{1}{2}\tilde{\sigma}^{T} : \mathbf{C}_{\mathrm{e}}^{-1} : \tilde{\sigma} = \frac{1}{2}\sigma^{\mathrm{T}} : \tilde{\mathbf{C}}_{\mathrm{e}}^{-1} : \sigma \end{aligned} \tag{46}$$

In Eq. (46), the effective elastic stiffness tensor $\tilde{\mathbf{C}}_e$ is defined as:

$$\tilde{\mathbf{C}}_{\mathrm{e}} = \mathbf{\Phi}^{-1} : \mathbf{C}_{\mathrm{e}} : \mathbf{\Phi}^{-\mathrm{T}} \tag{47}$$

4.2 Thermodynamic Formulation of the Constitutive Equation

Free energy

We postulated in Eq. (24) that strain can be decomposed into elastic and plastic components which are mutually independent. In plastic damage problem, as the same manner, the hardening effect of a plastic deformation can be assumed to have no coupling effects with any other internal variables. Then, the free energy density potential ψ can be dealt with a decomposed form of elastic and plastic terms[11, 16]:

$$\psi(\varepsilon_{\mathrm{e}},\ \mathrm{p},\ \mathbf{A},\ \mathbf{\Omega}) = \psi_{\mathrm{e}}\ (\varepsilon_{\mathrm{e}},\ \mathbf{\Omega}) + \psi_{\mathrm{p}}(\mathrm{p},\ \mathbf{A}) \tag{48}$$

Introducing the effective elastic stiffness tensor $\tilde{\mathbf{C}}_{\mathrm{e}}$ of Eq. (47), ψ_{e} can be expressed as:

$$\rho\psi_{\mathrm{e}} = \frac{1}{2}\varepsilon_{\mathrm{e}} : \tilde{\mathbf{C}}_{\mathrm{e}} : \varepsilon_{\mathrm{e}} \tag{49}$$

Then, the relation between the elastic strain and stress can be derived as:

$$\begin{aligned} \sigma &= \rho\frac{\partial\psi}{\partial\varepsilon_{\mathrm{e}}} = \rho\frac{\partial\psi_{\mathrm{e}}}{\partial\varepsilon_{\mathrm{e}}} = \tilde{\mathbf{C}}_{\mathrm{e}} : \varepsilon_{\mathrm{e}} \\ &= \mathbf{\Phi}^{-1} : \mathbf{C}_{\mathrm{e}} : \mathbf{\Phi}^{-T} : \varepsilon_{\mathrm{e}} \end{aligned} \tag{50}$$

that is,

$$\tilde{\sigma} = \mathbf{C}_{\mathrm{e}}\ \tilde{\varepsilon}_{\mathrm{e}} \tag{51}$$

where

$$\tilde{\varepsilon}_e = \mathbf{\Phi}^{-T} : \varepsilon_e \tag{52}$$

And according to Eq. (38), thermodynamic conjugated forces are defined by functions of internal variables:

$$\begin{aligned} -R &= \rho\frac{\partial\psi}{\partial p} = \rho\frac{\partial\psi_p}{\partial p} \\ -\alpha &= \rho\frac{\partial\psi}{\partial \mathbf{A}} = \rho\frac{\partial\psi_p}{\partial \mathbf{A}} \\ -\mathbf{Y} &= \rho\frac{\partial\psi}{\partial \mathbf{\Omega}} = \rho\frac{\partial\psi_e}{\partial \mathbf{\Omega}} \\ &= \frac{1}{2}\varepsilon_e : \frac{\partial\tilde{\mathbf{C}}}{\partial\mathbf{\Omega}} : \varepsilon_e \end{aligned} \tag{53}$$

Here, $-\mathbf{Y}$ of Eq. (53) means the strain energy release rate due to damage evolution [11, 16, 17].

Plastic-damage flow rule

As shown in Eq. (48), the complementary potential φ^* of Eq. (41) also can be expressed as a sum of the energy dissipation terms $\varphi^*{}_p$ due to plastic flow and $\varphi^*{}_d$ due to damage evolution:

$$\varphi^*(\sigma,\ \alpha,\ R,\ \mathbf{Y}) = \varphi^*{}_p\ (\sigma,\ \alpha,\ R) + \varphi^*{}_d(\mathbf{Y}) \tag{54}$$

Hence, according to Eq. (42), the evolution equations of flux variables can be written as:

$$\begin{aligned} \dot{\varepsilon}_p &= \frac{\partial\varphi^*{}_p}{\partial\sigma} \\ -\dot{\mathbf{A}} &= \frac{\partial\varphi^*{}_p}{\partial\alpha} \\ -\dot{p} &= \frac{\partial\varphi^*{}_p}{\partial R} \\ -\dot{\mathbf{\Omega}} &= \frac{\partial\varphi^*{}_d}{\partial\mathbf{Y}} \end{aligned} \tag{55}$$

When a meterial undergoes a plastic deformation at temperature lower than 1/4 of their absolute melting temperature, the time-dependent viscous effect can be considered negligible. In case of a time-independent instantaneous plastic deformation, the potential φ^* is not differentiable and Eq (55) can be expressed by a multiplicative form of partial derivatives of the equipotential surface function F and a multiplier $\dot{\lambda}$ [13]. In the same manner of φ^* of Eq. (52), the equipotential surface function F can also be decomposed into plastic and damage-related components as:

$$F(\sigma, \alpha, R, \mathbf{Y}) = F_p(\sigma, \alpha, R) + F_d(\mathbf{Y}) \tag{56}$$

Under the hypothesis of the associated plasticity, F_p can be regarded as the identical form of a yield function f and plastic-damage flow rule finally derived as:

$$\dot{\varepsilon}_p = \dot{\lambda}\frac{\partial F_p}{\partial\sigma} = \dot{\lambda}\frac{\partial f}{\partial\sigma}$$

$$\begin{aligned} -\dot{\alpha} &= \dot{\lambda}\frac{\partial \mathrm{F_p}}{\partial \mathbf{A}} = \dot{\lambda}\frac{\partial \mathrm{f}}{\partial \mathbf{A}} \\ -\dot{\mathrm{p}} &= \dot{\lambda}\frac{\partial \mathrm{F_p}}{\partial R} = \dot{\lambda}\frac{\partial \mathrm{f}}{\partial R} \\ -\dot{\mathbf{\Omega}} &= \dot{\lambda}\frac{\partial \mathrm{F_d}}{\partial \mathbf{Y}} \end{aligned} \tag{57}$$

If we postulate that the yield function of a damaged material maintains an identical form of an undamaged state and can be expressed by a function of the effective stress $\tilde{\sigma}$ instead of σ not considering damage, the general form of von Mises' yield function in a damaged material can be expressed as:

$$\mathrm{f}(\tilde{\sigma},\ \tilde{\alpha},\ R) = \tilde{\sigma}_{\mathrm{eq}} - R(\mathrm{p}) \tag{58}$$

where

$$\begin{aligned} \tilde{\sigma}_{\mathrm{eq}} &= \{\frac{3}{2}(\tilde{\sigma}-\tilde{\alpha})^{\mathrm{D}} : (\tilde{\sigma}-\tilde{\alpha})^{\mathrm{D}}\}^{1/2} \\ \tilde{\alpha} &= \mathbf{\Phi} : \alpha \end{aligned} \tag{59}$$

and $(\ )^{\mathrm{D}}$ means the deviatoric component of a tensor. According to the Ramberg-Osgood equation, the isotropic hardening parameter R is given as:

$$R(\mathrm{p}) = \sigma_{\mathrm{y}}(1 + \mathrm{c}\cdot\mathrm{p})^{\mathrm{n}} \tag{60}$$

where c and n are material constants. Inserting Eqns. (58)-(60) into Eq. (57) gives:

$$\begin{aligned} \dot{\varepsilon}_{\mathrm{p}} &= \dot{\lambda}\frac{\partial \mathrm{F}}{\partial \sigma} = \dot{\lambda}\frac{\partial \mathrm{f}}{\partial \tilde{\sigma}}\frac{\partial \tilde{\sigma}}{\partial \sigma} \\ &= \frac{3}{2}\frac{\dot{\lambda}}{\tilde{\sigma}_{\mathrm{eq}}}\mathbf{\Phi} : (\tilde{\sigma}-\tilde{\alpha})^{\mathrm{D}} \\ \dot{\mathbf{A}} &= -\dot{\lambda}\frac{\partial \mathrm{f}}{\partial \alpha} = -\dot{\lambda}\frac{\partial \mathrm{f}}{\partial \alpha} : \frac{\partial \tilde{\alpha}}{\partial \alpha} \\ &= \dot{\varepsilon}_{\mathrm{p}} \\ \dot{\mathrm{p}} &= -\dot{\lambda}\frac{\partial \mathrm{f}}{\partial R} \\ &= \dot{\lambda} \end{aligned} \tag{61}$$

And now, the effective plastic strain $\dot{\tilde{\varepsilon}}_{\mathrm{p}}$ is introduced as:

$$\begin{aligned} \dot{\tilde{\varepsilon}}_{\mathrm{p}} &= \mathbf{\Phi}^{-T} : \dot{\varepsilon}_{\mathrm{p}} = \dot{\lambda}\frac{\partial \mathrm{f}}{\partial \tilde{\sigma}} \\ &= \frac{3}{2}\frac{\dot{\lambda}}{\tilde{\sigma}_{\mathrm{eq}}}(\tilde{\sigma}-\tilde{\alpha})^{D}. \end{aligned} \tag{62}$$

4.3 Damage Evolution Equation

If the damage potential $\mathrm{F_d}$ of Eq. (57) is assumed as a linear function of energy release rate $\mathbf{Y}$[13], the damage evolution rate $\dot{\mathbf{\Omega}}$ can be derived as:

$$\dot{\mathbf{\Omega}} = -\dot{\lambda}\frac{\partial \mathrm{F_d}}{\partial \mathbf{Y}} = -\dot{\lambda}\mathrm{F}_{\Omega}\mathbf{Q} \tag{63}$$

where $\mathbf{Q}$ expresses anisotropic characteristics of a damage evolution which depends on the principal direction of effective stress tensor. Therefore all nonlinearities in the damage evolution process can be expressed in a form of function F_Ω.

If we postulate that the evolution of isotropic damage is proportional to the hydrostatic components of damage and stress tensor and that of anisotropic components are proportional to the deviatoric stress and magnitude of damage in principal directions $\nu^{(i)}$ of effective stress tensor, Eq. (63) can be expressed as:

$$\dot{\mathbf{\Omega}} = \mathbf{V}_\Omega \dot{p} \tag{64}$$

where

$$\mathbf{V}_\Omega = C_1(1 - \beta_1 \Omega_m) \langle \tilde{\sigma}_m \rangle \mathbf{I} + C_2 \sum_{i=1}^{3} (\mathbf{1} - \beta_2 \mathbf{\Omega}_{\nu(i)}) \langle \tilde{\sigma}'_i \rangle (\nu^{(i)} \otimes \nu^{(i)}) \tag{65}$$

$$\begin{aligned} < a > &= 1 \quad \text{when} a > 0 \\ &= 0 \quad \text{when} a \leq 0 \\ \Omega_m &= (\Omega_{11} + \Omega_{22} + \Omega_{33})/3 \qquad (66) \\ \tilde{\sigma}_m &= (\tilde{\sigma}_{11} + \tilde{\sigma}_{22} + \tilde{\sigma}_{33})/3 \qquad (67) \end{aligned}$$

$C_1, C_2, \beta_1, \beta_2$: material parameters
$\tilde{\sigma}'_i$: effective stress components in principal directions
$\nu^{(i)}$: direction cosine vectors of principal directions of effective stress tensor

$$\tilde{\sigma}'_i = \tilde{\sigma}_i - \tilde{\sigma}_m \tag{68}$$

$$\mathbf{\Omega}_{\nu(i)} = \nu^{(i)} \cdot \mathbf{\Omega} \cdot \nu^{(i)} \tag{69}$$

In Eq. (65), the first term of the right hand side is related to the isotropic damage evolution and the second term is related to the anisotropic damage evolution in the principal direction of the effective stress tensor.

4.4 Relation between Rates of Stress and Strain

When a yield function f satisfies following conditions, the plastic deformation and damage start to evolve:

$$f(\tilde{\sigma},\ \tilde{\alpha},\ R) = 0 \tag{70}$$

$$\dot{f} = \frac{\partial f}{\partial \tilde{\sigma}} \dot{\tilde{\sigma}} + \frac{\partial f}{\partial \tilde{\alpha}} \dot{\tilde{\alpha}} + \frac{\partial f}{\partial R} \dot{R} = 0 \tag{71}$$

Using Eqns. (58)-(61), Eq. (71) is transformed to:

$$\frac{\partial f}{\partial \tilde{\sigma}} \dot{\tilde{\sigma}} - \frac{\partial f}{\partial \tilde{\sigma}} \dot{\tilde{\alpha}} - \frac{\partial R}{\partial p} \dot{p} = 0 \tag{72}$$

which is called consistency condition. And $\dot{p}(= \dot{\lambda})$ is derived as:

$$\dot{p} = \dot{\lambda} = \left(\frac{\partial f}{\partial \tilde{\sigma}} \dot{\tilde{\sigma}} - \frac{\partial f}{\partial \tilde{\sigma}} \dot{\tilde{\alpha}}\right) / \frac{\partial R}{\partial p} \tag{73}$$

For the further formulation, the effective plastic strain $\dot{\tilde{\varepsilon}}_p$ is decomposed as:

$$\dot{\tilde{\varepsilon}} = \dot{\tilde{\varepsilon}}_e + \dot{\tilde{\varepsilon}}_p \tag{74}$$

Inserting Eqns. (51) and (74) into the time derivative form of Eq. (51) yields:

$$\begin{aligned} \dot{\tilde{\sigma}} &= \mathbf{C}_e : \dot{\tilde{\varepsilon}}_e \\ &= \mathbf{C}_e : \dot{\tilde{\varepsilon}} - \mathbf{C}_e : \frac{\partial f}{\partial \tilde{\sigma}} \dot{p} \end{aligned} \tag{75}$$

Premultiplying the both side of Eq. (75) by $\partial f/\partial\tilde{\sigma}$, we can have

$$\frac{\partial f}{\partial \tilde{\sigma}} : \dot{\tilde{\sigma}} = \frac{\partial f}{\partial \tilde{\sigma}} : \mathbf{C}_e : \dot{\tilde{\varepsilon}} - \frac{\partial f}{\partial \tilde{\sigma}} : \mathbf{C}_e : \frac{\partial f}{\partial \tilde{\sigma}} \dot{p} \tag{76}$$

and inserting Eq. (76) into the consistency condition of Eq. (72) gives the following equation:

$$\frac{\partial R}{\partial p}\dot{p} = \frac{\partial f}{\partial \tilde{\sigma}} : \mathbf{C}_e : \dot{\tilde{\varepsilon}} - \frac{\partial f}{\partial \tilde{\sigma}} : \dot{\tilde{\alpha}} - \frac{\partial f}{\partial \tilde{\sigma}} : \mathbf{C}_e : \frac{\partial f}{\partial \tilde{\sigma}} \dot{p} \tag{77}$$

Here, the isotropic and kinematic hardening modulus H'_I and H'_K are defined as:

$$\frac{\partial R}{\partial p}\dot{p} = H'_I \dot{p}, \quad \frac{\partial f}{\partial \tilde{\sigma}} \dot{\tilde{\alpha}} = H'_K \dot{p} \tag{78}$$

Inserting Eq. (78) into Eq. (77) gives

$$\dot{p} = \frac{(\partial f/\partial \tilde{\sigma}) : \mathbf{C}_e : \dot{\tilde{\varepsilon}}}{[H'_I + H'_K + (\partial f/\partial \tilde{\sigma}) : \mathbf{C}_e : (\partial f/\partial \tilde{\sigma})]} \tag{79}$$

and by inserting Eq. (79) into Eq. (75), the relation between incremental rates of the effective stress and strain is expressed as:

$$\begin{aligned} \dot{\tilde{\sigma}} &= [\mathbf{C}_e - \frac{\{\mathbf{C}_e : (\partial f/\partial \tilde{\sigma})\} \otimes \{(\partial f/\partial \tilde{\sigma}) : \mathbf{C}_e\}}{H'_I + H'_K + (\partial f/\partial \tilde{\sigma}) : \mathbf{C}_e : (\partial f/\partial \tilde{\sigma})}] : \dot{\tilde{\varepsilon}} \\ &= \mathbf{C}_{ep} : \dot{\tilde{\varepsilon}} \end{aligned} \tag{80}$$

Using Eq. (52) and (62), Eq. (73) leads to:

$$\dot{\tilde{\varepsilon}} = \frac{\partial \mathbf{\Phi}^{-T}}{\partial \mathbf{\Omega}} : \varepsilon_e : \dot{\mathbf{\Omega}} + \mathbf{\Phi}^{-T} : \dot{\varepsilon} \tag{81}$$

and by inserting Eq. (81) into Eq. (80), finally the relation between incremental rates of stress, strain, and damage is formulated as:

$$\dot{\sigma} = \tilde{\mathbf{C}}_{ep} : \dot{\varepsilon} - \mathbf{C}_d : \dot{\mathbf{\Omega}} \tag{82}$$

where

$$\tilde{\mathbf{C}}_{ep} = \mathbf{\Phi}^{-1} : \mathbf{C}_{ep} : \mathbf{\Phi}^{-T} \tag{83}$$

$$\mathbf{C}_d = (\tilde{\mathbf{C}}_{ep} : \frac{\partial \mathbf{\Phi}}{\partial \mathbf{\Omega}} : \mathbf{C}_e^{-1} - \frac{\partial \mathbf{\Phi}^{-1}}{\partial \mathbf{\Omega}}) : \tilde{\sigma} \tag{84}$$

5. NUMERICAL CALCULATIONS AND DISCUSSIONS

5.1 Derivation of Material Parameters

Chow and Wang[4, 5] carried out simple uniaxial tension tests for aluminium alloy 2024-T3 specimens and investigated the effects of damage evolution to the plastic deformation in ductile materials. The test process is shown briefly in Fig. 3. To estimate the material properties for the plastic deformation and damage evolution, the stress-strain history of a uniaxial tension specimen was recorded and effective elastic modulus and Poisson's ratio were measured during the repeated loading and unloading processes.

The detailed process to estimate material parameters related to the anisotropic damage evolution from the measured data is as follows. In simple uniaxial tension specimen, $\sigma_{22} = \sigma_{33} = \sigma_{12} = \sigma_{23} = \sigma_{31} = 0$, $\varepsilon_{12} = \varepsilon_{23} = \varepsilon_{32} = 0$, $\Omega_{12} = \Omega_{23} = \Omega_{31} = 0$ and from Eq. (51) (52) (44), following relations are derived:

$$\begin{aligned}
\varepsilon_{11}^{e} &= \frac{\sigma_{11}}{E(1-\Omega_{11})^2} = \frac{\sigma_{11}}{\tilde{E}} \\
\varepsilon_{22}^{e} &= -\frac{\nu\sigma_{11}}{E\,(1-\Omega_{11})(1-\Omega_{22})} = \frac{\tilde{\nu}_{12}}{\tilde{E}}\sigma_{11} \\
\varepsilon_{33}^{e} &= -\frac{\nu\sigma_{11}}{E(1-\Omega_{11})(1-\Omega_{33})} = \frac{\tilde{\nu}_{13}}{\tilde{E}}\,\sigma_{11}
\end{aligned} \tag{85}$$

where the effective elastic modulus and the effective Poisson's ratio are defined as:

$$\begin{aligned}
\tilde{E} &= E(1-\Omega_{11})^2 \\
\tilde{\nu}_{12} &= \nu(1-\Omega_{11})/(1-\Omega_{22}) \\
\tilde{\nu}_{13} &= \nu(1-\Omega_{11})/(1-\Omega_{33})
\end{aligned} \tag{86}$$

From Eq. (86), the components of the anisotropic damage tensor Ω at each deformation step are calculated as:

$$\begin{aligned}
\Omega_{11} &= 1-(\tilde{E}/E)^{1/2} \\
\Omega_{22} &= 1-(1-\Omega_{11})\nu/\tilde{\nu}_{12} \\
\Omega_{33} &= 1-(1-\Omega_{11})\nu/\tilde{\nu}_{13}.
\end{aligned} \tag{87}$$

In a simple uniaxial tension problem, ε_{22} is identical to ε_{33}, thus relations of $\tilde{\nu}_{12} = \tilde{\nu}_{13}$, $\Omega_{22} = \Omega_{33}$ are valid. Chow and Wang[4] measured the variations of $\tilde{E}$ and $\tilde{\nu}$ in uniaxial tension specimens as shown in Fig. 4 and Fig. 5, and calculated the damage variables based on the measured data and Eq. (87). The damage evolution curves and effective stress-strain relation are displayed in Fig. 6 and Fig. 7.

By fitting the points of Fig. 6, the material parameters C_1, C_2, β_1 and β_2 in the damage evolution equation of (65) are determined and the other parameters σ_y, c and n related to the plastic deformation in Eq. (60) are calculated from Fig. 7. The derived material parameters are summarized in Table 3. A kinematic hardening effect for reverse loading is not considered here.

Table 3: Material constants for plastic-damage analysis

Material properties		Value	
Elastic modulus	E	73600	MPa
Poisson's ratio	n	0.35	
Yield stress	σ_y	350	MPa
Isotropic hardening parameters	c	35	
	n	0.3	
Damage evolution parameters	C_1	0.45	
	C_2	0.5	
	β_1	7	
	β_2	16	

5.2 Verification of the Proposed Constitutive Model

To verify the validity of the proposed constitutive model of Eq. (62), a nonlinear large strain problem of a uniaxial tension in a plane stress condition shown in Fig. 4 is analyzed. The process of damage evolution and effective stress-strain curve are calculated and compared with the phenomenological constitutive model. Fig. 5.7 shows that both results are nearly identical. Also, we can observe that the analyzed effective stress curve is similar to the hardening curve of the material within 2% error range. The Cauchy stress ignoring the damage effect shows about 10% lower values than effective stress. The nominal stress not considering the area reduction effect in tension shows a peak value at elongation $\delta = 0.2$ cm. After the peak, the stress diminishes gradually, which implies that an unstable necking phenomenon takes place.

CONCLUSIONS

In this paper, an elastio-plastic-damage constitutive equation is derived for ductile materials within the framework of thermodynamic principles. Due to microstructural changes of a material in large deformation, suitable internal state variables should be presented to explain the irreversible energy dissipation process. Using continuum damage mechanics approach, a three-dimensional anisotropic damage which can represent micro-cracks and micro-voids is defined as a measure of an effective area reduction rate. The proposed constitutive model shows good agreement with experiments. The model will give rational approach to the material damage and more realistic result in large deformation analysis.

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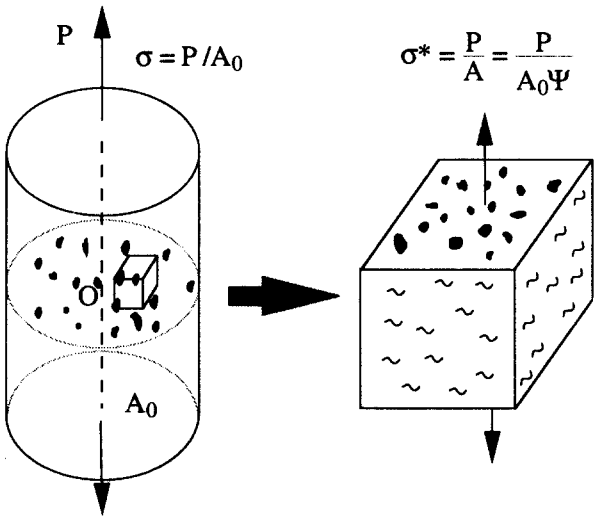

Figure 1: Damage model of Kachanov[3]

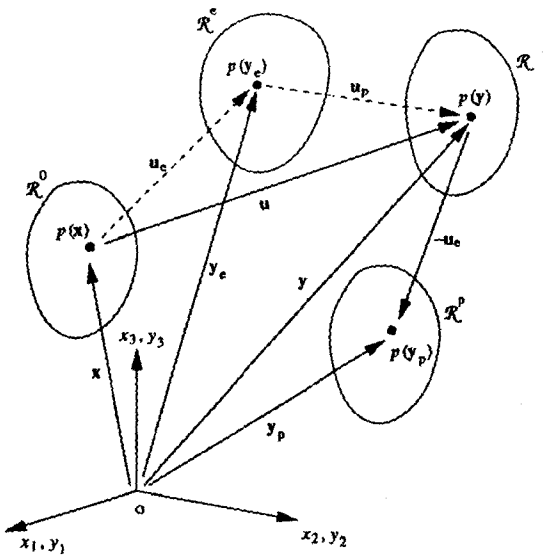

Figure 2: Configurations in elasto-plastic deformations

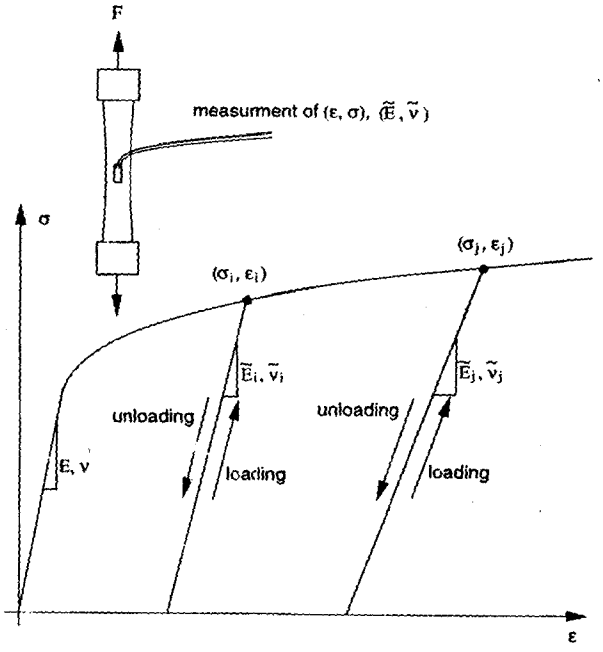

Figure 3: Measurement process of anisotropic damage

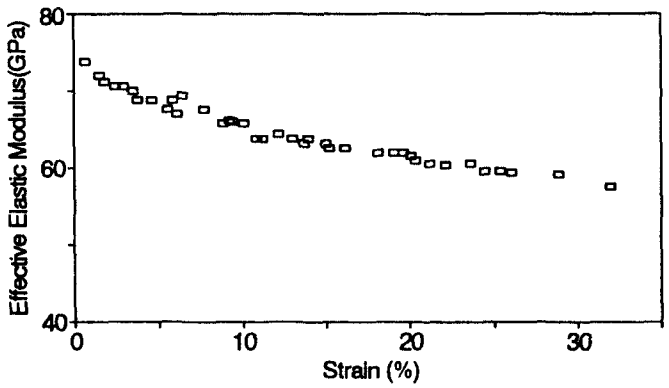

Figure 4: Variation of the effective elatic modulus

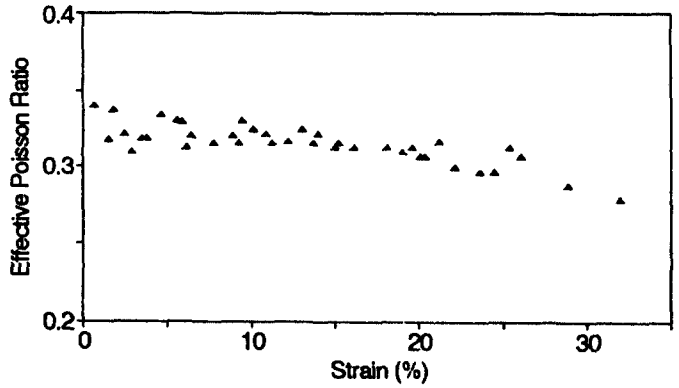

Figure 5: Variation of the effective Poisson's ratio

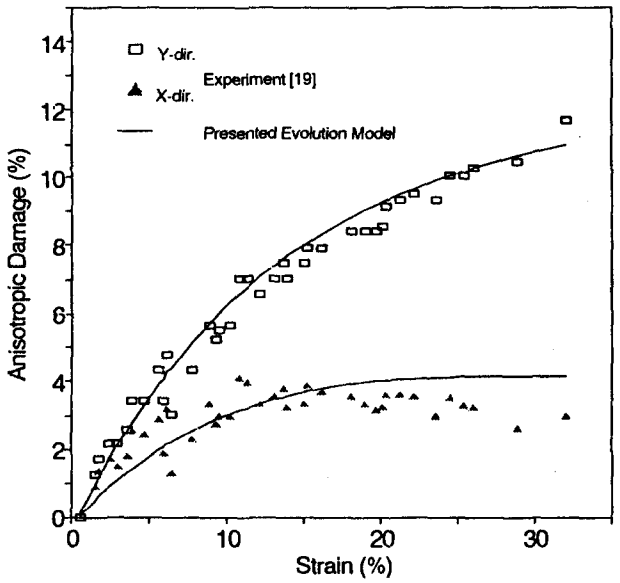

Figure 6: Measured anisotropic damages and proposed evolution curves

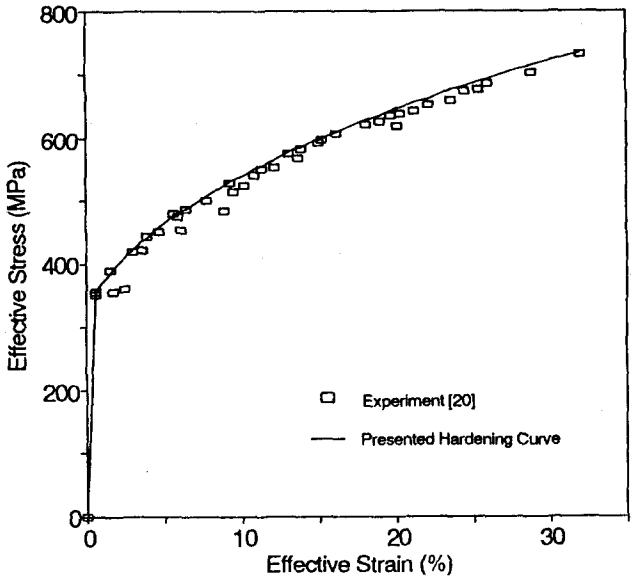

Figure 7: Relation between effective stress and strain: Measured data and fitted hardening curve

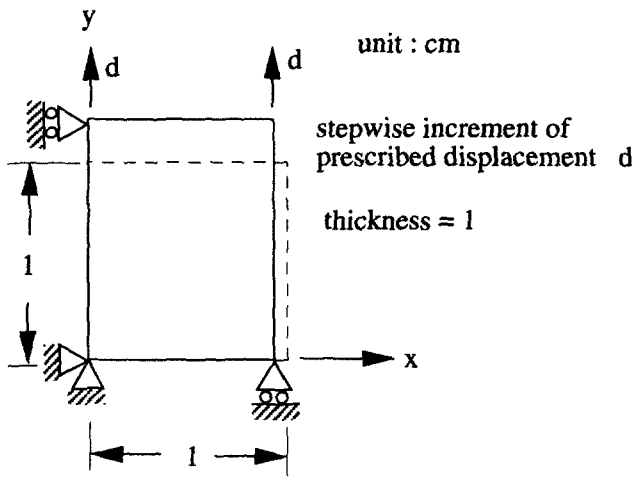

Figure 8: Uni-axial tension of a plane stress element

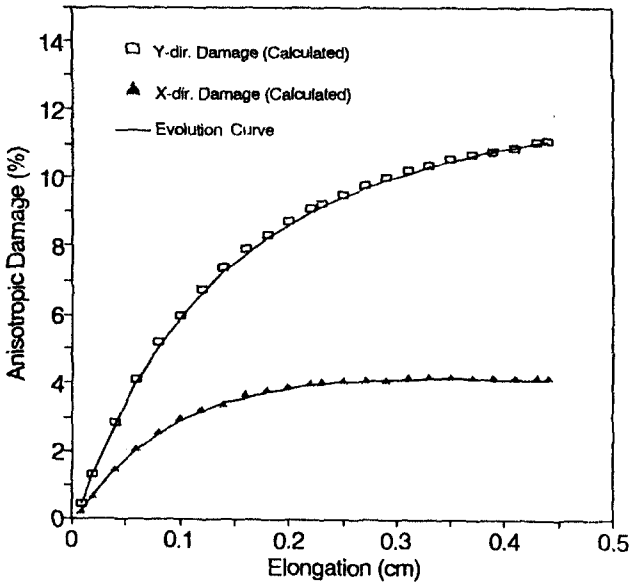

Figure 9: Calculated anisotropic damages

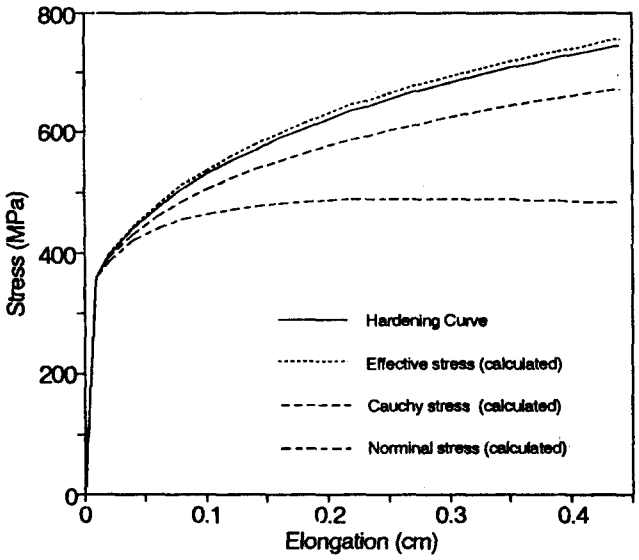

Figure 10: Calculated stress-displacement curves