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# CONFORMAL DEFORMATION OF WARPED PRODUCT METRICS

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## 1. Introduction

Let $(M, g)$ be an $n+1 (n \geq 3)$-dimensional differential manifolds with Lorentzian metric $g$. We say that another Lorentzian metric $\overline{g}$ on the given manifold is conformal to $g$ if there exists a positive function $u$ on $M$ such that $\overline{g} = u^{\frac{4}{n-1}} g$. If $H$ is a function on $M$, then we naturally ask : Can we find a function $u$ on $M$ such that $\overline{g}$ is conformal to $g$ and $H$ is the scalar curvature of $\overline{g}$? This question is equivalent to the problem of solving the hyperbolic nonlinear partial differential equation

$$(1.1) \qquad \frac{4n}{n-1} \Box_g u - hu + H u^{\frac{n+3}{n-1}} = 0, u > 0,$$

where $\Box_g$ is the D'Albertian in the $g$- metric and $h$ is the scalar curvature of $g$ (cf. [K.W.1,2,3], [A,p.126], or [N],etc.).

In particular, in this paper, we consider the conformal deformation of metrics on some warped product manifold $M = B \times_v F$ (cf. Definition 2.1). The concept of the warped product manifold is important not only in Riemannian geometry but also in Lorentzian geometry. In Riemannian geometry, the warped product is used for studying manifolds with various curvatures (cf.[B.O.], [D.G.], [D.D.], [Eb], [Ej], [K.K.P.], [M.M.]). And there is other application for the cohomology theory (cf.

[Z]). And in Lorentzian geometry, for example, Minkowski spacetime, Schwarzschild spacetime and the Robertson-Walker spacetime are well-known examples of Lorentzian warped product manifolds(cf.[B.E.],[B.E.P.],[O.]).

Although throughout this paper we will assume that all data ($M$, metric $g$, and curvature, etc.) are smooth, this is merely for convenience. Our proofs go through with little or no change if one makes minimal smoothness hypotheses. For example, without changing any proofs we need only assume that the given data are Holder continuous.

## 2. Preliminaries on a warped product manifold

In this section, we briefly recall some results on a warped product manifold. Complete details may be found in [B.E.], [B.O.], or [O].

On a Lorentzian product manifold $B \times F$, let $\pi$ and $\sigma$ be the first and second projections of $B \times F$ onto $B$ and $F$, respectively, and let $v > 0$ be a smooth function on $B$.

**DEFINITION 2.1.** *The warped product manifold $M = B \times_v F$ is the product manifold $M = B \times F$ furnished with metric tensor*

$$g = \pi^*(g_B) + (v \circ \pi)\sigma^*(g_F),$$

*where $g_B$ and $g_F$ are metric tensors of $B$ and $F$, respectively. In other words, if $X$ is tangent to $M$ at $(p,q)$, then*

$$g(X,X) = g_B(d\pi(X), d\pi(X)) + v(p)g_F(d\sigma(X), d\sigma(X)).$$

Here $B$ is called the base of $M$ and $F$ the fiber. And we denote the metric $g$ by $<,>$ . In view of the following (1) in [Remark 2.2] and Lemma 2.3, we also denote the metric $g_B$ by $<,>$ and the metric $g_F$ by $(,)$.

*[REMARK 2.2].*. Now we list some elementary properties of the warped product manifold $M = B \times_v F$. (For details, see [B.E.], [B.O.] or [O]).

(1) For each $q \in F$, the map $\pi|_{\sigma^{-1}(q)=B\times q}$ is an isometry onto $B$

(2) For each $p \in B$, the map $\sigma|_{\pi^{-1}(p)=p\times F}$ is a positive homothetic map onto $F$ with homothetic factor $1/v(p)$.

(3) For each $(p,q) \in M$, the horizontal leaf $B \times q$ and the vertical fiber $p \times F$ are orthogonal at $(p,q)$.

(4) The horizontal leaf $\sigma^{-1}(q) = B \times q$ is a totally geodesic submanifold of $M$ and the vertical fiber $\pi^{-1}(p) = p \times F$ is a totally umbilic submanifold of $M$.

(5) If $\phi$ is an isometry of $F$, then $1 \times \phi$ is an isometry of $M$. And if $\psi$ is an isometry of $B$ such that $v = v \circ \psi$, then $\psi \times 1$ is an isometry of $M$.

Recall that vectors tangent to leaves are called horizontal and vectors tangent to fibers are called vertical. From now on, we will often use a natural identification $T_{(p,q)}(B \times_f F) \cong T_{(p,q)}(B \times F) \cong T_p B \times T_q F$. The decomposition of vectors into horizontal and vertical parts plays a role in our proofs. If $X$ is a vector field on $B$, we define $\overline{X}$ at $(p,q)$ by setting $\overline{X}(p,q) = (X_p, 0_q)$. Then $\overline{X}$ is $\pi$-related to $X$ and $\sigma$-related to the zero vector field on $F$. Similarly, if $Y$ is a vector field on $F$, $\overline{Y}$ is defined by $\overline{Y}(p,q) = (0_p, Y_q)$.

**LEMMA 2.3.** *If $h$ is a smooth function on $B$, then the gradient of the lift $h \circ \pi$ of $h$ to $M$ is the lift to $M$ of gradient of $h$ on $B$.*

*Proof* See Lemma 7.34 in [O].

In view of Lemma 2.3, letting $grad, grad_B$ and $grad_F$ denote the gradient vector fields on $(M,g),(B,g_B)$ and $(F,g_F)$ respectively, it follows that for smooth functions $\phi_1 : B \to \mathbf{R}$ and $\phi_2 : F \to \mathbf{R}$,

$$grad(\phi_1 \circ \pi)(p,q) = (grad_B \phi_1(p), 0_q)$$

and

$$grad(\phi_2 \circ \sigma)(p,q) = (0_p, \frac{1}{v(p)} grad_F \phi_2(q)),$$

where $0_p$ and $0_q$ denote the zero tangent vectors of $T_p(B)$ and $T_q(F)$ respectively.

**PROPOSITION 2.4.** *Let $\Phi : B \times_v F \to \mathbf{R}$ be a smooth function of the form $\Phi = (\phi_1 \circ \pi)(\phi_2 \circ \sigma)$, where $\phi_1 : B \to \mathbf{R}$ and $\phi_2 : F \to \mathbf{R}$ are smooth. Then*

$$\Delta\Phi(p,q) = [\Box^B \phi_1(p) + \frac{dim F}{2v(p)}(grad_B \phi_1(p))(v)]\phi_2(q) + \frac{\phi_1(p)}{v(p)}\Delta\phi_2(q),$$

*where $\Box^B$ denotes the d'Alembertian of $(B, g_B)$ and $\Delta$ denotes the Laplacian of $(F, g_F)$.*

*Proof* See Theorem 5.4 in [B.E.P.].

## 3. Main result

In this section, we restrict our results to the case that $B = (a, b)$ is an open connected subset of $R$ with the negative definite metric $-dt^2$ and $-\infty \le a < b \le +\infty$. In [E.J.K.], P.E.Ehrlich, Y.T.Jung and S.B.Kim have studied the existence of a warping $v(t)$ such that the resulting warped metric admits a constant scalar cvurvature on $(a, b) \times_v F$.

From now on, we assume that for the warped product $M = B \times_v F, B = (a, b)$ with $-\infty \le a < b \le +\infty$ and $F$ is a compact $n(\ge 2)$-dimensional Riemannian manifold and the resulting warped metric admits constant scalar curvature $k$.

Naturally we ask the following problem:

*Problem (A):.* Assume that $(M = B \times_v F, g)$ satisfies the above assumptions. And let $H(t, x)$ be a smooth function on $M$. Then does there exist a new metric $\overline{g}$ on $M$ such that $\overline{g}$ is conformal to $g$(i.e.,$\overline{g} = u(t, x)^{\frac{4}{n-1}} g$) and $H(t, x)$ is a scalar curvature of $\overline{g}$?

Recalling that $dimM = n + 1$, according to the equation (1.1), Question (A) is equivalent to the problem of solving the elliptic nonlinear partial differential equation

$$\frac{4n}{n-1}\Box_g u - ku + Hu^{\frac{n+3}{n-1}} = 0, u > 0, \tag{3.1}$$

where $\Box_g$ is the D'Albertian in the $g$-metric on $M = B \times_v F$ and $k$ is the constant scalar curvature of $g$.

Let $\Phi : M = B \times_v F \to R$ be a smooth function of the form $\Phi = \phi_1(t)\phi_2(x)$, where $\phi_1 : (a, b) \to R$ and $\phi_2 : F \to R$ are smooth. Then, in view of Proposition 2.4, recalling that $\Box^B \phi_1(t) = -\phi_1''(t)$ and $grad_B \phi_1(t)(v) = -\phi_1'(t)v'(t)$,

$$\Box_g \Phi = -[\phi_1''(t) + \frac{n}{2v(t)}\phi_1'(t)v'(t)]\phi_2(x) + \frac{\phi_1(t)}{v(t)}\Delta_{g_F}\phi_2(x).$$

**THEOREM 3.2.** *There exists a positive $\phi_2(x)$ solution of the equation $\Delta_{g_F}\phi_2(x) + K(x)\phi_2(x)^{\frac{n+3}{n-1}} = 0$ if and only if either $K(x) \equiv 0$ or $\int_F K(x)dV < 0$ and $K(x)$ changes sign.*

*Proof* See Theorem 5 and Theorem 7 in [J.]

If $\phi_1(t)$ is the solution of a linear ordinary differential equation

$$-\frac{4n}{n-1}\phi_1''(t) - \frac{2n^2 v'(t)}{(n-1)v(t)}\phi_1'(t) - k\phi_1(t) = 0,$$

then we can find $\phi_1(t)$ which is positive. This is possible because ,by the elementary ordinary differential equation theorem, we can choose the suitable initial conditions about $\phi_1(x_0)$ and $\phi_1'(x_0)$ for some point $x_0 \in (a,b)$ so that $\phi_1(x)$ is positive in (a,b), maybe, in a smaller domain. If so, at first, we choose the domain as ours. Therefore, we have the following theorem.

**THEOREM 3.3.** *Let $\phi_1(t)$ be the positive solution of a linear ordinary differential equation $-\frac{4n}{n-1}\phi_1''(t) - \frac{2n^2 v'(t)}{(n-1)v(t)}\phi_1'(t) - k\phi_1(t) = 0$ and let $H(t,x)$ be a function of the form $\frac{K(x)}{\phi_1(t)^{\frac{4}{n-1}}v(t)}$, where $K(x)$ changes sign and $\int_F K(x)dV < 0$. Then there exists a new metric $\overline{g}$ on $M = (a,b)\times_v F$ such that $\overline{g}$ is conformal to $g$ and $H(t,x)$ is a scalar curvature of $\overline{g}$.*

*Proof* We have only to show that there exists a solution of the equation (3.1). Put $u(t,x) = \phi_1(t)\phi_2(x)$. Theorem 3.2 and Theorem 3.3 imply the following:

$$\begin{aligned}
&\frac{4n}{n-1}\Box_g u(t,x) - ku(t,x) + H(t,x)u(t,x)^{\frac{n+3}{n-1}}\\
&= \frac{4n}{n-1}\Box_g(\phi_1(t)\phi_2(x)) - k\phi_1(t)\phi_2(x) + H(t,x)(\phi_1(t)\phi_2(x))^{\frac{n+3}{n-1}}\\
&= \frac{4n}{n-1}[-\phi_1''(t) - \frac{n}{2v(t)}\phi_1'(t)v'(t)]\phi_2(x) + \frac{4n}{n-1}\frac{\phi_1(t)}{v(t)}\Delta_{g_F}\phi_2(x) - k\phi_1(t)\phi_2(x)\\
&+ H(t,x)(\phi_1(t)\phi_2(x))^{\frac{n+3}{n-1}}\\
&= -\frac{4n}{n-1}[\phi_1''(t) + \frac{n}{2v(t)}\phi_1'(t)v'(t) - \frac{k(n-1)}{4n}\phi_1(t)]\phi_2(x)\\
&+ \frac{\phi_1(t)}{v(t)}[\frac{4n}{n-1}\Delta_{g_F}\phi_2(x) + K(x)\phi_2(x)^{\frac{n+3}{n-1}}]\\
&= 0.
\end{aligned}$$

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