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A NOTE ON THE FUNCTION SPACE $\mathcal{M}$

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1. Introduction

We consider the set of all real valued β- measurable functions defined on (X, β, μ) and indentify μ- equivalent β measurable functions. This means that we deal with a set $\mathcal{M} \equiv \mathcal{M}(X, \beta, \mu)$ of real valued measurable functions which contains exactly one representative for each μ- equivalence class. Thus the set $\mathcal{M}$ is the set of all non μ- equivalent real valued β- measurable functions on (X, β, μ). Also $\mathcal{M}$ is a vector space over the real field under the pointwise addition and the pointwise scalar multiplication.

Now we shall give the topology $\mathcal{T}$ on $\mathcal{M}$ determined by a family of pseudometric on $\mathcal{M}$, $\mathcal{D} = \{d_E : E \in \beta, \mu(E) < \infty\}$; that is, a subbasis for the topology is formed by the sets

$$B_E(f, \delta) = \{g \in \mathcal{M} : d_E(f, g) < \delta\}, f \in \mathcal{M}, \ \delta > 0, \ d_E \in \mathcal{D}$$

This topology $\mathcal{T}$ on $\mathcal{M}$ will be called the topology of convergenc in measure on the measurable subsets of X whose measure is finite.

In this paper we investigate some topological structure $\mathcal{T}$ of $\mathcal{M}$. Indeed, $(\mathcal{M}, \mathcal{T})$ becomes a topological vector space over $\mathbb{R}$, and then the convergence of a sequence (f_n) to a function f in $\mathcal{M}$ relative to the topology $\mathcal{T}$ is equivalent to that of (f_n) to f with respect to d_E for every $d_E \in \mathcal{D}$.

Lastly we show that if a measure space (X, β, μ) is a σ-finite, one can define a complete invariant metric d on $\mathcal{M}$ which is compatible with the topology $\mathcal{T}$ on $\mathcal{M}$, and hence $(\mathcal{M}, \mathcal{T})$ becomes a F-space over $\mathbb{R}$.

2. Topological structures of $\mathcal{M}$

In this section we shall topologize the set $\mathcal{M}$ by a family of pseudometrics on M. And then it will be seen that M is in fact a topological vector space over the real field. We also examine a relationship between the convergence of a sequence (f_n) in $\mathcal{M}$ with respect to the topology on $\mathcal{T}$ and that of (f_n) in $\mathcal{M}$ with respest to pseudometric on $\mathcal{M}$ which induced $\mathcal{T}$.

Let (X, β, μ) be a measure space and $\mathcal{M}$ be the vector space of all real valued measurable functions defined on (X, β, μ) and let $E \in \beta$ with $\mu(E) < \infty$

Define $d_E : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ by

$$d_E(f, g) = \int_E \frac{|f - g|}{1 + |f - g|}\, d\mu$$

Then d_E is an invariant pseudometric on $\mathcal{M}$.

A sequence (f_n) in $\mathcal{M}$ converges locally in measure μ to $f \in \mathcal{M}$ if and only if $d_E(f_n, f) \to 0$ as $n \to \infty$ for all $d_E \in \mathcal{D}$.

Let (f_n) be a sequence in $\mathcal{M}$ such that $d_E(f_n, f) \to 0$ as $n, m \to \infty$ for every $d_E \in D$. If (X, β, μ) is a σ-finite measure space, then it follows from Theorem 7.6 (1,p,69) and the fact that there exists a sequence (E_n) in β such that $(E_n) < \infty$ and $E_n\acute{s}$ are pairwise disjoint, that there exists a function $f \in \mathcal{M}$ such that $d_E(f_n, f) \to 0$ as $n \to \infty$ for every $d_E \in \mathcal{D}$.

DEFINITION 2-1. *Let $\mathcal{D} = \{d_E : E \in \beta, \mu(E) < \infty\}$ be the family of pseudometrics on E. Then we provide the topology $\mathcal{T}$ on $\mathcal{M}$ determined by $\mathcal{D}$; that is, a subbasis for the topology is formed by the sets $B_E(f, \varepsilon) = \{g \in \mathcal{M} : d_E(f, g) < \varepsilon\}, f \in \mathcal{M}, \varepsilon > 0 \;\; d_E \in \mathcal{D}$.*

This topology $\mathcal{T}$ on $\mathcal{M}$ will be called the topology of convergence in measure on every measurable subsets of X whose measure is finite.

We note that a basic open neighborhood of f in the topology $\mathcal{T}$ is of the form

$$\begin{aligned} U(f; \varepsilon; d_{E_1}, d_{E_2}, d_{E_3}, \dots, d_{E_n}) &= \{g \in \mathcal{M} : d_{E_k}(f, g) < \varepsilon, k = 1, 2, 3 \dots, n\} \\ &= \bigcap_{k=1}^{n} B_{E_k}(f, \varepsilon) \end{aligned}$$

where $d_{E_1}, d_{E_2}, d_{E_3}, \dots, d_{E_n} \in \mathcal{D}$ and $\varepsilon > 0$.

THEOREM 2-2. *The topological space $(\mathcal{M}, \mathcal{T})$ is topological vector space over $\mathbb{R}$.*

Proof For any $f, g \in \mathcal{M}$ and $\lambda \in \mathbb{R}$, since $f+g$ and λf are clearly β-measurable functions, We have $f+g \in \mathcal{M}$ and $\lambda f \in \mathcal{M}$

Thus $\mathcal{M}$ is a vector space over $\mathbb{R}$, Now it remains only to show that the vector operations are continuous. First, We show that the addition $+$ is continuous.

Let $f_0, g_0 \in \mathcal{M}$ and $\varepsilon > 0$, and consider the open neighborhood

$$U(f_0, g_0, ; \varepsilon; d_{E_1}, d_{E_2}, d_{E_3}, \ldots, d_{E_n}) \text{ of } f_0 + g_0 \text{ in } \mathcal{T}.$$

If U denotes the open neighborhood

$$U(f_0; \frac{\varepsilon}{2}; d_{E_1}, d_{E_2}, d_{E_3}, \ldots, d_{E_n}) \times U(g_0; \frac{\varepsilon}{2}; d_{E_1}, d_{E_2}, d_{E_3}, \ldots, d_{E_n})$$

in the product topology on $\mathcal{M} \times \mathcal{M}$, then clearly $(f, g) \in U$ implies that

$$\begin{aligned} d_{E_k}(f+g, f_0+g_0) &= \int_{E_k} \frac{|(f+g)-(f_0+g_0)|}{1+|(f+g)-(f_0+g_0)|}\, d\mu \\ &\le \int_{E_k} \frac{|f-f_0|+|g-g_0|}{1+|f-f_0|+|g-g_0|}\, d\mu \\ &\le \int_{E_k} \frac{|f-f_0|}{1+|f-f_0|}\, d\mu + \int_{E_k} \frac{|g-g_0|}{1+|g+g_0|}\, d\mu \\ &= d_{E_k}(f, f_0) + d_{E_k}(g, g_0) \\ &\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad (k = 1, 2, \ldots, n) \end{aligned}$$

This shows that addition is continuous. Next we show that scalar multiplication is continuous. Let $f_0 \in \mathcal{M}$ and $\lambda_0 \in \mathbb{R}$ be fixed for any $d_E \in \mathcal{D}$.

$$\begin{aligned} d_E(\lambda f, \lambda_0 f_0) &\le d_E(\lambda f, \lambda f_0) + d_E(\lambda f_0, \lambda_0 f_0) \\ &= \int_E \frac{|\lambda f - \lambda f_0|}{1+|\lambda f - \lambda f_0|}\, d\mu + \int_E \frac{|\lambda f_0 - \lambda_0 f_0|}{1+|\lambda f_0 - \lambda_0 f_0|}\, d\mu \\ &= \int_E \frac{|\lambda||f-f_0|}{1+|\lambda||f-f_0|}\, d\mu + \int_E \frac{|\lambda-\lambda_0||f_0|}{1+|\lambda-\lambda_0||f_0|}\, d\mu \\ &\le (1+|\lambda_0|) \int_E \frac{|f-f_0|}{1+|f-f_0|}\, d\mu + \int_E \frac{|\lambda-\lambda_0||f_0|}{1+|\lambda+\lambda_0||f_0|}\, d\mu \\ &= (1+|\lambda_0|)\, d_E(f, f_0) + d_E(|\lambda-\lambda_0|f_0, 0) \qquad (*) \end{aligned}$$

Provided $|\lambda - \lambda_0| < 1$. Now we see that Lebesgue Dominated Convergence Theorem (1, p.44) implies

$$\lim_{\delta \to 0} \int_E \frac{\delta |f_0|}{1 + \delta |f_0|} = \lim_{\delta \to 0} d_E(\delta f_0, 0) = 0 \qquad (**)$$

Let $\varepsilon > 0$. For any $d_{E_1}, d_{E_2}, d_{E_3}, \ldots, d_{E_n}$ in $\mathcal{D}$ it follows from (**) that there exit positive real numbers $\delta_1, \delta_2, \delta_3, \ldots, \delta_n$ in $(0,1)$ such that $0 < \delta < \delta_k$ implies $|d_{E_k}(\delta f_0, 0)| < \frac{\varepsilon}{2}$

Let $\delta_0 = min\{\delta_1, \delta_2, \delta_3, \ldots, n\}$, then $0 < \delta < \delta_0$ implies

$$|d_{E_k}(\delta_0, 0)| < \frac{\varepsilon}{2} \quad \text{for all } k = 1, 2, 3, 4, \ldots, n.$$

Now consider the open neighborhood

$$U(\lambda_0 f_0; \varepsilon; d_{E_1}, d_{E_2}, d_{E_3}, \ldots, d_{E_n}) \text{ of } \lambda_0 f_0 \text{ in } \mathcal{T}.$$

If U denotes the open neighborhood

$$\{\lambda \in \mathbb{R} : |\lambda - \lambda_0| < \delta_0\} \times U(f_0; (\frac{\varepsilon}{2})(1 + |\lambda_0|); d_{E_1}, d_{E_2}, d_{E_3}, \ldots, d_{E_n})$$

in the product tolology on $\mathbb{R} \times \mathcal{M}$, then $\lambda f \in U$ and $(*)$ imply that

$$\begin{aligned} d_{E_k}(\lambda f, \lambda_0 f_0) &\leq (1 + |\lambda_0|) d_E(f, f_0) + d_{E_k}(|\lambda - \lambda_0| f_0, 0) \\ &< (1 + |\lambda_0|)(\frac{\varepsilon}{2})(1 + |\lambda_0|) + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for every $k = 1, 2, \ldots, n$.
This shows that scalar multiplication is continuous.

THEOREM 2-3. *A sequence (f_n) in $\mathcal{M}$ converges to $f \in \mathcal{M}$ in the topology $\mathcal{T}$ if and only if for any $d_E \in \mathcal{D}$, $d_E(f_n, f) \to 0$ as $n \to \infty$.*

Proof $(\Longrightarrow)$ Let $\varepsilon > 0$ be given. Then for each $d_E \in \mathcal{D}$, the neighborhood $U(f; \varepsilon; d_E)$ is an open neighborhood of f in $\mathcal{T}$. Since (f_n) converges to f in $(\mathcal{M}, \mathcal{T})$, there exists some N such that if $n > N$, then $f_n \in U(f; \varepsilon; d_E)$, that is $d_E(f_n, f) < \varepsilon$. Thus $\lim_{n \to \infty} d_E(f_n, f) = 0$

($\Longleftarrow$) Let U be an open set containing f in the topology $\mathcal{T}$. Then by the definition of $\mathcal{T}$, there exit $d_{E_1}, d_{E_2}, d_{E_3}, \ldots, d_{E_n} \in \mathcal{D}$. such that

$$U(f; \varepsilon; d_{E_1}, d_{E_2}, d_{E_3}, \ldots, d_{E_n}) \subset U$$

Since $\lim_{n\to\infty} d_E(f_n, f) = 0$ for all $d_E \in \mathcal{D}$, for each $d_{E_1}, d_{E_2}, d_{E_3}, \ldots, d_{E_n}$, there exist some N_k, $k = 1, 2, 3, 4, \ldots, n$ such that if $n > N_k$ $k = 1, 2, 3, \ldots, n$ then $d_{E_k}(f_n, f) < \varepsilon$.

Now let $N = max\{N_1, N_2, N_3, \ldots, N_n\}$, then for all $n > N$, $d_{E_k}(f_n, f) < \varepsilon$ for all $k = 1, 2, 3, \ldots, n$. Thus $f_n \in U(f; \varepsilon; d_{E_1}, d_{E_2}, d_{E_3}, \ldots, d_{E_n})$ for all $n > N$. Hence (f_n) converges to f in the topology $\mathcal{T}$.

For any two functions $f, g \in \mathcal{M}$, let $d : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ be defined by

$$d(f,g) = \sum_{n=1}^{\infty} 2^{-n} \frac{d_{E_n}(f,g)}{1 + d_{E_n}(f,g)}$$

where

$$d_{E_n}(f,g) = \int_{E_n} \frac{|f-g|}{1+|f-g|}\, d\mu \;\; n = 1, 2, 3, \cdots.$$

Then it easily follow that d is an invariant metric on $\mathcal{M}$. Indeed, we shall show that it is possible to define a complete invariant metric on $\mathcal{M}$ which is compatible with the topology.

THEOREM 2-4. *The function space $(\mathcal{M}, d)$ is a complete metric space.*

The metric topology $\mathcal{T}_d$ on $(\mathcal{M}, d)$ determined by d coincides with the topology $\mathcal{T}_1$ determined by a family of pseudometrics, $\{d_{E_n} : n = 1, 2, 3, 4, 5, \ldots\}$. Consequently $\lim_{n\to\infty} d(f_n, f) = 0$ if and only if $\lim_{n\to\infty} d_E(f_n, f) = 0$ for all $n = 1, 2, 3, \cdots$.

Proof Let (f_n) be a Cauchy sequence in $(\mathcal{M}, d)$. Then $d(f_n, f) \to 0$ as $m, n \to \infty$. For any $k \geq 1$, we note that $d_{E_k}(f_m, f_n) \leq 2^k\, d(f_m, f_n)$ for all $m, n = 1, 2, 3, \ldots$. Thus $d_{E_k}(f_m, f_n) \to 0$ for every k as $m, n \to \infty$, so that (f_n) converges in $\mathcal{M}$ as E_k to a function $f \in \mathcal{M}$. Since

$$\sum_{i=1}^{k} 2^{-i} \frac{d_i(f_n, f)}{1 + d_i(f_n, f)}$$

converges uniformly in n, it follows from the iterated limit theorem (2, p.143) that

$$\begin{aligned}\lim_{n\to\infty}\sum_{k=1}^{\infty}2^{-k}\frac{d_k(f_n,f)}{1+d_k(f_n,f)} &= \lim_{n\to\infty}\lim_{k\to\infty}\sum_{i=1}^{k}2^{-i}\frac{d_i(f_n,f)}{1+d_i(f_n,f)}\\ &= \lim_{k\to\infty}\sum_{i=1}^{k}2^{-i}\lim_{n\to\infty}\frac{d_i(f_n,f)}{1+d_i(f_n,f)}\\ &= 0\end{aligned}$$

Hence $\lim_{n\to\infty} d(f_n,f)=0$
Therefore d is a complete metric on $\mathcal{M}$.

Now we shall show that $\mathcal{T}_d=\mathcal{T}_1$. To show that $\mathcal{T}_d\subset\mathcal{T}_1$, it is enough to show that for any $f\in\mathcal{M}$ and for any subbasic open neighborhood of f relative to $\mathcal{T}_d$ of the form $B(f,\varepsilon)=\{g\in\mathcal{M}|d(f,g)<\varepsilon\}$, there exists a sufficiently large positive integer m such that

$$B_{E_m}(f,1/2^m)=\{g|d_{E_m}(f,g)<1/2^m\}\subset B(f,\varepsilon).$$

Choose a positive integer k such that $1/2^k<\varepsilon$. If $g\in B_{E_m}(f,1/2^m)$, then $d_{E_k}(f,g)<(1/2^m)$, and hence

$$d_{E_1}(f,g)\leq d_{E_2}(f,g)\leq d_{E_3}(f,g)\cdots\leq d_{E_m}(f,g)<1/2^m$$

Moreover, since

$$\frac{d_{E_i}(f,g)}{1+d_{E_i}(f,g)}\leq d_{E_i}(f,g)\ \ \text{for every } i=1,2,\cdots,$$

we see that

$$
\begin{aligned}
d(f,g) = \sum_{i=1}^{\infty} \frac{d_{E_i}(f,g)}{2^i(1+d_{E_i}(f,g))} &= \sum_{i=1}^{m} \frac{d_{E_i}(f,g)}{2^i(1+d_{E_i}(f,g))} + \sum_{i=m+1}^{\infty} \frac{d_{E_i}(f,g)}{2^i(1+d_{E_i}(f,g))} \\
&\leq \frac{1}{2^m}\left(\sum_{i=1}^{m} \frac{1}{2^i} + \sum_{i=m+1}^{\infty} \frac{1}{2^i}\right) \\
&< \frac{1}{2^m}\left(\sum_{i=1}^{\infty} \frac{1}{2^i} + \sum_{i=1}^{\infty} \frac{1}{2^i}\right) \\
&= \frac{1}{2^m - 1}
\end{aligned}
$$

Now let $m = k+1$, then $d(f,g) < 1/2^k$, and hence

$$B_{E_{k+1}}(f, 1/2^{k+1}) \subset B(f, 1/2^k) \subset B(f, \varepsilon)$$

This implies that $\mathcal{T}_d \subset \mathcal{T}_1$. Next, to show that $\mathcal{T}_1 \subset \mathcal{T}_d$, it is enough to show that for any $f \in \mathcal{M}$ and for any subbasic open neighborhood of f relative to $\mathcal{T}_1$ of the form

$$B_{E_m}(f,\varepsilon) = \{g \in \mathcal{M} | d_{E_m}(f,g) < \varepsilon\},$$

there exists a sufficiently large positive integer ℓ such that

$$B(f, 1/2^{\ell}) \subset B_{E_m}(f,\varepsilon).$$

Choose a positive integer k such that $1/2^k < \varepsilon$. If $g \in B(f, 1/2^{\ell})$ then

$$d_{E_i}(f,g) = \sum_{i=1}^{\infty} \frac{d_{E_i}(f,g)}{2^i(1+d_{E_i}(f,g))} < \frac{1}{2^{\ell}}$$

and hence we have

$$\frac{d_{E_m}(f,g)}{2^m(1+d_{E_m}(f,g))} < \frac{1}{2^{\ell}}$$

If we solve this inequality for $d_{E_m}(f,g)$, we obtain $d_{E_m}(f,g) < \frac{1}{2^{\ell-m}-1}$. Now let $\ell = k+m+1$, then $d_{E_m}(f,g) < \frac{1}{2^{k+1}+1} < \frac{1}{2^k}$ and hence $B(f, 1/2^{k+m+1}) \subset d_{E_m}(f, 1/2^k) \subset B_{E_m}(f,\varepsilon)$. This implies that $\mathcal{T}_1 \subset \mathcal{T}_d$.

COROLLARY 2-5. *The metric topology $\mathcal{T}_d$ on $\mathcal{M}$ coincides with the topology $\mathcal{T}$ in $\mathcal{M}$ convergence on each measurable subset of X whose measure is finite. Consequently the topological vector space $(\mathcal{M}, \mathcal{T})$ becomes a F-space.*

Proof We recall that the topology $\mathcal{T}_1$ on $\mathcal{M}$ is topology determined by $\mathcal{D} = \{d_E : E \in \beta, \mu(E) < \infty\}$

Since $\{d_{E_n} : n = 1, 2, 3, 4, \cdots\} \subset \mathcal{D}$, it follows that $\mathcal{T}_d \subset \mathcal{T}$.

Now we show that $\mathcal{T} \subset \mathcal{T}_d$. To show this, it is enough to show that for any $f \in \mathcal{M}$ and for any subbasic open neighborhood relative to $\mathcal{T}$ of f of the form $B_E(f, \delta)$, there exists a subbasic neighborhood relative to $\mathcal{T}_d$ of f, $B_n(f, \varepsilon) = \{g : d_{E_n}(f, g) < \varepsilon\}$ such that $B_{E_n}(f, \varepsilon) \subset B_{E_n}(f, \delta)$. Since $E \in \beta$ and $\mu(E) < \infty$, we can sufficiently large n such taht $\mu(E) < \mu(E_n)$. Hence we have

$$\int_E \frac{|f-g|}{1+|f-g|}\, d\mu \leq \int_{E_n} \frac{|f-g|}{1+|f-g|}\, d\mu$$

so that $B_{E_n}(f, \delta) \in B_E(f, \delta)$. Therefore we have $\mathcal{T} = \mathcal{T}_d$. As we have just shown above, $\mathcal{T}$ is induced by a complete invariant metric d. Therefore $(\mathcal{M}, \mathcal{T})$ is a F-space.

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