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PETTIS INTEGRABILITY IN TERMS OF OPERATORS

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ABSTRACT. A weakly measurable function $f : \Omega \to X$ is said to be determined by an operator T_f defined by $T_f(x^*) = x^* f$ if for each $x^* \in X^*, x^* f \in L_1(\lambda)$ and $T_f : X^* \to L_1(\lambda)$.

For a given Dunford integrable function $f : \Omega \to X^*$ with an additive indefinite integral we show that f is Pettis integral if and only if $T_f : X^{**} \to L_1(\lambda)$ is a weakly compact operator iff $\tilde{T}_f(K(F,\epsilon) \cap X)$ is weakly dense in $T_f(K(F,\epsilon))$ for each finite set $F \subseteq X^*$ and each ϵ.

1. Introduction

Since the invention of the Pettis integral the problem of recognizing the Pettis integrability of a function against an individual condition has been much studied [7], [8], [9], [13].

In spite of the R.F. Geitz (1982) and M. Talagrand's (1984) characterization of Pettis integrability, there is often trouble in recognizing when a function is or is not Pettis integrable.

Suppose that X is a Banach space with continuous dual X^* and $(\Omega, \Sigma, \lambda)$ is a finite measure space. Given any Dunford integrable function $f : \Omega \to X$, T_f always denotes the operator $T_f : X^* \to L_1(\lambda), x^* \longmapsto x^* f$.

Recently, in a series of papers, [2], [4] and [10] Bator, Huff, Lewis and Race effectively used properties of the operator T_f to determine Pettis integrability.

In this paper we are going to show the Pettis integrability of the Dunford integrable function $f : \Omega \to X^*$ by the operator $T_f : X^{**} \to L_1(\lambda), x^{**} \longmapsto x^{**} f$.

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And for a bounded and weakly measurable function $f : \Omega \to X^*$ we further characterize Pettis integrability in terms of the operator T_f.

2. Definitions and Preliminaries

We present some necessary notations and terminology which are needed in our subsequent section. Insofar as possible, we adopt the definitions and notations of [5] and [6].

The triple $(\Omega, \Sigma, \lambda)$ will always be a finite measure space. Throughout this paper X and Y are real Banach spaces with duals X^* and Y^* , respectively. By an operator T from X to Y we shall mean a continuous linear transformation $T : X \to Y$; the adjoint of T will be denoted by T^* and by the unit ball, B_X (respectively B_{X^*}), of X (resp. X^*), we will mean the closed unit ball.

DEFINITION 2-1. *A function $f : \Omega \to X$ is called simple if there exist the x_i's in X and the E_i's in Σ such that*

$$f = \sum_{i=1}^{n} x_i \chi_{E_i},$$

*where $\chi_{E_i}(w) = 1$ if $\omega \in E_i$ and $\chi_{E_i}(\omega) = 0$ if $\omega \notin E_i$. A function $f : \Omega \to X$ is called **countably valued** if it can be represented in the form*

$$f = \sum_{i=1}^{\infty} x_i \chi_{E_i}$$

*where the x_i's are distinct elements in X and the E_i's are disjoint elements in Σ. A function $f : \Omega \to X$ is called **strongly measurable** if there exists a sequence (f_n) of simple functions with $\lim_n \|f_n(t) - f(t)\| = 0$ for almost all t in Ω. A function $f : \Omega \to X$ is called **weakly measurable** if for all x^* in X^* the scalar-valued x^*f is measurable. A function $f : \Omega \to X^*$ is called **weak* measurable** if for each x in X xf is measurable. Let $f, g : \Omega \to X$ be two weakly measurable functions. They are said to be **weakly equivalent** if for all $x^* \in X^*, x^*f = x^*g$ almost everywhere.*

While we can see that if X has a weak* seprarable dual space and $f : \Omega \to X$ is weakly measurable then f is weakly equivalent to a strongly measurable if and only if f is strongly measurable, we note that even though the function is not weakly equivalent to a strongly measurable function, it does have one of the desired properties of a strongly measurable function, namely that there is one sequence (f_n) of simple functions such that for all linear functionals $x^*, x^*f = \lim_n x^*f_n$.

DEFINITION 2-2. *A strongly measurable function $f : \Omega \to X$ is called Bochner integrable if there exists a sequence (f_n) of simple functions such that*

$$\lim_n \int_\Omega \|f_n - f\| d\lambda = 0.$$

In this case, we define the integral of f over any set E in Σ by the equation

$$\int_E f d\lambda = \lim_n \int_E f_n d\lambda.$$

The function $\nu : \Sigma \to X, E \longmapsto \int_E f d\lambda$ is the indefinite integral of f.

PROPOSTION. *A strongly measurable function $f : \Omega \to X$ is Bochner integrable if and only if $\int_\Omega \|f\| d\lambda < \infty$.*

The proposition described above is a concise characterization of Bochner integrable functions, and it shows that the Bochner integral is a straightforward abstraction of the Lebesgue integral. We now list further basic properties of Bochner integrable functions, some which will be need later. The proofs of these properties can be found in [5].

PROPOSTION 2-4. *If $f : \Omega \to X$ is a Bochner integrable function, then*

(a) The vector measure $F(: \Sigma \to X)$ defined by $F(E) = \int_E f d\lambda$ is λ-continuous, that is, $\lim_{\lambda(E)\to 0} F(E) = 0$.

(b) $\| \int_E f d\lambda\| \leq \int_E \|f\| d\lambda$, for all $E \in \Sigma$.

(c) The vector measure $F : \Sigma \to X$ in (a) is countably additive in norm topology of X.

(d) The vector measure F in (a) is of bounded variation and $|F|(E) = \int_E \|f\| d\lambda$ for all $E \in \Sigma$.

The next Proposition exhibits a strong property of Bochner integration that has no analogue in the theory of Lebesgue integration.

PROPOSITION 2-5. *(Hille) [5]. Let T be a closed linear operator defined inside X and having values in a Banach space Y. If f and Tf are Bochner integrable with respect to λ, then $T(\int_E f d\lambda) = \int_E T f d\lambda$ for all $E \in \Sigma$. If the domain of T is X, then T is a bounded operator.*

A theory of integration similar to the Bochner integral is impossible for weakly measurable functions which are not strongly measurable. Moreover, it is impossible to use the Bochner integral theory directly to integrate a function f if $\|f\|$ is not integrable. Nevertheless, there are rather simple methods available to integrate some such functions.

The Lemma below provides the basis for integrating weakly measurable functions in general.

LEMMA 2-6. *Suppose $f:\Omega\to X$ is weakly measurable and $x^*f\in L_1(\lambda)$ for all $x^*\in X^*$. Then for each $E\in\Sigma$ there exists an element $x_E^{**}\in X^{**}$ such that*

$$x_E^{**}(x^*)=\int_E x^*fd\lambda \text{ for all } x^*\in X^*.$$

Proof Define $T:X^*\to L_1(\lambda)$ by $T(x^*)=x^*(f\chi_E)$. Then T is closed. Indeed, if $\lim_n x_n^*=x^*$ and $\lim_n T(x_n^*)=g$ exist in $L_1(\lambda)$, then some subsequence $(x_{n_j}^*(f\chi_E))=(T(x_{n_j}^*))$ tends to almost everywhere to g. But $\lim_n x_n^*(f\chi_E)=x^*(f\chi_E)$, a.e. Hence $x^*f=g$, a.e. and T is a closed linear operator. An appeal to Banach's closed graph theorem shows that T is continuous. Hence

$$\|x^*(f)\|_1\le\|T(x^*)\|\le\|T\|\|x^*\|.$$

Since the operation of integration over E is a continuous linear functional of norm at most 1, it follows that

$$\left|\int_E x^*fd\lambda\right|\le\|T\|\|x^*\|.$$

Hence the mapping $x^*\to\int_E x^*fd\lambda$ defines a continuous linear functional on X^* and, as such, defines a member x_E^{**} of X^{**}. With the help of the preceding results the Dunford integral can be defined very simply.

DEFINITION 2-7. *A weakly measurable function $f:\Omega\to X$ is said to Dunford integrable if $x^*f\in L_1(\lambda)$ for all $x^*\in X^*$. In this case, the Dunford integral of f over $E\in\Sigma$ is defined to be equal to the x_E^{**} in the previous Proposition 2-6,and we write $x_E^{**}=(D)-\int_E fd\lambda$.*

In the case that $(D)-\int_E fd\lambda$ is a member of X for all $E\in\Sigma$, f is called *Pettis integrable* and we write $(P)-\int_E fd\lambda$ instead of $(D)-\int_E fd\lambda$.

In the case that $(D) - \int_E f d\lambda$ is a member of X for all $E \in \Sigma$, f is called *Pettis integrable* and we write $(P) - \int_E f d\lambda$ instead of $(D) - \int_E f d\lambda$.

The function $\nu : \Sigma \longmapsto X^{**}, E \longmapsto \int_E f d\lambda$ is called the indefinite integral of f.

And a subset K of $L_1(\lambda)$ is called *uniformly integrable* if $\lim_{\lambda(E)\to 0} \int_E f d\lambda = 0$ uniformly for each $f \in K$.

EXAMPLE 2-8. A Dunford integrable function which is not Pettis integrable.

Define $f : [0,1] \to C_0$ by the equation

$$\begin{aligned} f(t) &= (n \cdot \chi_{(0,1/n]}(t)) \\ &= (\chi_{(0,1]}(t), 2\chi_{(0,1/2]}(t), \cdots, n\chi_{(0,1/n]}(t), \cdots) \end{aligned}$$

for $t \in [0,1]$.

If $x^* = (\alpha_n) = (\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots) \in C_0^* = l_1$ then $x^* f = \sum_{n=1}^{\infty} \alpha_n n \chi_{(0,1/n]}$, a function which is certainly Lebesgue integrable. However, if λ is the Lebesgue measure on $[0,1]$, then

$$\int_{(0,1]} x^* f d\lambda = \sum_{n=1}^{\infty} \alpha_n$$

and the mapping $x^* = (\alpha_n) \longmapsto \sum_{n=1}^{\infty} \alpha_n$ is a linear functional on l_1 corresponding to $(1,1,\cdots,1,\cdots) \in l_\infty \setminus C_0$. Hence, $(D) - \int_{(0,1]} f d\lambda = (1,1,\cdots,1,\cdots) \in l_\infty \setminus C_0$, so f is a Dunford integrable function that is not Pettis integrable.

The above example shows that the indefinite Dunford integral need not be countably additive. The following Lemma tells us exactly when countable additivity holds.

LEMMA 2-9. *[10] Let $f : \Omega \to X$ be Dunford integrable and let T be the operator $T : X^* \to L_1(\lambda), x^* \longmapsto x^* f$. The following statements are equivalent:*

(a) T is weakly compact.

(b) $\{x^* f : x^* \in B_{X^*}\}$ is uniformly integrable.

(c) The indefinite integral of f is countably additive.

Condition (b) of the above Lemma is clearly satisfied when the function f satisfies the following condition: There exists a constant M such that for each x^* in B_{X^*}, $|x^* f| \leq M$ almost everywhere.

DEFINITION 2-10. *Let $f : \Omega \to X$ be weakly measurable. It is said to be weakly bounded if there exists a constant M such that for all $x^* \in X^*, |x^* f| \leq M\|x^*\|$ a.e. And we say that a function $f : \Omega \to X$ is weak* equivalent to a function $g : \Omega \to X^{**}$ if for all $x^* \in X^*, x^* f(\cdot) = g(\cdot)x^*$ a.e.*

By the above definitions we note that if $f : \Omega \to X$ is weakly measurable, weakly bounded, and weak* equivalent to a strongly measurable function $g : \Omega \to X^{**}$, then g is bounded off a set of measure zero.

If X is a locally convex space, the weak topology on X, denoted by $\sigma(X, X^*)$, is the topology defined by the family of seminorms $\{P_{x^*} : x^* \in X^*\}$, where $P_{x^*}(x) = |< x, x^* >|$. The weak* topology on X^*, denoted by $\sigma(X^*, X)$, is the topology defined by the family of seminorms $\{P_x : x \in X\}$, where $P_x(x^*) = |< x^*, x >|$.

We say that f is *scalar measurable* with respect to λ if $x^* f$ is λ-measurable for $x^* \in X^*$, and say that f belongs to weak - $L_1(\lambda, X)$ if $x^* f \in L_1(\lambda)$ for all $x^* \in X^*$.

If $f \in$ *weak-* $L_1(\lambda, X)$, then we define the operator $T_f : X^* \to L_1(\lambda)$ by $T_f(x^*) = x^* f$, and we say that f is a *λ-Pettis integrable* if T_f^* maps $L_\infty(\lambda)$ into the canonical image of X in X^{**}.

DEFINITION 2-11. *An operator $T : X^* \to Y$ is said to be (w^*, w)-continuous provided that $(T(x_\alpha^*))$ converges to $T(x^*)$ in the weak topology of Y whenever (x_α^*) is a net which converges to x^* in the weak* topology of X^*.*

If F is a finite subset in X and $\epsilon > 0$, set

$$K(F, \epsilon) = \{x^* \in X^* : \|x^*\| \leq 1 \text{ and } x^*(x) \leq \epsilon \text{ for all } \quad x \text{ in } F\}.$$

Then $K(F, \epsilon)$ is convex and weak* compact for all finite F in X and $\epsilon > 0$. So $TK(F, \epsilon)$ is a closed and convex subset of $L_1(\lambda)$ for all $K(F, \epsilon)$.

DEFINITION 2-12. *If X and Y are Banach spaces and $T : X \to Y$ is a linear transformation, then T is compact if the closure of $T(B_X)$ is compact in Y. And an operator T in $\beta(X, Y)$ is weakly compact operator if the closure of $T(B_X)$ is weakly compact.*

It is easy to see that compact operators are bounded. For operators on Hilbert space, we note that the following concept is equivalent to compactness: If X and Y are Banach spaces and $T \in \beta(X, Y)$, then T is *completely continuous* if for any sequence (x_n) in X such that $x_n \to x$ weakly it follows that $\|Tx_n - Tx\| \to 0$.

We state a following result in [10] which effectively used the operator T to determine Pettis integrability.

LEMMA 2-13. *A Dunford integrable function f is Pettis integrable if and only if the operator T is weak*-to-weak continuous. Moreover, if f is Pettis integrable, then T is necessarily a weakly compact operator.*

The following Proposition further characterizes Pettis integrability in terms of the operator T.

PROPOSITION 2-14. *[10] If $f : \Omega \to X$ is Dunford integrable, then the following statements are equivalent:*

(a) *f is Pettis integrable.*
(b) *T is weakly compact and $\cap\{T(K(F,\epsilon)) : F \subseteq X,\ F$ finite, and $\epsilon > 0\} = \{0\}$.*

Proof (a) $\Rightarrow$ (b). If f is Pettis integrable, T is weakly compact by Lemma 2-13. Suppose g is in $\cap\{T(K(F,\epsilon) : F \subseteq X, F$ finite, and $\epsilon > 0\}$. For each (F,ϵ) choose $x^*_{(F,\epsilon)}$ in $K(F,\epsilon)$ such that $g = T(x^*_{(F,\epsilon)})$. But $(x^*_{(F,\epsilon)})_{(F,\epsilon)}$ is naturally a net in X^* which converges weak* to zero. Hence, $g = T(x^*_{(F,\epsilon)}) \to 0$.

$(b) \Rightarrow (a)$ Let $B_{X^*} = \{x^* : \|x^*\| \leq 1\}$. Suppose a net (x^*_α) in $\frac{1}{2}B_{X^*}$ converges to x^*. Then $(x^* - x^*_\alpha)$ is a net in B_{X^*} and for all (F,ϵ) it is eventually in $K(F,\epsilon)$. Let g be any weak cluster point of $(T(x^* - x^*_\alpha))$. Then $g \in \cap\{T(K(F,\epsilon)) : F \subseteq X, F$ finite, and $\epsilon > 0\}$, so g is equal to zero. Thus $T(x^*_\alpha) \to T(x^*)$ weakly in $L_1(\lambda)$. It follows that T is (w^*, w)-continuous.

3. The Main Results

Let X be a Banach space with dual X^* and $(\Omega, \Sigma, \lambda)$ be a finite measure space.

DEFINITION 3-1. *If $f : \Omega \to X^*$ is bounded and weakly measurable, i.e., if $x^{**}f$ is measurable for every $x^{**} \in X^{**}$, then it can easily be shown that*
(i) for every $E \in \Sigma$, there exists $X^*_E \in X^*$ such that, for all $x \in X$,

$$x^*_E(x) = \int_E xf\, d\lambda$$

and (ii) for every $E \in \Sigma$, there exists $x^{***}_E \in X^{***}$ such that, for all $x^{**} \in X^{**}$,

$$x^{***}_E(x^{**}) = \int_E x^{**}f\, d\lambda.$$

The element x_E^* is called the *weak* (or Gel'fand) integral* of f over E, denoted by $(w^*)-\int_E f\,d\lambda$, and x_E^{***} is called the *Dunford integral* of f over E, denoted by $(D)-\int_E f\,d\lambda$. By Definition 2-7, f is Pettis integrable if and only if $x_E^{***}(=(D)-\int_E f\,d\lambda)\in X^*$.

A Banach space Y is said to have the λ*-Pettis Integral Property* (or $\lambda-PIP$) if every bounded weakly measurable function $f:\Omega\to Y$ is Pettis integrable. Clearly, we see that X^* has the λ-PIP if and only if for every $f:\Omega\to X^*$ that is bounded and weakly measurable, $(w^*)-\int_E f\,d\lambda=(D)-\int_E f\,d\lambda$ for every $E\in\Sigma$. The following Lemma is essentially a reformation of the definition of Pettis integral.

LEMMA 3-2. *Let $f:\Omega\to X^*$ be Dunford integrable and $T_f:X^{**}\to L_1(\lambda)$ be operator defined by $x^{**}\mapsto x^{**}f$. Then f is Pettis integrable if and only if $T_f:X^{**}\to L_1(\lambda)$ is (w^*,w)-continuous.*

Proof For $x_E\in(L_1(\lambda))^*=L_\infty(\lambda)$,

$$x_E^{**}(x^{**})=\int_E x^{**}f\,d\lambda=<T_f(x^{**}),x_E>.$$

If $T_f:X^{**}\to L_1(\lambda)$ is (w^*,w)-continuous, then x_E^{***} defined by $x_E^{***}(x^{**})=<T_f(x^{**}),x_E>$ is weak*-continuous functional. Hence $x_E^{***}\in X^*$ and f is Pettis integrable.

Conversely, if f is Pettis integrable, then for subset S of all simple functions in dual $L_\infty(\lambda)$ of $L_1(\lambda)$, $T_f^*(S)\subset X^*$. Hence $T_f^*(L_\infty(\lambda))\subset X^*$. Therefore T_f is (w^*,w)-continuous.

Suppose that X is a Banach space, F is a finite subset of X^* and $\epsilon>0$. We again define $K(F,\epsilon)$ by

$$\begin{aligned}K(F,\epsilon)&=\{x^{**}\in X^{**}:\|x^{**}\|\le 1 \quad\text{and}\quad x^{**}(x)\le\epsilon \quad\text{for each}\quad x^*\in F\}\\&=\{x^{**}\in B_{X^{**}}:|\,x^{**}(x^*)\,|\le\epsilon \quad\text{for each}\quad x\in F\}.\end{aligned}$$

THEOREM 3-3. *If $f:\Omega\to X^*$ is Dunford integrable, the followings are equivalent:*

(a) *f is Pettis integrable.*
(b) *$T_f:X^{**}\to L_1(\lambda)$ is a weakly compact operator, $T_f(K(F,\epsilon)$ is closed for all (F,ϵ) and $\cap_{(F,\epsilon)}T_f(K(F,\epsilon))=\{0\}$.*

Proof (a) $\Rightarrow$ (b) If f is Pettis integrable, then by Lemma 3-2, T_f is a weakly compact operator. Suppose that g is in $\cap_{(F,\epsilon)} T_f(K(F,\epsilon))$. Let $x^{**}_{(F,\epsilon)} \in K(F,\epsilon)$ so that $T(x^{**}_{(F,\epsilon)}) = g$ for each (F,ϵ). But $(x^{**}_{(F,\epsilon)})$ forms a net in the obvious ordering, and certainly it converges weak* to zero. Since T_f is (w^*, w)-continuous, $T_f(x^{**}_{(F,\epsilon)}) \xrightarrow{w} 0$ and thus $g = 0$.

(b) $\Rightarrow$ (a) Suppose that a net (x^{**}_α) in $\frac{1}{2}B_{X^{**}}$ converges weak* to x^{**}. Then $x^{**}_\alpha - x^{**}$ is in $B_{X^{**}}$ for each α, and $(x^{**}_\alpha - x^{**})_\alpha$ is eventually in $K(F,\epsilon)$ for each pair (F,ϵ).

Now $T_f(x^{**}_\alpha - x^{**})_\alpha \subsetneq T_f(B_{X^{**}})$, a relatively weakly compact subset of $L_1(\lambda)$. Suppose that y is a weak cluster point of $T_f(x^{**}_\alpha - x^{**})_\alpha$. Therefore $y \in w$-closure $(T_f(K(F,\epsilon)))$. But $T_f(K(F,\epsilon))$ is convex and norm closed. Consequently, $y \in T_f(K(F,\epsilon))$ for each pair (F,ϵ). Thus $y = 0$ and net $T_f(x^{**}_\alpha)$ converges weakly to $T_f(x^{**})$. Therefore T_f is (w^*, w)-continuous and f is Pettis integrable.

PROPOSITION 3-4. *An operator $T : X \to Y$ is weakly compact if and only if its adjoint $T^* : Y^* \to X^*$ is (w^*, w)- continuous.*

Proof Let $T^* : Y^* \to X^*$ be a (w^*, w)-continuous, and let $x^{**}_0 \in X^{**}$. Then if $y^*_\alpha y \to y^* y$ for each $y \in Y$, then

$$x^{**}_0 T^*(y^*_\alpha) = T^{**}(x^{**}_0) y^*_\alpha \to T^{**}(x^{**}_0) y^*.$$

Thus $T^{**}(x^{**}_0)$ in Y^{**} is a weak* continuous functional on Y^*. Hence $T^{**}(x^{**}_0) \in Y$, so $T^{**}(X^{**}) \subset Y$. Therefore T is a weakly compact operator.

Conversely, if T is a weakly compact operator, for each x^{**} in X^{**}, there is y in Y with $T^{**}(X^{**}) \subset Y$ such that $x^{**} T^*(y^*) = T^{**}(x^{**}) y^* = y^* y$ for all $y^* \in Y^*$. Thus if (y^*_α) converges with weak* to y^*, then $(T^*(y^*_\alpha))$ converges weakly to $T^*(y^*)$ in X^*. Hence T^* is (w^*, w)-continuous.

THEOREM 3-5. *Let $f : \Omega \to X^*$ be Dunford integrable and let $T_f : X^{**} \to L_1(\lambda)$ and let $\widetilde{T}_f = T_f | X$. Then f is Pettis integrable if and only if $T_f = \widetilde{T}^{**}_f$.*

Proof If f is Pettis integrable, then T_f is (w^*, w)-continuous by Lemma 3-2. Let x^{**} be in $B_{X^{**}}$. Since B_X is $\sigma(X^{**}, X^*)$ dense in $B_{X^{**}}$, we can choose a net (x_α) from B_X such that (x_α) converges weak* to x^{**}. Then

$$T_f(x_\alpha) = \widetilde{T}_f(x_\alpha) = \widetilde{T}^{**}_f(x_\alpha)$$

for each α. Moreover, by hypothesis $(T_f(x_\alpha))$ converges weakly to $T_f(x^{**})$, and since the operator $\widetilde{T}_f^{**}$ is (w^*, w)-continuous, $(\widetilde{T}_f^{**}(x_\alpha))$ converges weakly to $\widetilde{T}_f^{**}(x^{**})$. Therefore $\widetilde{T}_f^{**}(x^{**}) = T_f(x^{**})$, and so $T_f = \widetilde{T}_f^{**}$.

Conversely, suppose that $T_f = \widetilde{T}_f^{**}$. Then $\widetilde{T}_f^{**}(x^{**}) = T_f(x^{**}) \subset L_1(\lambda)$, since $\widetilde{T}_f^{**} : X^{**} \to L_1(\lambda)$ is a weakly compact operator. Thus T_f is a weakly compact operator. Hence $\widetilde{T}_f^{**} : X^{**} \to L_1(\lambda)$ is (w^*, w)-continuous. Consequently, by hypothesis T_f is (w^*, w)-continuous, and so f is Pettis integrable.

Before proceeding to next characterization of (w^*, w)-continuity, we remark the following Lemma which has been proved in [4].

LEMMA 3-6. *If F is a finite subset of X^* and $\epsilon > 0$, then $K(F, \epsilon) \cap X$ is weak*-dense in $K(F, \epsilon)$.*

THEOREM 3.7. *Let $f : \Omega \to X^*$ be Dunford integrable and $T_f : X^{**} \to L_1(\lambda)$ be weakly compact operator defined by $T_f(x^{**}) = x^{**}f$. Then f is Pettis integrable iff $\widetilde{T}_f(K(F, \epsilon) \cap X)$ is weakly dense in $T_f(K(F, \epsilon))$ for each finite set $F \subseteq X^*$ and each $\epsilon > 0$.*

Proof Suppose that $\widetilde{T}_f(K(F, \epsilon) \cap X)$ is weakly dense in $T_f(K(F, \epsilon))$ for each pair (F, ϵ). Since $\widetilde{T}_f^{**}$ is (w^*, w)-continuous, $\widetilde{T}_f(K(F, \epsilon) \cap X)$ is norm dense in $T_f(K(F, \epsilon))$ for each pair (F, ϵ). Since T_f^{**} is (w^*, w)-continuous, $\widetilde{T}_f^{**}(K(F, \epsilon))$ is closed and thus $T_f(K(F, \epsilon)) \subseteq \widetilde{T}_f^{**}(K(F, \epsilon))$. Hence $\cap_{(F,\epsilon)} T_f(K(F, \epsilon)) = \{0\}$. Therefore f is Pettis integrable by Theorem 3-3.

Conversely, suppose that f is Pettis integrable, then T_f is (w^*, w)-continuous. Let $x^{**} \in K(F, \epsilon)$ and let (x_α) be a net from $K(F, \epsilon) \cap X$ so that x_α converges weak* to x^{**}. Then $T_f(x_\alpha)$ converges weakly to $T_f(x^{**})$, and $\widetilde{T}_f(x_\alpha) = T_f(x_\alpha)$ for each α.

It is known that if $f : \Omega \to X^*$ is bounded and weakly measurable, then f is weak*-integrable and Dunford integrable.

THEOREM 3.8. *A subset K of $L_1(\lambda)$ is relatively weakly compact iff it is bounded and uniformly integrable.*

Proof Suppose that K is bounded and uniformly integrable. Let (f_n) be a sequence in K. Then there is a countable field $\mathcal{F}$ such that each f_n is measurable relative to the σ-field Σ_1, generated by $\mathcal{F}$. By Cantor diagonalization, select a subsequence (f_{n_j}) such that for vector measure $F : \Sigma_1 \to X$,

$$\lim_j \int_E f_{n_j} d\lambda = F(E)$$

exists for all $E \in \mathcal{F}$. Also, since K is uniformly integrable, it follows that F is λ-continuous. Thus there exists $f \in L_1(\Sigma_1, \lambda)$ such that

$$\lim_j \int_E f_{n_j} d\lambda = \int_E f \, d\lambda$$

for each $E \in \Sigma$. From this point, it is a simple argument to verify that

$$\lim_j \int_\Omega f_{n_j} g \, d\lambda = \int_\Omega f \, d\lambda$$

for each $g \in L_1(\Sigma_1, \lambda)$. Hence (f_{n_j}) converges weakly to f in $L_1(\Sigma_1, \lambda)$. But $L_1(\Sigma_1, \lambda)$ is a closed linear subspace of $L_1(\lambda)$. Hence (f_n) converges weakly to f in $L_1(\lambda)$ and K is relatively weakly compact.

Conversely, let $K \subset L_1(\lambda)$ be relatively weakly compact. Then K is bounded and if (f_n) is a sequence in K, then (f_n) has a weakly convergent subsequence by Eberlein's Theorem. Hence there is a subsequence (f_{n_j}) such that

$$\lim_j \int_E f_{n_j} d\lambda$$

exists for all $E \in \Sigma$. By Vitali-Hahn-Saks Theorem, (f_n) is uniformly integrable. Hence every sequence in K has a uniformly integrable subsequence. Consequently, K is uniformly integrable.

THEOREM 3-9. *Let $f : \Omega \to X^*$ be a bounded weakly measurable function and $T_f : X^{**} \to L_1(\lambda)$ be an operator defined by $x^{**} \mapsto x^{**} f$ and $\widetilde{T}_f = T_f \mid X$.*

If there exist a net (x_α) in X such that $(T_f(x_\alpha))$ converges weakly to $T_f(x^{**})$ and (x_α) converges weak to x^{**}, for each element x^{**} in $B_{X^{**}}$, then f is Pettis integrable.*

Proof Since f is bounded, we can assume that $f : \Omega \to B_{X^*}$ and operator T_f defined by $x^{**} \mapsto x^{**} f$ is well defined. Since for x^{**} in $B_{X^{**}}$

$$\|T_f(x^{**})\| = \int \mid x^{**} f \mid d\lambda < \|f\|_\infty \lambda(X),$$

$T_f(B_{X^{**}})$ is norm bounded and since for x^{**} in $B_{X^{**}}$

$$\int_E | x^{**} f | \, d\lambda = \int_E \|f\| d\lambda,$$

$T_f(B_{X^{**}})$ is uniformly integrable. By above Theorem 3-8, $T_f(B_{X^{**}})$ is weakly compact set. Hence T_f is a weakly compact operator. Since $T_f : X^{**} \to L_1(\lambda)$ is weakly compact operator, $\widetilde{T}_f$ is a weakly compact operator and $T_f^{**}(x^{**}) \subset L_1(\lambda)$. Thus $\widetilde{T}_f^{**}$ is (w^*, w)-continuous.

Let $x^{**} \in B_{X^{**}}$. By hypothesis, if we choose a net (x_α) in B_X such that (x_α) converges weak* to x^{**} and $(T_f(x_\alpha))$ converges weakly to $T_f(x^{**})$, then $T_f(x_\alpha) = \widetilde{T}_f(x_\alpha) = \widetilde{T}_f^{**}(x_\alpha)$ for each α. Furthermore, by Theorem 3-5 $(\widetilde{T}_f^{**}(x_\alpha))$ converges to $\widetilde{T}_f^{**}(x^{**})$. Hence $\widetilde{T}_f^{**}(x^{**}) = T_f(x^{**})$ and $\widetilde{T}_f^{**} = T_f$, so T_f is (w^*, w)-continuous. Therefore f is Pettis integrable.

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