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Equivalence-Singularity Dichotomies of Gaussian and Poisson Processes from The Kolmogorov's Zero-One Law

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ABSTRACT

Let P and Q be probability measures on a measurable space $(\Omega, \mathcal{F})$, and $\{\mathcal{F}_n\}_{n\geq 1}$ be a sequence of increasing sub σ-fields which generates $\mathcal{F}$. For each $n \geq 1$, let P_n and Q_n be the restrictions of P and Q to $\mathcal{F}_n$, respectively. Under the assumption that $Q_n \ll P_n$ for every $n \geq 1$, a zero-one condition is derived for P and Q to have the dichotomy, i.e., either $Q \ll P$ or $Q \perp P$. Then using this condition and the Kolmogorov's zero-one law, we give new and simple proofs of the dichotomy theorems for a pair of Gaussian measures and of Poisson processes with examples.

KEYWORDS: Absolute Continuity, Dichotomy, Gaussian Measures, Kolmogorov's Zero-one Law, Poisson Processes, Singularity

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1. INTRODUCTION

Suppose that two probability measures P and Q are given on a measurable space $(\Omega, \mathcal{F})$ with an increasing sequence of σ-fields $\{\mathcal{F}_n\}_{n\geq 1}$ where $\mathcal{F} = \sigma(\cup_{n=1}^{\infty}\mathcal{F}_n)$, the σ-field generated by $\cup_{n=1}^{\infty}\mathcal{F}_n$. Q is called *absolutely continuous* with respect to P (denoted by $Q \ll P$) if $Q(A) = 0$ whenever $P(A) = 0$ for $A \in \mathcal{F}$. If $Q \ll P$ and $P \ll Q$, these measures are called *equivalent* $(Q \sim P)$. We say that Q and P are (mutually) *singular* or *orthogonal* $(Q \perp P)$ if there exists a set $B \in \mathcal{F}$ such that $Q(B) = 0$ and $P(B^c) = 0$. Let Q_n and P_n be the restrictions of Q and P to $\mathcal{F}_n$ for $n \geq 1$, respectively.

Throughout this paper, it is assumed that $Q_n \ll P_n$ for each $n \geq 1$. Under this assumption, there are some cases that Q and P are either absolutely continuous or singular: the *dichotomy*. The main purpose of this paper is to find the conditions which guarantee the dichotomy and to prove some dichotomy theorems using those conditions and the well-known Kolmogorov's zero-one law.

It is easy to see that $Q \ll P$ imply $Q_n \ll P_n$ for all $n \geq 1$. However the converse is not true. $Q_n \ll P_n$ for all n does not always imply $Q \ll P$. Note that $Q_n \perp P_n$ for some $n \geq 1$ imply $Q \perp P$. It is of course possible that Q and P are neither absolutely continuous nor singular. For example, let Q and P be singular on $(\Omega, \mathcal{F})$ with $Q_n \ll P_n$ for all $n \geq 1$, and let $R = (P+Q)/2$. Then $R_n \ll P_n$ for all n, but neither $R \ll P$ nor $R \perp P$.

Kakutani (1948) obtained the dichotomy theorem for infinite product probability measures using the Hellinger distance. Kakutani's theorem was extended to the case of non-product measures by many authors.

Hajek (1958) and Feldman (1958) independently proved the dichotomy property for Gaussian measures using the information theory and the reproducing kernel Hibert space theory, respectively. Gihman and Skorohod (1966), and Brown (1971) obtained the dichotomy for Poisson processes. Park (1993) obtained the dichotomy for a pair of stationary and ergodic measures, one of which need not be ergodic, using the ergodic decomposition theorem. Park

(1993) generalized the existing dichotomy theorem (see Breiman (1968), Corollary 6.24, for example) for two different stationary and ergodic measures by eliminating the ergodicity assumption from one of the two measures.

For arbitrary probability measures, Kabanov, Liptser and Shiryayev (1977, abbreviatedly KLS) established some necessary and sufficient conditions for absolute continuity and singularity using martingale theory. Lepage and Mandrekar (1972) obtained a zero-one law type condition for the dichotomy. A survey of some important results on this topic, along with relations to contiguity and separability, is presented in Prakasa Rao (1987).

In this paper, firstly, we slightly extended Lepage-Mandrekar's zero-one condition for the dichotomy using a result of KLS. Then, using this condition and the Kolmogorov's zero-one law, we give new and simple proofs of the dichotomy theorems for a pair of Gaussian measures (Hajek-Feldman theorem), and of Poisson processes (Gihman-Skorohod-Brown theorem). Examples are given for Poisson processes in the last section.

2. PRELIMINARIES

Let Z_n be the Radon-Nikodym derivative dQ_n/dP_n (the likelihood ratio in Statistics) under the assumption $Q_n \ll P_n$ for each $n \geq 1$. Then clearly $\{Z_n, \mathcal{F}_n, P\}$ forms a nonnegative martingale with $E(Z_n) = 1$. (In this paper, the expectation is taken with respect to P unless a specific notation is used). Hence, from the martingale convergence theorem, there exists a random variable $Z \geq 0$ such that $E|Z| < \infty$ and $\lim_{n\to\infty} Z_n = Z$ a.e. $[P]$.

Therefore we have $P(0 \leq Z < \infty) = 1$ and $0 \leq E(Z) \leq \liminf_{n\to\infty} E(Z_n) = 1$ by Fatou's lemma. It is notable (see Lemma 5 of KLS) that

$$\lim_{n\to\infty} Z_n = Z \qquad \text{a.e. } [Q]. \tag{2.1}$$

By the Lebesgue decomposition theorem, there exist unique sub-probability measures Q^c and Q^s such that $Q = Q^c + Q^s$ with $Q^c \ll P$ and $Q^s \perp P$. By the uniqueness of Lebesgue decomposition, it is easy to show that

$$Q^s \equiv 0 \quad \text{iff} \quad Q \ll P \;,$$

$$Q^c \equiv 0 \quad \text{iff} \quad Q \perp P. \tag{2.2}$$

Since it is known that (see Gihman and Skorohod (1974), p.442, Theorem 1)

$$Z = dQ^c/dP \qquad \text{a.e. } [P], \tag{2.3}$$

we have, for $A \in \mathcal{F}$,

$$Q(A) = \int_A Z\, dP + Q^s(A). \tag{2.4}$$

Hence, using (2.4) and the properties of uniform integrability, we have the well known fact that $Q \ll P$, $E(Z) = 1$, and uniform integrability of $\{Z_n, \mathcal{F}_n, P\}$ are equivalent. Moreover, from (2.2)–(2.4) and the nonnegativity of Z a.e. $[P]$, it follows that

$$Z = 0 \qquad \text{a.e. } [P] \qquad \text{iff } Q \perp P. \tag{2.5}$$

The following lemma, proved in Lemma 6 of KLS, plays a key role in the present paper (for the proof of Theorem 1).

Lemma 1. Under the assumption that $Q_n \ll P_n$, for every $n \geq 1$,

$$Q \ll P \qquad \text{iff} \qquad Q(Z < \infty) = 1,$$

$$Q \perp P \qquad \text{iff} \qquad Q(Z < \infty) = 0.$$

3. ZERO-ONE CONDITION FOR THE DICHOTOMY

The following Theorem, which is a slight extension of the result of Lepage and Mandrekar (1972), plays a key role to determine whether given measures enjoy the dichotomy or not. Kakutani's dichotomy theorem is easily proved using the following Theorem 1.

Let $\beta_n = Z_n Z_{n-1}^{-1}$. Since $Q(0 < Z_n < \infty) = 1$, $Q(0 < \beta_n < \infty) = 1$ for every $n \geq 2$ so that $\{\beta_n\}_{n\geq 2}$ is Q-a.e. well defined. However, on the set $\{Z_{n-1} = 0\}$, we take $\beta_n = 0$. Let $\mathcal{F}^* = \cap_{k=1}^{\infty} \sigma(\beta_n |\ n > k)$.

Theorem 1. Assume that $Q_n \ll P_n$ for every $n \geq 1$. If the tail σ-field $\mathcal{F}^*$ has zero-one property with respect to Q, then the dichotomy arises : either $Q \ll P$ or $Q \perp P$.

Proof. Since $\lim_{n\to\infty} Z_n = Z$ a.e. $[Q]$ by (2.1), $\{Z < \infty\} = \{\lim_{n\to\infty} Z_n < \infty\}$ a.e. $[Q]$. Now we have a.e. $[Q]$,

$$\begin{aligned}
\{\lim_{m\to\infty} Z_m < \infty\} &= \{\lim_{m\to\infty} \log Z_m < \infty\} \\
&= \Big\{\lim_{m\to\infty} \sum_{n=k+1}^{m} (\log Z_n - \log Z_{n-1}) < \infty\Big\} \\
&\in \sigma\Big\{\log (Z_n Z_{n-1}^{-1}) \mid n > k\Big\} = \sigma\{\ \beta_n \mid n > k\}
\end{aligned}$$

for every $k \geq 1$. Note that the second equality holds Q-a.e. because $Q(Z_n > 0) = 1$ or $Q(\log Z_n > -\infty) = 1$. Since $Q(Z < \infty) = 0$ or 1 by assumption, the proof is completed from Lemma 1.

Remark 1. Lepage and Mandrekar (1972) obtained a theorem similar to Theorem 1 by considering the set $\{Z > 0\} \in \mathcal{F}^*$. Their assumption, however, is the equivalence of Q_n and P_n which is stronger than our absolute continuity assumption $Q_n \ll P_n$: assuming $Q_n \sim P_n$, if $\mathcal{F}^*$ has zero-one property with respect to P, then either $Q \perp P$ or $P \ll Q$.

When $Q_n \sim P_n$ are assumed for every $n \geq 1$, by interchanging the roles of P and Q, we simply obtain the following.

Corollary 1. Assuming $Q_n \sim P_n$ for every $n \geq 1$, if $\mathcal{F}^*$ has zero-one property with respect to Q and P, then either $Q \perp P$ or $Q \sim P$.

For the application of Theorem 1, some appropriate zero-one laws for $\mathcal{F}^*$ are crucial. In this paper, the Kolmogorov zero-one law is mainly used by inducing a sequence of $Q-$independent random variables $\{\beta_n\}_{n\geq 1}$ or $\{\log\ \beta_n\}_{n\geq 1}$.

Let $P = \prod_{i=1}^{\infty} P^{(i)}$ and $Q = \prod_{i=1}^{\infty} Q^{(i)}$ are infinite product measures on $(\mathbf{R}_\infty, \mathcal{F}) = \prod_{i=1}^{\infty}(R^{(i)}, \mathcal{F}^{(i)})$, where $P^{(i)}$ and $Q^{(i)}$ are probability measures on $(R^{(i)}, \mathcal{F}^{(i)})$. For every n, let

$$\mathcal{F}_n = \prod_{i=1}^{n} \mathcal{F}^{(i)} \times \prod_{i=n+1}^{\infty} \{\emptyset, R^{(i)}\}. \tag{3.1}$$

Then, under the assumption that $Q_n \ll P_n$ for every $n \geq 1$, Kakutani's dichotomy theorem is obtained by Theorem 1 and the Kolmogorov's zero-one law. This is because that $\beta_n = dQ^{(n)}/dP^{(n)} = q^{(n)}/p^{(n)}$ for $n \geq 1$, where $q^{(n)}$ and $p^{(n)}$ are the n-th marginal probability density functions, form a sequence of independent random variables with respect to Q (and P).

4. THE DICHOTOMY FOR GAUSSIAN MEASURES

The Hajek-Feldman's dichotomy theorem for Gaussian measures also can be proved by Theorem 1 and the well-known Kolmogorov's zero-one law while Lepage and Mandrekar (1972) applied the operator theory and the Kallianpur's zero-one law for Gaussian measures in their proof. Thus our proof is much easier to understand than the proofs of Feldman and of Lepage-Mandrekar.

We mainly use, in the proof, the fact that the Radon-Nikodym derivative for any pair of normal distributions can be described by pair of sequences of independent normal distributions.

For each $n \geq 1$, finite dimensional Gaussian measures P_n and Q_n are either singular or equivalent (because any nondegerate Gaussian measures P_n and Q_n are equivalent). We don't need the usual assumption $Q_n \ll P_n$ for every $n \geq 1$ in the following theorem.

Theorem 2. Two Gaussian measures P and Q on $(\mathbf{R}_\infty, \sigma(\mathbf{R}_\infty))$ are either singular or equivalent.

Proof. Let $\mathbf{X} = (X_1, \ldots, X_n)$ be $N(a_n, \Sigma_n)$ under Q and be $N(b_n, \Lambda_n)$ under P, where Σ_n and Λ_n are nonsingular positive definite matrices. Then clearly either $P_n \sim Q_n$ or $P_n \perp Q_n$ for every $n \geq 1$. If $P_n \perp Q_n$ for some n, then $P \perp Q$. Thus we assume $P_n \sim Q_n$ for all $n \geq 1$. Then

$$\frac{dQ_n}{dP_n}(\mathbf{x}) = \frac{|\Lambda_n|^{1/2}}{|\Sigma_n|^{1/2}} \exp\Big\{-\frac{1}{2}(\mathbf{X} - a_n)' \sum_n^{-1} (\mathbf{X} - a_n) \\ \frac{1}{2}(\mathbf{X} - b_n)'\Lambda_n^{-1}(\mathbf{X} - b_n)\Big\}. \tag{4.1}$$

Now, as done by Hajek (1958), take a nonsingular matrix V such that $\Sigma_n = VV'$ and an orthogonal matrix U such that $U'V'\Lambda_n^{-1}VU$ is a diagonal matrix (say the diagonal entries σ_i^2 for every $i \geq 1$). And take transformations for vectors $\mathbf{Y} = (Y_1, \ldots, Y_n)'$ and $\mu = (\mu_1, \ldots, \mu_n)'$ by $\mathbf{X} - a_n = VU\mathbf{Y}$ and $b_n - a_n = VU\mu$. Then clearly a random vector $\mathbf{Y}$ is normally distributed with mean 0 and variance $\mathbf{I}$ under Q, and is independent normally distributed with mean μ and variance $(\sigma_1^2, \ldots, \sigma_n^2)'$ under P. Also we have

$$(\mathbf{X} - a_n)'\Sigma_n^{-1}(\mathbf{X} - a_n) = \mathbf{Y}'U'V'\Sigma_n^{-1}VU\mathbf{Y} \\ = \mathbf{Y}'\mathbf{Y} \quad = \sum_{i=1}^{n} Y_i^2, \tag{4.2}$$

$$(\mathbf{X} - b_n)'\Lambda_n^{-1}(\mathbf{X} - b_n) = (\mathbf{Y} - \mu)'U'V'\Lambda_n^{-1}VU(\mathbf{Y} - \mu) \\ = \sum_{i=1}^{n} \frac{(Y_i - \mu_i)^2}{\sigma_i^2}. \tag{4.3}$$

Thus, after taking account of Jacobian of the transformation, it follows from (4.1) that

$$\frac{dQ_n}{dP_n}(\mathbf{x}) = \frac{dQ_n}{dP_n}(\mathbf{y}) \\ = \frac{|\Lambda_n|^{1/2}}{|\Sigma_n|^{1/2}} \exp\Big\{-\frac{1}{2}\sum_{i=1}^{n} Y_i^2 + \frac{1}{2}\sum_{i=1}^{n} \frac{(Y_i - \mu_i)^2}{\sigma_i^2}\Big\}. \tag{4.4}$$

Therefore

$$\log \beta_n = \log Z_n - \log Z_{n-1} = c_n - \frac{1}{2}Y_n^2 + \frac{1}{2}\frac{(Y_n - \mu_n)^2}{\sigma_n^2},$$

where c_n is a finite constant. Since $\{Y_n^2\}_{n\geq 1}$ is an independent sequence under Q, $\{Y_n^2 - (Y_n - \mu_n)^2/\sigma_n^2\}_{n\geq 1}$ is also an independent sequence under Q (even though Y_n^2 and $(Y_n - \mu_n)^2/\sigma_n^2$ need not be independent for a fixed n).

Therefore, by the Kolmogorov's zero-one law and Theorem 1, we have the dichotomy : either $Q \perp P$ or $Q \ll P$. By interchanging the roles of P and Q, it follows that $P \perp Q$ or $P \ll Q$, which completes the proof.

5. THE DICHOTOMY FOR POISSON PROCESSES

In this section, the dichotomy theorem (Gihman-Skorohod-Brown) for two probability measures induced by a Poisson process is proved by using Theorem 1 and the Kolmogorov's zero-one law. In the proof, the approach due to Brown (1971) and the property that a Poisson process has stationary and independent increments are mainly used. The following construction of Poisson measures is based on Brown (1971).

Consider a random nonnegative integer valued set function N on a measurable space $(A, \mathcal{A})$ having the property that for any $k \geq 1$,

(1) $N(C_1), N(C_2), \ldots, N(C_k)$ are independent when $C_1, C_2, \ldots, C_n \in \mathcal{A}$ are disjoint, and

(2) for each $C \in \mathcal{A}$ and $n \geq 0$,

$$P\{N(C) = n\} = \frac{e^{-\mu(C)}\ \mu(C)^n}{n!}, \tag{5.1}$$

where μ is a σ-finite measure over a measurable space $(A, \mathcal{A})$.

A collection $\{N(C_1), N(C_2), \ldots\}$ of random variables having the above properties is called a Poisson process with a mean measure μ. For the notational convenience, let the sequence $(C_1, C_2, \ldots)$ stands for any sequence of disjoint sets in $\mathcal{A}$ such that $A = \cup_{k=1}^{\infty} C_k$ and $\mu(C_k) < \infty$ for each $k \geq 1$.

Each realization of a Poisson process is, according to Brown (1971), of the form $N(C, \omega) = \sum_{t_i' \subset C} N(t_i'(\omega), \omega)$ where $\{t_i',\ i = 1, 2, \ldots\}$ is a random countable collection of *chunks* of A (a chunk of A is a set $C \in \mathcal{A}$ such that $C' \subset C$, $C' \in \mathcal{A}$ implies $C' = C$ or $C' = \emptyset$), and $N(t_i'(\omega), \omega)$ is a positive finite integer. Thus the probability measure (say Poisson measure and denote P_μ) induced by a Poisson process with a mean measure μ over $(A, \mathcal{A})$,

$$P_\mu \;=\; P\Big\{(\omega_1, \omega_2, \dots) \;:\; N(C_1, \omega_1) = n_1,\; N(C_2, \omega_2) = n_2, \dots\Big\}, \tag{5.2}$$

is defined over $(\Omega, \mathcal{F})$, where Ω is the set of all countable subsets of chunks of A (multiple occurrences of chunks are permitted), and $\mathcal{F} = \sigma(N(C_1), N(C_2), \dots)$. For more general discussion on Poisson measures, see Daley and Vere-Jones (1988). Let $\mathcal{F}_n = \sigma(N(C_1), N(C_2), \dots, N(C_n))$ and $P_{\mu,n}$ be the restriction of P_μ to $\mathcal{F}_n$, for each $n \geq 1$. Then $P_{\mu,n}$ is a Poisson measure over $(\Omega, \mathcal{F}_n)$ with a mean measure μ over a measurable space $(A_n, \mathcal{A}_n)$, where $A_n = \cup_{k=1}^n C_k$ and $\mathcal{A}_n = \mathcal{A} \cap A_n$. Also Poisson measures P_ν and $P_{\nu,n}$ with a σ-finite mean measure ν are defined similarly.

Now, for the Radon-Nikodym derivative of Poisson measures, the following lemma is useful (see Brown for the proof).

Lemma 3. If P_μ and P_ν are Poisson measures with finite mean measures μ and ν over $(A, \mathcal{A})$, respectively, and $\mu \ll \nu$ with $f = d\mu/d\nu$, then $P_\mu \ll P_\nu$ with

$$dP_\mu/dP_\nu \;=\; \exp\Big[-\Big\{\mu(A) - \nu(A)\Big\}\Big] \prod_{i=1}^{N(A)} f(t_i). \tag{5.3}$$

Theorem 3. Let P_μ and P_ν be Poisson measures with σ-finite mean measures μ and ν over $(A, \mathcal{A})$, and $\mu \ll \nu$, then the dichotomy arises: either $P_\mu \perp P_\nu$ or $P_\mu \ll P_\nu$.

Proof. Since $\mu(A_n) = \sum_{k=1}^n \mu(C_k) < \infty$ and $\nu(A_n) < \infty$ on $\mathcal{A}_n$ for all finite $n \geq 1$, we have, by Lemma 3, $P_{\mu,n} \ll P_{\nu,n}$ with

$$Z_n \;=\; dP_{\mu,n}/dP_{\nu,n} \;=\; \prod_{k=1}^n Y_k, \tag{5.4}$$

where $Y_k = \exp\,[-\{\mu(C_k) - \nu(C_k)\}]\ \prod_{i=1}^{N(C_k)} f(t_i)$ for $t_i \in C_k$.

Note that $\beta_n = Z_n Z_{n-1}^{-1} = Y_n$ for $n = 2, 3, \ldots$ form a sequence of independent (with respect to P_μ and P_ν) random variables by the property (1) of Poisson processes. Thus, again by the Kolmogorov's zero-one law and Theorem 1, the dichotomy holds, which completes the proof.

6. EXAMPLES

In this section, firstly, a precise condition for absolute continuity and singularity is considered after assuming that the dichotomy holds. Then two examples for Poisson processes which enjoy the dichotomy are given. More precise conditions for absolute continuity and singularity, and examples of dichotomous Gaussian measues, including the following theorem, are given in Park and Jeon (1983).

Theorem 4. Assume that the dichotomy holds. Then

$$\lim_{n\to\infty} E(Z_n^\alpha) > 0 \text{ for some } \alpha \in (0,1) \text{ iff } Q \ll P,$$

and

$$\lim_{n\to\infty} E(Z_n^\alpha) = 0 \text{ for some } \alpha \in (0,1) \text{ iff } Q \perp P.$$

Proof. Since $\frac{1}{\alpha} > 1$ and $E\{(Z_n^\alpha)^{1/\alpha}\} = E(Z_n) = 1$, $\{Z_n^\alpha, \mathcal{F}_n, P\}$ is uniformly integrable. Thus $\lim_{n\to\infty} E(Z_n^\alpha) = 0$ implies $Z = 0$ a.e. $[P]$. Therefore $Q \perp P$ by (2.5), and the proof is completed by the dichotomy assumption.

Example 1. Let $\{N_{t_1}, N_{t_2}, \ldots\}$ be an ordinary Poisson process with the intensities α under Q, and β under P, where $0 = t_0 < t_1 < t_2 < \ldots < t_k < \ldots$. Then

$$Z_k^{1/2} = \exp\Big\{-\frac{t_k}{2}(\alpha - \beta)\Big\}\ (\alpha/\beta)^{\sum_{j=1}^k n_j/2}. \tag{6.1}$$

Since $\sum_{j=1}^k N_j$ is a Poisson random variable with intensity $t_k\beta$ under P, we have

$$\lim_{k\to\infty} E(Z_k^{1/2}) = \lim_{k\to\infty} \exp\Big\{-\frac{t_k}{2}(\alpha + \beta - 2\sqrt{\alpha\beta})\Big\}. \tag{6.2}$$

Since $\alpha + \beta - 2\sqrt{\alpha\beta} \geq 0$ with equality iff $\alpha = \beta$, it follows that $Q = P$ iff $\alpha = \beta$, and $Q \perp P$ iff $\alpha \neq \beta$ by Theorem 4.

Example 2. Let P and Q be the probability measures induced by Poisson processes with σ-finite mean measures μ and ν, respectively. Under the same notations as in Section 5,

$$Z_k^{1/2} = \exp\Big\{-\frac{1}{2}\sum_{j=1}^{k}\mu(C_j) + \frac{1}{2}\sum_{j=1}^{k}\nu(C_j)\Big\} \times \prod_{j=1}^{k}\Big\{\mu(C_j)/\nu(C_j)\Big\}^{n_j/2}. \quad (6.3)$$

Since it is assumed that $Q_n \ll P_n$ for every $n \geq 1$, we only consider the sets C_j's such that $\nu(C_j) > 0$ in (6.3), without loss of generality. Since N_j is an independent Poisson random variable with mean $\nu(C_j)$ under P for each $j \geq 1$, a simple calculation shows that

$$\lim_{k\to\infty} E(Z_k^{1/2}) = \lim_{k\to\infty} \exp\Big\{-\frac{1}{2}\sum_{j=1}^{k}(\mu(C_j) + \nu(C_j)) + \sum_{j=1}^{k}\sqrt{\mu(C_j)\nu(C_j)}\Big\}. \quad (6.4)$$

If μ and ν are finite, then (6.4) is positive which leads $Q \ll P$ by Theorem 4.

By noting that $\mu(C_j) + \nu(C_j) \geq 2\sqrt{\mu(C_j)\nu(C_j)}$ with equality iff $\mu(C_j) = \nu(C_j)$ for every $j \geq 1$, we have $Q = P$ when $\mu = \nu$, whatever those are finite or not. If μ or ν is infinite (but $\mu \neq \nu$), then (6.4) is zero which leads to $Q \perp P$.

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