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An LIL Via Self-Normalization for Sequences of Sign-Invariant Random Variables

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ABSTRACT

Some extensions of the law of the iterated logarithm via self-normalization are obtained for sequences of sign-invariant random variables.

KEYWORDS: LIL, Self-normalization, Sign-invariant random variables.

1. INTRODUCTION

Let $X_1, X_2, \ldots$ be a sequence of random variables and let

$$S_n = X_1 + \cdots + X_n \qquad (n \geq 1),$$

$$V_n^2 = X_1^2 + \cdots + X_n^2 \qquad (n \geq 1).$$

The first result on the law of the iterated logarithm (LIL) via self-normalization was obtained by Marcinkiewicz(1937) who observed that for any symmetric distribution

$$\limsup_{n\to\infty} \frac{S_n}{(2V_n^2 L_2 V_n^2)^{1/2}} \leq 1 \quad \text{a.s.}, \tag{1.1}$$

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where $Lx = \max(1, \log_e x)$ and $L_2x = L(Lx)$. For this result the random variables need not be identically distributed, just independent and $V_n^2 \to \infty$ with probability 1. Later, results were obtained with a refinement of (1.1) by Griffin and Kuelbs(1991). They also treated identically distributed case. The purpose of this paper is to generalize Griffin and Kuelbs' LIL via self-normalizations to sequences of "sign-invariant" random variables. We also consider LIL via self-normalization for a sequence of exchangeable and sign-invariant random variables.

2. EXTENSTIONS TO SIGN-INVARIANT SEQUENCES

Let (X_n) be a sequence of random variables on a probability space $(\Omega, \mathcal{F}, P)$ such that every finite dimensional distribution function(d.f.) of the sequence is invariant under any changes in the sign of (X_n). Such random variables were called "sign-invariant" (Berman,1962,1965). It is obvious that a sequence of independent random variables with distribution functions $F_n(x)$ of X_n is sign-invariant if and only if $F_n(x)$ is symmetric, i.e., every one-dimensional d.f. is invariant under changes in signs.

Lemma 1. Let $X_1, X_2, \ldots$ be sign-invariant random variables on a probability space $(\Omega, \mathcal{F}, P)$. Then there exists a regular conditional distribution, say P^ω, for $X = (X_1, X_2, \ldots)$ given $\sigma(|X_n|, n \geq 1)$ such that for each $\omega \in \Omega$ the coordinate random variables $\{\xi_n, n \geq 1\}$ of probability space $(\mathcal{R}^\infty, \mathcal{B}^\infty, P^\omega)$ are independent and $P^\omega\Big(\xi_n = X_n(\omega)\Big) = \frac{1}{2} = P^\omega\Big(\xi_n = -X_n(\omega)\Big)$ for all n.

Proof. Let $B = \Big\{\xi_1 > t_1, \ldots, \xi_n > t_n\Big\}$, $t_i \geq 0$, $i = 1, 2, \ldots, n$. Note that

$$P^\omega(B) = \begin{cases} \frac{1}{2^n}, & \text{if } |X_i(\omega)| > t_i, \quad i = 1, \ldots, n \\ 0, & \text{if not} \end{cases},$$

and

$$P\Big(|X_1| > t_1, \ldots, |X_n| > t_n\Big) = 2^n P\Big(X_1 > t_1, \ldots, X_n > t_n\Big).$$

Then

$$\int P^\omega\Big(\xi_1 > t_1, \ldots, \xi_n > t_n\Big) dP(\omega)$$

$$= 2^{-n} P\Big(|X_1| > t_1, \ldots, |X_n| > t_n\Big)$$

$$= 2^{-n} 2^n P\Big(X_1 > t_1, \ldots, X_n > t_n\Big)$$

$$= P\Big(X_1 > t_1, \ldots, X_n > t_n\Big).$$

This proves the Lemma.

The sequences (X_n) is "exchangeable" if the joint d.f. of $(X_1, X_2, \ldots, X_n)$ say $G_n(x_1, \ldots, x_n)$, is a symmetric function for each n. According to the fundamental theorem of de Finetti, there exists a sub-σ-field of the σ-field $\mathcal{F}$ and a conditional d.f. $G^\omega(x)$ such that the (X_n) are conditionally independent given $\mathcal{G}$ with the common conditional d.f. $G^\omega(x)$. More specifically, one may write,

$$G_n(x_1, \ldots, x_n) = \int_\Omega G^\omega(x_1) \cdots G^\omega(x_n) dP(\omega), \tag{2.1}$$

where $G^\omega(x)$ is a d.f. for each $\omega \in \Omega$, and an $\mathcal{F}$-measurable function for each x. In general, for any set $H \in \mathcal{F}$,

$$P(H) = \int_\Omega P^\omega(H) dP(\omega), \tag{2.2}$$

where $P^\omega(H)$ is the conditional probability of H computed under the assumption that the X_n are mutually independent with the common conditional d.f. $G_\omega(x)$.

The following lemma, due to Berman(1962), is used.

Lemma 2. If (X_n) are exchangeable and sign-invariant random variables, then in the representation (2.1), for almost all ω

$$G^\omega(x) = 1 - G^\omega(-x),$$

for all x, in the continuity set of $G^\omega(x)$.

Theorem 1. Let (X_n) be exchangeable and sign-invariant with $E(X_1^2) < \infty$ and let $\Delta = \{\omega : G_\omega(\cdot) \text{ is degenerate at } 0\}$. If ϕ is eventually nondecreasing and positive, then

$$P(S_n > V_n\phi(n) \quad \text{i.o.}) = 0\big(= 1 - P(\Delta)\big)$$

according as

$$J(\phi) < \infty (= \infty),$$

where

$$J(\phi) = \sum_{n=1}^{\infty} \frac{\phi(n)}{n} e^{-\phi^2(n)/2}. \tag{2.3}$$

Proof. According to (2.2),

$$P(S_n > V_n\phi(n) \quad \text{i.o.}) = \Big(\int_\Delta + \int_{\Omega-\Delta}\Big) P^\omega\big(S_n > V_n\phi(n) \quad \text{i.o.}\big) dP(\omega).$$

For $\omega \in \Delta$, $S_n = V_n = 0$ a.s. for all n, hence $P^\omega(S_n > V_n\phi(n) \quad \text{i.o.}) = 0$; on the other hand, by Theorem 2(Griffin and Kuelbs, 1991) and Lemma 2, for almost all $\omega \notin \Delta$,

$$P^\omega\big(S_n > V_n\phi(n) \quad \text{i.o.}\big) = 0\big(= 1\big)$$

according as

$$J(\phi) < \infty\big(= \infty\big).$$

Hence the theorem follows.

Theorem 2. Let $X_1, X_2, \ldots$ be sign-invariant random variables with $V_n^2 \to \infty$ w.p. 1. If ϕ is nondecreasing and positive eventually and $J(\phi) < \infty$ where $J(\phi)$ is as in (2.3), then

$$P\big(S_n > V_n\phi(V_n^2) \quad \text{i.o.}\big) = 0.$$

Proof. We set $T_n = \xi_1 + \cdots + \xi_n$, $W_n^2 = \xi_1^2 + \cdots + \xi_n^2$. By Lemma 1 $\xi_1, \xi_2, \ldots$ is independent and symmetric with respect to P^ω for all ω and by the assumption $V_n^2 \to \infty$ w.p.1, $P^\omega\Big(W_n^2 \to \infty\Big) = 1$ w.p. 1. Hence it follows from Theorem 1(Griffin and Kuelbs, 1991) that

$$P\Big(P^\omega\Big(T_n > W_n\phi(W_n^2) \quad \text{i.o}\Big) = 0\Big) = 1.$$

Therefore we have

$$\begin{aligned}
&P\Big(S_n > V_n\phi(V_n^2) \quad \text{i.o.}\Big) \\
&= \int P\Big(S_n > V_n\phi(V_n^2) \quad \text{i.o.} \mid |X_1|, |X_2|, \ldots\Big) dP \\
&= \int P^\omega\Big(T_n > W_n\phi(W_n^2) \quad \text{i.o.}\Big) dP = 0.
\end{aligned}$$

This completes proof.

Theorem 3. Let $X, X_1, X_2, \ldots$ be stationary, ergodic and sign-invariant with $0 < E(X^2) < \infty$. If ϕ is eventually nondecreasing and positive, then

$$P\Big(S_n > V_n\phi(n) \quad \text{i.o}\Big) = 0\Big(= 1\Big) \tag{2.4}$$

according as

$$J(\phi) < \infty\Big(= \infty\Big),$$

where $J(\phi)$ is as in (2.3).

To prove this we need the following lemmas.

Lemma 3. Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables with $P\{X_n = \pm x_n\} = 1/2$, and let $|x_n| \leq n^{1/2}$ for all large n and $(x_1^2 + \cdots + x_n^2)/n$ converges to a positive constant. If

$$\sum_{n=1}^{\infty} X_{n+1}^2 I\Big(|X_{n+1}| \leq C\Big) \frac{\phi(n)}{n} e^{-\phi^2(n)/2}$$

diverges with probabilty 1 for some $C > 0$ and $\phi(n) \uparrow \infty$, then

$$P\Big(S_n > (x_1^2 + \cdots + x_n^2)\phi(n) \quad \text{i.o}\Big) = 1.$$

Proof. Just follow the step 2 and step 3 in the proof Theorem 2 (Griffin and Kuelbs, 1991).

Lemma 4. Let ϕ be eventually nondecreasing and positive such that $L_2 n \le \phi^2(n) \le 3L_2 n$ for large n, and let $\sum_{n=1}^{\infty} f(n) = \infty$, where $f(n) = \frac{\phi(n)}{n} e^{-\phi^2(n)/2}$. Then if $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} a_i = \alpha(0 < \alpha < 1)$ where $a_i = 0$ or 1, we have

$$\sum_{n=1}^{\infty} a_n f(n) = \infty.$$

Proof. Let $\{a_{n_k}\}$ be the subsequence of $\{a_n\}$ such that $a_{n_k} = 1$ for all k, then $\lim_{n\to\infty} \frac{n_k}{k} = 1/\alpha$ and hence for large k, $n_k \le ([\frac{1}{\alpha}]+1)k$ where $[x]$ stands for the integer part of x. Let $g(n) = \frac{(L_2 n)^{1/2}}{n} e^{-\phi^2(n)/2}$ and let $[\frac{1}{\alpha}] + 1 = \beta$, then g is eventually nonincreasing, and hence $g(n_k) \ge g(\beta k)$ for large k. Now we note that $3^{-1/2} f(n) \le g(n)$ and hence $\sum_{n=1}^{\infty} g(\beta k) = \infty$, since $\sum_{n=1}^{\infty} f(n) = \infty$. Therefore we have

$$\sum_{n=1}^{\infty} a_n f(n) = \sum_{k=1}^{\infty} f(n_k) \ge \sum_{k=1}^{\infty} g(n_k) \ge c + \sum_{k=1}^{\infty} g(\beta k)$$

for some c, which proves the lemma.

Proof of theorem. As in the proof Theorem 2(Griffin and Kuelbs, 1991) it is standard argument to show that if Theorem 3 holds for $\phi(n)$ nondecreasing, positive and such that

$$L_2 n \le \phi^2(n) \le 3L_2 n \tag{2.5}$$

for large n, then Theorem 3 holds without the restriction (2.5). Hence we will assume that (2.5) holds. The proof that $J(\phi) < \infty$ implies the probability in (2.4) is zero just follows the proof of Theorem 2 (Griffin and Kuelbs, 1991) using Theorem 2 and $V_n^2 \sim nEX^2$ which comes from the Brikhoff's ergodic theorem. Now we show that $J(\phi) = \infty$ implies $P\big(S_n > V_n\phi(n) \;\; \text{i.o.}\big) = 1$. Since $E\big(X^2 I|X| \le C\big) > 0$, for some $C > 0$, we can choose $B > 0$ such that

$P\Big(B \le |X| \le C\Big) > 0$. Now

$$\sum_{n=1}^{\infty} X_{n+1}^2 I\Big(|X_{n+1}| \le C\Big)\frac{\phi(n)}{n}e^{-\phi^2(n)/2}$$
$$\ge \sum_{n=1}^{\infty} B^2 I\Big(B \le |X_{n+1}| \le C\Big)\frac{\phi(n)}{n}e^{-\phi^2(n)/2}.$$

By the ergodic theorem,

$$\sum_{k=1}^{n} I(B \le |X_{k+1}| \le C) \sim nP(B \le |X| \le C)$$

hence by Lemma 4,

$$P\Bigg(E\Big(\sum_{n=1}^{\infty} X_{n+1}^2 I\Big(|X_{n+1}| \le C\Big)\frac{\phi(n)}{n}e^{-\phi^2(n)/2}\Big||X_1|, |X_2|, \ldots\Big) = \infty\Bigg) = 1.$$

Also we can easily check by Borel-Cantelli lemma and ergodic theorem that $P\Big(|X_j| > j^{1/2} \quad \text{i.o.}\Big||X_1|, |X_2|, \ldots\Big) = 0$ and $P\Big(V_n^2/n \to EX^2\Big||X_1|, |X_2|, \ldots\Big) = 1$. Hence by Lemma 3, $P\Big(S_n > V_n\phi(n) \quad \text{i.o.}\Big||X_1|, |X_2|, \ldots\Big) = 1$ a.s. Therefore we have

$$P\Big(S_n > V_n\phi(n) \quad \text{i.o.}\Big)$$
$$= \int P\Big(S_n > V_n\phi(n) \quad \text{i.o.}\Big||X_1|, |X_2|, \ldots\Big)dP$$
$$= 1,$$

which completes the proof.

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