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Distribution of Votaw's λ_1(mvc) Criterion

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ABSTRACT

In this paper, distribution of Votaw's λ_1(mvc) criterion has been obtained using inverse Mellin transform, residue theorem and properties of special functions.

KEYWORDS: Likelihood ratio criterion, Null distribution, Inverse Mellin transform, Residue theorem, Meijer's G-function.

1. INTRODUCTION

Let Π be the normal t-variate population and $X_i (i = 1, \ldots, t, t \geq 3)$ be the i-th variate. Let the set of variates $X_1, \ldots, X_t$ be partitioned into q mutually exclusive subsets of which, say, ℓ subsets contain exactly one variate and remaining $q - \ell = h$ subsets $(h \geq 1)$ contain $n_1, \ldots, n_h$ variates respectively, where $n_\eta \geq 2 \left(\eta = 1, \ldots, h;\ \ell + \sum_{\eta=1}^{h} n_\eta = t\right)$. No generality is lost in assuming that the t variates are ordered such that the first ℓ subsets contain one variate each and next n_1 variates belong to $(\ell + 1)$-th subset, ..., the last n_h variates

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belong to q-th subset, where $n_1 \le n_2 \le \cdots \le n_h$. Let $(1^\ell, n_1, \ldots, n_h)$ represent such a partition of variates $X_1, \ldots, X_t$ into q subsets.

Let $H_1(\text{mvc})$ be the hypothesis that within each subset of variates the means are equal, the variances are equal and the covariances are equal and that between any two distinct subsets of variates the covariances are equal. When this hypothesis is true, the t-variate Gaussian distribution is said to have compound symmetry of type I. Votaw (1948) defined various hypotheses of compound symmetry and has illustrated their use in psychometric experiment and medical research experiment (also see Votaw et al., 1950). Such models also arise in the study of symmetries in animals and plants.

Let $X_{i\gamma}$ be the γ-th independent observation on X_i for $i = 1, \ldots, t$ and $\gamma = 1, \ldots, N$. The likelihood ratio statistic $\lambda_1(\text{mvc})$, for testing the hypothesis $H_1(\text{mvc})$, is known to be (Votaw, 1948)

$$\lambda_1(\text{mvc}) = \Big[|v_{ij}|\Big\{\prod_{\eta=1}^{h}(v'_\eta - w'_\eta)^{-(n_\eta-1)}\Big\}|v''_{\delta\delta'}|^{-1}\Big]^{\frac{N}{2}}, \tag{1.1}$$

where the elements of the matrices $\|v_{ij}\|$ $(i, j = 1, \ldots, t)$ and $\|v''_{\delta\delta'}\|$ $(\delta, \delta' = 1, \ldots, \ell + h)$ are given by

$$\begin{aligned}
v_{ij} &= \sum_{\gamma=1}^{N}(X_{i\gamma} - X_{i\cdot})(X_{j\gamma} - X_{j\cdot}), \quad X_{i\cdot} = \frac{1}{N}\sum_{\gamma=1}^{N} X_{i\gamma}, \\
v''_{ss'} &= v_{ss'}, \\
v''_{s,\ell+\eta} &= u'_{s\eta}\sqrt{n_\eta}, \\
v''_{\ell+\eta,\ell+\eta} &= v_\eta + (n_\eta - 1)w'_\eta, \\
v''_{\ell+\eta,\ell+\eta'} &= \sqrt{n_\eta n_{\eta'}}\, z'_{\eta\eta'}, \quad \eta \ne \eta'
\end{aligned}$$

with

$$v_{ss'} = \sum_{\gamma=1}^{N}(X_{s\gamma} - X_{s\cdot})(X_{s'\gamma} - X_{s'\cdot}), \quad X_{s\cdot} = \frac{1}{N}\sum_{\gamma=1}^{N} X_{s\gamma}$$

$$u'_{s\eta} = \frac{1}{n_\eta} \sum_{\gamma=1}^{N} \sum_{i_\eta=\ell+\bar{n}_\eta+1}^{\ell+\bar{n}_{\eta+1}} (X_{s\gamma} - X_{s\cdot})(X_{i_\eta\gamma} - \tilde{X}_{\eta\cdot}),$$

$$\tilde{X}_{\eta\cdot} = \frac{1}{Nn_\eta} \sum_{\gamma=1}^{N} \sum_{i_\eta=\ell+\bar{n}_\eta+1}^{\ell+\bar{n}_{\eta+1}} X_{i_\eta\gamma},$$

$$v'_\eta = \frac{1}{n_\eta} \sum_{\gamma=1}^{N} \sum_{i_\eta=\ell+\bar{n}_\eta+1}^{\ell+\bar{n}_{\eta+1}} (X_{i_\eta\gamma} - \tilde{X}_{\eta\cdot})^2,$$

$$w'_\eta = \frac{1}{n_\eta(n_\eta - 1)} \sum_{\gamma=1}^{N} \sum_{\substack{i_\eta, j_\eta=\ell+\bar{n}_\eta+1 \\ i_\eta \neq j_\eta}}^{\ell+\bar{n}_{\eta+1}} (X_{i_\eta\gamma} - \tilde{X}_{\eta\cdot})(X_{j_\eta\gamma} - \tilde{X}_{\eta\cdot}),$$

$$z'_{\eta\eta'} = \frac{1}{n_\eta n_{\eta'}} \sum_{\gamma=1}^{N} \sum_{i_\eta=\ell+\bar{n}_\eta+1}^{\ell+n_{\eta+1}} \sum_{i_{\eta'}=\ell+\bar{n}_{\eta'}+1}^{\ell+n_{\eta'+1}} (X_{i_\eta\gamma} - \tilde{X}_{\eta\cdot})(X_{i_{\eta'}\gamma} - \tilde{X}_{\eta'\cdot}), \quad \eta \neq \eta',$$

where $s, s' = 1, \ldots, \ell$; $\bar{n}_\eta = n_1 + \ldots + n_{\eta-1}$, $\bar{n}_1 = 0$; $\eta, \eta' = 1, \ldots, h$.

The d-th moment of $V = \left[\lambda_1(\text{mvc})\right]^{2/N}$, where $H_1(\text{mvc})$ is true, is given by Votaw (1948) as

$$E(V^d) = \prod_{i=1+q}^{t} \left\{ \frac{\Gamma\left[\frac{1}{2}(N-i) + d\right]}{\Gamma\left[\frac{1}{2}(N-i)\right]} \right\} \prod_{\eta=1}^{h} \left\{ \frac{(n_\eta - 1)^{(n_\eta - 1)d} \Gamma\left[\frac{1}{2}(n_\eta - 1)N\right]}{\Gamma\left[(n_\eta - 1)\left\{\frac{1}{2}N + d\right\}\right]} \right\}, \quad (1.2)$$

for $\text{Re}(d) > -\frac{1}{2}(N-t)$, $\text{Re}(\cdot)$ is the real part $(\cdot)$.

In this paper distribution of V in terms of Meijer's G-function as well in series form suitable for computation has been derived (see Consul, 1969; Bagai, 1972; Mathai and Saxena, 1973; Nagar, Jain, and Gupta, 1985; and Gupta and Nagar, 1987). First in Section 2, the density is expressed in Meijer's G-function. In Section 3, the density is given in series involving psi function and generalized Riemann zeta function.

2. DENSITY OF V IN G-FUNCTION

On substituting $p_\eta = n_\eta - 1$, $\eta = 1, \dots, h$, $p = p_1 + \cdots p_h$ and $d = \tau - 1$ and using Gauss-Legendre multiplication formula (Luke, 1969, p. 11), the expression (1.1) simplifies to

$$E(V^{\tau-1}) = K(p_1, \dots, p_h; t) \frac{\prod_{j=1}^{p} \Gamma\left[\frac{1}{2}(N - t - 1 + j) + \tau - 1\right]}{\prod_{\eta=1}^{h} \prod_{r_\eta=0}^{p_\eta - 1} \Gamma\left[\frac{1}{2}N + \tau - 1 + \frac{r_\eta}{p_\eta}\right]}, \tag{2.1}$$

where

$$K(p_1, \dots, p_h; t) = (2\pi)^{(p-h)/2} \frac{\prod_{\eta=1}^{h} \left\{ p_\eta^{-(Np_\eta - 1)/2} \Gamma\left[\frac{1}{2} p_\eta N\right] \right\}}{\prod_{j=1}^{p} \Gamma\left[\frac{1}{2}(N - t - 1 + j)\right]}. \tag{2.2}$$

Now using inverse Mellin transform, the density of V, denoted by $f(v)$ is derived as

$$f(v) = K(p_1, \dots, p_h; t)(2\pi i)^{-1} \cdot \int_C \frac{\prod_{j=1}^{p} \Gamma\left[\frac{1}{2}(N - t - 1 + j) + \tau - 1\right]}{\prod_{\eta=1}^{h} \prod_{r_\eta=0}^{p_\eta - 1} \Gamma\left[\frac{1}{2}N + \tau - 1 + \frac{r_\eta}{p_\eta}\right]} v^{-\tau}\, d\tau, \quad 0 < v < 1, \tag{2.3}$$

where $i = \sqrt{-1}$ and C is a suitable contour. Substituting $\frac{1}{2}(N - t - 1) + \tau - 1 = s$, the density (2.3) is restated as

$$f(v) = K(p_1, \dots, p_h; t) v^{(N-t-3)/2} (2\pi i)^{-1} \int_{C_1} \Delta(s) v^{-s}\, ds, \quad 0 < v < 1, \tag{2.4}$$

where

$$\Delta(s) = \frac{\prod_{j=1}^{p} \Gamma\left(s + \frac{1}{2}j\right)}{\prod_{\eta=1}^{h} \prod_{r_\eta=0}^{p_\eta - 1} \Gamma\left[s + \frac{1}{2}(t + 1) + \frac{r_\eta}{p_\eta}\right]} \tag{2.5}$$

and C_1 is a suitable contour. A contour C_1 exists for which (2.4) can be represented as a G-function and it can be evaluated as a sum of residues at the poles of the integrand. Properties of G-functions and other details are available in Luke (1969) and Mathai and Saxena (1973). Using the definition of Meijer's G-function (Luke, 1969, p. 143), the above density can be put as

$$f(v) = K(p_1, \ldots, p_h; t) v^{(N-t-3)/2}$$
$$\cdot\, G_{p,p}^{p,0}\left[v \middle| \begin{matrix} \left\{\frac{t+1}{2} + \frac{r_\eta}{p_\eta},\ r_\eta = 0, 1, \ldots, p_\eta - 1,\ \eta = 1, \ldots, h\right\} \\ \left\{\frac{j}{2},\ j = 1, 2, \ldots, p\right\} \end{matrix} \right], \quad 0 < v < 1. \quad (2.6)$$

Using the result (Mathai and Saxena, 1973, p. 64)

$$G_{2,2}^{2,0}\left[z \middle| \begin{matrix} \alpha_1 + \beta_1 - 1, & \alpha_2 + \beta_2 - 1 \\ \alpha_1 - 1, & \alpha_2 - 1 \end{matrix} \right]$$
$$= \frac{z^{\alpha_2 - 1}(1-z)^{\beta_1 + \beta_2 - 1}}{\Gamma(\beta_1 + \beta_2)}\ {}_2F_1(\alpha_2 + \beta_2 - \alpha_1, \beta_1; \beta_1 + \beta_2; 1 - z), \quad |z| < 1,$$

the density for $n_1 = n_2 = 2$ (or $p_1 = p_2 = 1$) is given in a simple form

$$f(v) = K(1, 1; t) \frac{v^{(N-t-1)/2}(1-v)^{t-\frac{3}{2}}}{\Gamma\left(t - \frac{1}{2}\right)}\ {}_2F_1\left(\frac{1}{2}t, \frac{1}{2}t; t - \frac{1}{2}; 1 - v\right), \quad 0 < v < 1,$$

where ${}_2F_1$ is the Gauss' hypergeometric function.

3. DENSITY OF V IN SERIES FORM

For the computation of percentage points one needs an explicit representation for $f(v)$. In order to evaluate the density in computable form, that is as a sum of residues, we identify all the poles and their orders, of the integrand in (2.4). Since the alternate gamma functions coincide whereas the poles of adjacent gamma functions do not. So we separate the two types of poles and represent all the poles in two sets. Also some of the gamma functions in the numerator may cancel out with the gamma functions in denominator depending upon $p_1, \ldots, p_h$ and t. We therefore, without any loss of generality, assume that δp_η's are even and $(h-\delta)p_\eta$'s are odd and consider the eight cases, (i) p-even, t-even, $p/2 > h$ (ii) p-even, t-odd, $p/2 > h$ (iii) p-odd, t-even, $(p-1)/2 > h$ (iv) p-odd, t-odd, $(p-1)/2 > h$ (v) p-even, t-even, $\delta \le p/2 \le h$ (vi) p-odd, t-even, $\delta \le (p-1)/2 < h$ (vii) p-even, t-odd, $\delta \le p/2 \le h$ and (viii) p-odd, t-odd, $\delta \le (p-1)/2 < h$, for cancellation of gamma functions separately. Furthermore, by comparing the resulting expression of $\Delta(s)$ for these cases, we can see that $\Delta(s)$ has three different structures.

Type 1.

$$\Delta_1(s) = \frac{\prod_{j=1}^{a}\Gamma\left(s-\frac{1}{2}+j\right)\prod_{j=1}^{b}\Gamma\left(s+j\right)}{\prod_{j=a+1}^{a'}\left(s-\frac{1}{2}+j\right)^{a_j}\prod_{j=b+1}^{b'}(s+j)^{b_j}\prod_{\eta=1}^{h}\prod_{\substack{r_\eta-1\\ \neq p_\eta/2}}^{p_\eta-1}\Gamma\left[s+\frac{1}{2}(t+1)+\frac{r_\eta}{p_\eta}\right]} \tag{3.1}$$

where a, b, a', b' etc. have different values for different subcases:

(i) p-even, t-even, $p/2 > h$

$$\begin{aligned} a &= p/2-h & b &= p/2-\delta \\ a' &= t/2 & b' &= t/2 \\ a'' &= p/2-1 & b'' &= p/2-1 \\ a_j &= \begin{cases} j-a, & j=a+1,\ldots,a'' \\ h, & j=a''+1,\ldots,a' \end{cases} & b_j &= \begin{cases} j-b, & j=b+1,\ldots,b'' \\ \delta, & j=b''+1,\ldots,b' \end{cases}. \end{aligned} \tag{3.2}$$

(ii) p-even, t-odd, $p/2 > h$

$$\begin{aligned} a &= p/2-\delta & b &= p/2-h \\ a' &= (t+1)/2 & b' &= (t+1)/2-1 \\ a'' &= p/2-1 & b'' &= p/2-1 \\ a_j &= \begin{cases} j-a, & j=a+1,\ldots,a'' \\ \delta, & j=a''+1,\ldots,a' \end{cases} & b_j &= \begin{cases} j-b, & j=b+1,\ldots,b'' \\ h, & j=b''+1,\ldots,b' \end{cases}. \end{aligned} \tag{3.3}$$

(iii) p-odd, t-even, $(p-1)/2 > h$

$$\begin{aligned} a &= (p+1)/2 - h & b &= (p-1)/2 - \delta \\ a' &= t/2 & b' &= t/2 \\ a'' &= (p+1)/2 - 1 & b'' &= (p-1)/2 - 1 \\ a_j &= \begin{cases} j-a, \ j = a+1, \dots, a'' \\ h, \ j = a''+a, \dots, a' \end{cases} & b_j &= \begin{cases} j-b, \ j = b+1, \dots, b'' \\ \delta, \ j = b''+1, \dots, b' \end{cases}. \end{aligned} \tag{3.4}$$

(iv) p-odd, t-odd, $(p-1)/2 > h$

$$\begin{aligned} a &= (p+1)/2 - \delta & b &= (p-1)/2 - h \\ a' &= (t+1)/2 & b' &= (t-1)/2 \\ a'' &= (p+1)/2 - 1 & b'' &= (p-1)/2 - 1 \\ a_j &= \begin{cases} j-a, \ j = a+1, \dots, a'' \\ \delta, \ j = a''+1, \dots, a' \end{cases} & b_j &= \begin{cases} j-b, \ j = b+1, \dots, b'' \\ h, \ j = b''+1, \dots, b' \end{cases}. \end{aligned} \tag{3.5}$$

Type 2.

$$\begin{aligned} \Delta_2(s) \; &= \frac{\prod_{j=1}^{b} \Gamma(s+j)}{\prod_{j=b+1}^{b'} (s+j)^{b_j} \Gamma^c\left[s + \frac{1}{2}(t+1)\right] \prod_{j=1}^{c'} \left(s - \frac{1}{2} + j\right)^{c_j}} \\ &\cdot \frac{1}{\prod_{\eta=1}^{h} \prod_{\substack{r_\eta=1 \\ \neq p_\eta/2}}^{p_\eta-1} \Gamma\left[\alpha + \frac{1}{2}(t+1) + \frac{r_\eta}{p_\eta}\right]} \end{aligned} \tag{3.6}$$

(v) p-even, t-even, $\delta \le p/2 < h$

$$\begin{aligned} b &= p/2 - \delta & c &= h - p/2 \\ b' &= t/2 & c' &= t/2 \\ b'' &= p/2 - 1 & c'' &= p/2 - 1 \\ b_j &= \begin{cases} j - b, \ j = b+1, \dots, b'' \\ \delta, \ j = b''+1, \dots, b' \end{cases} & c_j &= \begin{cases} j, \ j = 1, \dots, c'' \\ p/2, \ j = c''+1, \dots, c' \end{cases} \end{aligned} \tag{3.7}$$

(vi) p-odd, t-even, $\delta \le (p-1)/2 < h$

$$\begin{aligned} b &= (p-1)/2 - \delta & c &= h - (p+1)/2 \\ b' &= t/2 & c' &= t/2 \\ b'' &= (p-1)/2 - 1 & c'' &= (p+1)/2 - 1 \\ b_j &= \begin{cases} j - b, \ j = b+1, \dots, b'' \\ \delta, \ j = b''+1, \dots, b' \end{cases} & c_j &= \begin{cases} j, \ j = 1, 2, \dots, c'' \\ (p+1)/2, \ j = c''+1, \dots, c' \end{cases} \end{aligned} \tag{3.8}$$

Type 3.

$$\Delta_3(s) = \frac{\prod_{j=1}^{a} \Gamma\left(s - \frac{1}{2} + j\right)}{\prod_{j=a+1}^{a'} \left(s - \frac{1}{2} + j\right)^{a_j} \Gamma^c\left[s + \frac{1}{2}(t+1)\right] \prod_{j=1}^{c'} (s+j)^{c_j}} \cdot \frac{1}{\prod_{\eta=1}^{h} \prod_{\substack{r_\eta=1 \\ \neq p_\eta/2}}^{p_\eta - 1} \Gamma\left[\alpha + \frac{1}{2}(t+1) + \frac{r_\eta}{p_\eta}\right]} \tag{3.9}$$

(vii) p-even, t-odd, $\delta \le p/2 \le h$

$$a = p/2 - \delta \qquad c = h - p/2$$

$$a' = (t+1)/2 \qquad c' = (t-1)/2$$

$$a'' = p/2 - 1 \qquad c'' = p/2 - 1 \tag{3.10}$$

$$a_j = \begin{cases} j - a, \ j = a+1, \ldots, a'' \\ \delta, \ j = a''+1, \ldots . a' \end{cases} \qquad c_j = \begin{cases} j, \ j = 1, \ldots, c'' \\ p/2, \ j = c''+1, \ldots, c' \end{cases}.$$

(viii) p-odd, t-odd, $\delta \le (p-1)/2 < h$

$$a = (p+1)/2 - \delta \qquad c = h - (p-1)/2$$

$$a' = (t+1)/2 \qquad c' = (t-1)/2$$

$$a'' = (p+1)/2 - 1 \qquad c'' = (p-1)/2 - 1$$

$$a_j = \begin{cases} j - a, \ j = a+1, \ldots, a' \\ \delta, \ j = a''+1, \ldots, a' \end{cases} \qquad c_j = \begin{cases} j, \ j = 1, \ldots, c'' \\ (p-1)/2, \ j = c''+1, \ldots, c' \end{cases}. \tag{3.11}$$

Now with the help of $\Delta_1(s), \Delta_2(s)$, and $\Delta_3(s)$, one can identify poles and their orders of the integrand for Case 1, Case2 and Case 3. The poles are available by equating to zero each factor of

$$\pi'\Big(s - \frac{1}{2} + i\Big)^{\alpha_i} \pi''(s+i)^{\beta_i} \tag{3.12}$$

where α_i and β_i give orders of the pole at $s = -i + \frac{1}{2}$ and $s = -i$ respectively. For Case 1 $\pi'(\cdot) = \prod_{i=1}^{\infty}(\cdot)$, $\pi''(\cdot) = \prod_{i=1}^{\infty}(\cdot)$, for Case 2 $\pi'(\cdot) = \prod_{i=1}^{c'}(\cdot)$, $\pi''(\cdot) = \prod_{i=1}^{\infty}(\cdot)$ and for Case 3 $\pi'(\cdot) = \prod_{i=1}^{\infty}(\cdot)$, $\pi''(\cdot) = \prod_{i=1}^{c'}(\cdot)$. The orders of the poles for the Case 1, Case 2, and Case 3 are given below :

Case 1.

$$\alpha_j = \begin{cases} j, \ j = 1, \ldots, a \\ a + a_j, \ j = a+1, \ldots, a' \\ a, \ j = a'+1, \ldots, \ldots \end{cases} \quad \beta_j = \begin{cases} j, \ j = 1, \ldots, b \\ b + b_j, \ j = b+1, \ldots, b' \\ b, \ j = b'+1, \ldots, \ldots \end{cases}. \tag{3.13}$$

Case 2.

$$\alpha_j = c_j,\ j = 1, \ldots, c' \qquad \beta_j = \begin{cases} j,\ j = 1, \ldots, b \\ b + b_j,\ j = b+1, \ldots, b' \\ b,\ j = b'+1, \ldots, \ldots \end{cases}. \quad (3.14)$$

Case 3.

$$\alpha_j = \begin{cases} j,\ j = 1, \ldots, a \\ a + a_j,\ j = a+1, \ldots, a' \\ a,\ j = a'+1, \ldots, \ldots \end{cases} \quad \beta_j = c_j,\ \ j = 1, \ldots, c'. \quad (3.15)$$

Now using the residue theorem from (2.4), we get the following result.

Theorem 3.1. The p.d.f. of $V = [\lambda_1(\text{mvc})]^{2/N}$, where $\lambda_1(\text{mvc})$ is the likelihood ratio criterion for testing Votaw's $H_1(\text{mvc})$ hypothesis, is given by

$$f(v) = K(p_1, \ldots, p_h; t) v^{(N-t-3)/2} \Big[\Sigma' R_{1i} + \Sigma'' R_{2i} \Big],\ 0 < v < 1, \quad (3.16)$$

where $K(p_1, \ldots, p_h; t)$ is definded in (2.2), R_{1i} and R_{2i} are the residues at the poles $s = -i + \frac{1}{2}$ and $s = -i$ respectively. Also

$$R_{1i} = \frac{1}{(\alpha_i - 1)!} \lim_{s \to -i + \frac{1}{2}} \frac{\partial^{\alpha_i - 1}}{\partial s^{\alpha_i - 1}} \Big[\Big(s - \frac{1}{2} + i \Big)^{\alpha_i} \Delta(s) v^{-s} \Big] \quad (3.17)$$

and

$$R_{2i} = \frac{1}{(\beta_i - 1)!} \lim_{s \to -i} \frac{\partial^{\beta_i - 1}}{\partial s^{\beta_i - 1}} \Big[\Big(s + i \Big)^{\beta_i} \Delta(s) v^{-s} \Big] \quad (3.18)$$

where for the Case 1, $\Sigma'(\cdot) = \sum_{i=1}^{\infty}(\cdot)$, $\Sigma''(\cdot) = \sum_{i=1}^{\infty}(\cdot)$, $\Delta(s) = \Delta_1(s)$ and α_i, β_i are given by (3.13), for the Case 2, $\Sigma'(\cdot) = \sum_{i=1}^{c'}(\cdot)$, $\Sigma''(\cdot) = \sum_{i=1}^{\infty}(\cdot)$, $\Delta(s) = \Delta_2(s)$ and α_i, β_i are given by (3.14) and for the Case 3, $\Sigma'(\cdot) = \sum_{i=1}^{\infty}(\cdot)$, $\Sigma''(\cdot) = \sum_{i=1}^{c'}$, $\Delta(s) = \Delta_3(s)$ and α_i, β_i are given by (3.15).

Clearly, the residues R_{1i} and R_{2i} will be different for the three cases considered above. Considering them separately one can derive the expressions for residues explicitly. Case 1 covers four subcases. For illustration, we derive the explicit expression for R_{1i} for Case 1 below. In this case the expression (3.17) is written as

$$R_{1i} = \frac{1}{(\alpha_i - 1)!} \lim_{s \to -i + \frac{1}{2}} \frac{\partial^{\alpha_i - 1}}{\partial s^{\alpha_i - 1}} \Big[A_{11i} v^{-s} \Big] \quad (3.19)$$

where

$$A_{11i} = \frac{\left(s-\frac{1}{2}+i\right)^{\alpha_i}\prod_{j=1}^{a}\Gamma\left(s-\frac{1}{2}+j\right)\prod_{j=1}^{b}\Gamma\left(s+j\right)}{\prod_{j=a+1}^{a'}\left(s-\frac{1}{2}+j\right)^{\alpha_j}\prod_{j=b+1}^{b'}(s+j)^{b_j}\prod_{\eta=1}^{h}\prod_{\substack{r_\eta=1\\ \neq p_\eta/2}}^{p_\eta-1}\Gamma\left[s+\frac{1}{2}(t+1)+\frac{r_\eta}{p_\eta}\right]}$$

$$= \frac{\Gamma^{\alpha_i'}\left(s-\frac{1}{2}+i+1\right)\prod_{j=\alpha_i'+1}^{a}\Gamma\left(s+\frac{1}{2}+j\right)\prod_{j=1}^{b}\Gamma(s+j)}{\sum_{\substack{j=1+a\\ \neq i}}^{a'}\left(s-\frac{1}{2}+j\right)^{\alpha_j}\prod_{j=1}^{\alpha_i'-1}\left(s-\frac{1}{2}+j\right)^{j}\prod_{j=\alpha_i'}^{i-1}\left(s-\frac{1}{2}+j\right)^{\alpha_i'}}$$

$$\cdot\frac{1}{\prod_{j=b+1}^{b'}(s+j)^{b_j}\prod_{\eta=1}^{h}\prod_{\substack{r_\eta=1\\ \neq p_\eta/2}}^{p_\eta-1}\Gamma\left[s+\frac{1}{2}(t+1)+\frac{r_\eta}{p_\eta}\right]} \tag{3.20}$$

$$\alpha_i' = \begin{cases} i, & i=1,2,\ldots,a \\ a, & i=a+1,a+2,\ldots \end{cases}. \tag{3.21}$$

By using a result from differential calculus it is possible to further develop the expression (3.19) as

$$R_{1i} = \frac{1}{(\alpha_i-1)!}\lim_{s\to -i+\frac{1}{2}} v^{-s}\sum_{k=0}^{\alpha_i-1}\binom{\alpha_i-1}{k}(-\ell n v)^{\alpha_i-1-k}A_{11i}^{(k)}. \tag{3.22}$$

Clearly

$$\begin{aligned} A_{11i}^{(k)} &= \frac{\partial^k}{\partial s^k}A_{11i} = \frac{\partial^{k-1}}{\partial s^{k-1}}\left(\frac{\partial}{\partial s}A_{11i}\right) \\ &= \frac{\partial^{k-1}}{\partial s^{k-1}}\left(A_{11i}\frac{\partial}{\partial s}\ell n A_{11i}\right) = \frac{\partial^{k-1}}{\partial s^{k-1}}(A_{11i}B_{11i}), \end{aligned} \tag{3.23}$$

where

$$\begin{aligned} B_{11i} &= \frac{\partial}{\partial s}\ell n A_{11i} \\ &= \alpha_i'\psi\left(s-\frac{1}{2}+i+1\right) + \sum_{j=\alpha_i'+1}^{a}\psi\left(s-\frac{1}{2}+j\right) \end{aligned}$$

$$+\sum_{j=1}^{b}\psi(s+j)-\sum_{\substack{j=a+1\\ \neq i}}^{a'}a_j\left(s-\frac{1}{2}+j\right)^{-1}$$

$$-\sum_{j=1}^{\alpha_i'-1}j\left(s-\frac{1}{2}+j\right)^{-1}-\alpha_i'\sum_{j=\alpha_i'}^{i-1}\left(s-\frac{1}{2}+j\right)^{-1}$$

$$-\sum_{j=b+1}^{b'}b_j(s+j)^{-1}-\sum_{\eta=1}^{h}\sum_{\substack{r_\eta=1\\ \neq p_\eta/2}}^{p_\eta-1}\psi\left(s+\frac{1}{2}(t+1)+\frac{r_\eta}{p_\eta}\right). \qquad (3.24)$$

Consequently all the derivatives of A_{11i} are available from the following recurrence relation

$$A_{11i}^{(k)}=\sum_{m=0}^{k-1}\binom{k-1}{m}A_{11i}^{(k-1-m)}B_{11i}^{(m)} \qquad (3.25)$$

with

$$\begin{aligned}B_{11i}^{(m)} &= \frac{\partial^m}{\partial s^m}B_{11i}\\ &= (-1)^{m+1}m!\left[\alpha_i'\zeta\left(m+1,s-\frac{1}{2}+i+1\right)+\sum_{j=\alpha_i'+1}^{a}\zeta\left(m+1,s-\frac{1}{2}+j\right)\right.\\ &+\sum_{j=1}^{b}\zeta(m+1,s+j)+\sum_{\substack{j=a+1\\ \neq i}}^{a'}a_j\left(s-\frac{1}{2}+j\right)^{-1-m}\\ &+\sum_{j=1}^{\alpha_i'-1}j\left(s-\frac{1}{2}+j\right)^{-1-m}+\alpha_i'\sum_{j=\alpha_i'}^{i-1}\left(s-\frac{1}{2}+j\right)^{-1-m}\\ &\left.+\sum_{j=b+1}^{b'}b_j(s+j)^{-1-m}-\sum_{\eta=1}^{h}\sum_{\substack{r_\eta=1\\ \neq p_\eta/2}}^{p_\eta-1}\zeta\left(m+1,s+\frac{1}{2}(t+1)+\frac{r_\eta}{p_\eta}\right)\right]. \qquad (3.26)\end{aligned}$$

In the expressions (3.24) and (3.26), $\psi(\cdot)$ and $\zeta(\cdot,\cdot)$ are well known psi function and generalized Riemann zeta function respectively (Abramowitz and Stegun, 1966, Magnus et. al., 1966).

From (3.22) it is possible to write

$$R_{1i} = \frac{v^{i-1/2}}{(\alpha_i - 1)!} \sum_{k=0}^{\alpha_i} \binom{\alpha_i - 1}{k} (-\ell n v)^{\alpha_i - 1 - k} A_{11io}^{(k)}, \tag{3.27}$$

where

$$A_{11io}^{(k)} = \sum_{m=0}^{k-1} \binom{k-1}{m} A_{11io}^{(k-1-m)} B_{11io}^{(m)} \tag{3.28}$$

with

$$\begin{aligned} A_{11io}^{(0)} &= A_{11io} \text{ at } s = -i + \frac{1}{2} \\ &= \frac{\prod_{j=\alpha_i'+1}^{a} \Gamma(j-i) \prod_{j=1}^{b} \Gamma\left(j - i + \frac{1}{2}\right)}{\prod_{\substack{j=a+1 \\ \neq i}}^{a'} (j-i)^{a_j} \prod_{j=1}^{\alpha_i'-1} (j-i)^j \prod_{j=\alpha_i'}^{i-1} (j-i)^{\alpha_i'}} \\ &\quad \cdot \frac{1}{\prod_{j=b+1}^{b'} \left(j - i + \frac{1}{2}\right)^{b_j} \prod_{\eta=1}^{h} \prod_{\substack{r_\eta=1 \\ \neq p_\eta/2}}^{p_\eta-1} \Gamma\left[\frac{1}{2}(t+2) - i + \frac{r_\eta}{p_\eta}\right]} \end{aligned} \tag{3.29}$$

$$\begin{aligned} B_{11io}^{(0)} &= B_{11i} \quad \text{at } s = -i + \frac{1}{2} \\ &= \alpha_i' \psi(1) + \sum_{j=\alpha_i'+1}^{a} \psi(j-i) + \sum_{j=1}^{b} \psi\left(j - i + \frac{1}{2}\right) - \sum_{j=a+1}^{a'} a_j (j-i)^{-1} \\ &\quad - \sum_{j=1}^{\alpha_i'-1} j(j-i)^{-1} - \alpha_i' \sum_{j=\alpha_i'}^{i-1} (j-i)^{-1} - \sum_{j=b+1}^{b} b_j \left(j - i + \frac{1}{2}\right)^{-1} \\ &\quad - \sum_{\eta=1}^{h} \sum_{\substack{r_\eta=1 \\ \neq p_\eta/2}}^{p_\eta-1} \psi\left(\frac{1}{2}(t+2) - i + \frac{r_\eta}{p_\eta}\right) \end{aligned} \tag{3.30}$$

$$B_{11io}^{(m)} = B_{11i}^{(m)} \text{ at } s = -i + \frac{1}{2}$$

$$= (-1)^{m+1} m! \Big[\alpha_i' \zeta(m+1,1) + \sum_{j=\alpha_i'+1}^{a} \zeta\Big(m+1, j-i\Big)$$

$$+ \sum_{j=1}^{b} \zeta\Big(m+1, j-i+\frac{1}{2}\Big) + \sum_{j=a+1}^{a'} a_j (j-i)^{-1-m}$$

$$+ \sum_{j=1}^{\alpha_i'-1} j(j-i)^{-1-m} + \alpha_i' \sum_{j=\alpha_i'}^{i-1} (j-i)^{-1-m} + \sum_{j=b+1}^{b'} b_j \Big(j-i+\frac{1}{2}\Big)^{-1-m}$$

$$- \sum_{\eta=1}^{h} \sum_{\substack{r_\eta=1 \\ \neq p_\eta/2}}^{p_\eta-1} \zeta\Big(m+1, \frac{1}{2}(t+2) - i + \frac{r_\eta}{p_\eta}\Big)\Big]. \tag{3.31}$$

Following exactly same procedure the residue at $s = -i$ is obtained as

$$R_{2i} = \frac{v^i}{(\beta_i - 1)!} \sum_{k=0}^{\beta_i - 1} \binom{\beta_i - 1}{k} (\ell n v)^{\beta_i - 1 - k} A_{12io}^{(k)} \tag{3.32}$$

where

$$A_{12io}^{(k)} = \sum_{m=0}^{k-1} \binom{k-1}{m} A_{12io}^{(k-1-m)} B_{12io}^{(m)} \tag{3.33}$$

with

$$A_{12io}^{(0)} = \frac{\prod_{j=\beta_i'+1}^{b} \Gamma(j-i) \prod_{j=1}^{a} \Gamma\Big(j-i-\frac{1}{2}\Big)}{\prod_{j=a+1}^{a'} \Big(j-i-\frac{1}{2}\Big)^{a_j} \prod_{\substack{j=b+1 \\ \neq i}}^{b'} (j-i)^{b_j} \prod_{j=1}^{\beta_i'-1} (j-i)^j}$$

$$\cdot \frac{1}{\prod_{j=\beta_i'}^{i-1} (j-i)^{\beta_i'} \prod_{\eta=1}^{h} \prod_{\substack{r_\eta=1 \\ \neq p_\eta/2}}^{p_\eta-1} \Gamma\Big[\frac{1}{2}(t+1) - i + \frac{r_\eta}{p_\eta}\Big]} \tag{3.34}$$

$$\beta_i' = \begin{cases} i, & i = 1, 2, \ldots, b \\ b, & j = b+1, b+2, \ldots \end{cases} \tag{3.35}$$

$$B_{12io}^{(0)} = \beta_i' \psi(1) + \sum_{j=\beta_i'+1}^{b} \psi(j-i) + \sum_{j=1}^{a} \psi\Big(j-i-\frac{1}{2}\Big)$$

$$- \sum_{j=a+1}^{a'} a_j\Big(j-i-\frac{1}{2}\Big)^{-1} - \sum_{\substack{j=b+1\\ \neq i}}^{b'} b_j(j-i)^{-1} - \sum_{j=1}^{\beta_i'-1} j(j-i)^{-1}$$

$$- \beta_i' \sum_{j=\beta_i'}^{i-1} (j-i)^{-1} - \sum_{\eta=1}^{h} \sum_{\substack{r_\eta=1\\ \neq p_\eta/2}}^{p_\eta-1} \psi\Big(\frac{1}{2}(t+1)-i+\frac{r_\eta}{p_\eta}\Big) \qquad (3.36)$$

$$B_{12io}^{(m)} = (-1)^{m+1} m! \Bigg[\beta_i'\zeta(m+1,1) + \sum_{j=\beta_i'+1}^{b} \zeta(m+1,j-i)$$

$$+ \sum_{j=1}^{a} \zeta\Big(m+1, j-i-\frac{1}{2}\Big) + \sum_{j=a+1}^{a'} a_j\Big(j-i-\frac{1}{2}\Big)^{-1-m}$$

$$+ \sum_{\substack{j=b+1\\ \neq i}}^{b'} b_j(j-i)^{-1-m} + \sum_{j=1}^{\beta_i'-1} j(j-i)^{-1-m}$$

$$+ \beta_i' \sum_{j=\beta_i'}^{i-1} (j-i)^{-1-m} - \sum_{\eta=1}^{h} \sum_{\substack{r_\eta=1\\ \neq p_\eta/2}}^{p_\eta-1} \zeta\Big(m+1, \frac{1}{2}(t+1)-i+\frac{r_\eta}{p_\eta}\Big)\Bigg]. \qquad (3.37)$$

Now substituting (3.27) and (3.32) in (3.16) we get the following result.

Theorem 3.2. The p.d.f. of $V = [\lambda_1(\text{mvc})]^{2/N}$, where $\lambda_1(\text{mvc})$ is the likelihood ratio criterion for testing $H_1(\text{mvc})$, for Case 1, is given by

$$f(v) = K(p_1, \ldots, p_h, ; t) v^{(N-t-3)/2}$$

$$\cdot \sum_{i=1}^{\infty} \Bigg[\frac{v^{i-1/2}}{(\alpha_i-1)!} \sum_{k=0}^{\alpha_i-1} \binom{\alpha_i-1}{k} (-\ell n v)^{\alpha_i-1-k} A_{11io}^{(k)}$$

$$+ \frac{v^i}{(\beta_i-1)!} \sum_{k=0}^{\beta_i-1} \binom{\beta_i-1}{k} (-\ell n v)^{\beta_i-1-k} A_{12io}^{(k)} \Bigg], \; 0 < v < 1, \quad (3.38)$$

where α_i, β_i, $K(p_1, \ldots, p_h; t)$, $A_{11io}^{(k)}$ and $A_{12io}^{(k)}$ are given by (3.13), (2.2), (3.28)–(3.31) and (3.32)–(3.37) respectively.

Similarly, evaluating R_{1i} and R_{2i} using (3.6) and (3.14), we get the density for Case 2 as follows.

Theorem 3.3. The p.d.f. of $V = [\lambda_1(\text{mvc})]^{2/N}$, where $\lambda_1(\text{mvc})$ is the likelihood ratio criterion for testing $H_1(\text{mvc})$, for Case 2, is given by

$$\begin{aligned} f(v) &= K(p_1,\ldots,p_h,;t)v^{(N-t-3)/2} \\ &\cdot\Bigg[\sum_{i=1}^{c'}\frac{v^{i-1/2}}{(\alpha_i-1)!}\sum_{k=0}^{\alpha_i-1}\binom{\alpha_i-1}{k}(-\ell n v)^{\alpha_i-1-k}A_{21io}^{(k)} \\ &+\sum_{i=1}^{\infty}\frac{v^{i}}{(\beta_i-1)!}\sum_{k=0}^{\beta_i-1}\binom{\beta_i-1}{k}(-\ell n v)^{\beta_i-1-k}A_{22io}^{(k)}\Bigg],\ 0<v<1, \end{aligned} \tag{3.39}$$

where α_i, β_i and $K(p_1,\ldots,p_h;t)$ are given by (3.14) and (2.2) respectively, and $A_{21io}^{(k)}$ and $A_{22io}^{(k)}$ are given by (3.40)–(3.46) below.

$$A_{2\varepsilon io}^{(k)} = \sum_{m=0}^{k-1}\binom{k-1}{m}A_{2\varepsilon io}^{(k-1-m)}B_{2\varepsilon io'}^{(m)},\quad \varepsilon=1,2, \tag{3.40}$$

$$\begin{aligned} A_{21io}^{(0)} &= \frac{\prod_{j=1}^{b}\Gamma\left(j-i+\frac{1}{2}\right)}{\prod_{j=b+1}^{b'}\left(j-i+\frac{1}{2}\right)^{b_j}\prod_{\substack{j=1\\ \neq i}}^{c'}(j-i)^{c_j}\Gamma^{c}\left[\frac{1}{2}(t+2)-i\right]} \\ &\cdot\frac{1}{\prod_{\eta=1}^{h}\prod_{\substack{r_\eta=1\\ \neq p_\eta/2}}^{p_\eta-1}\Gamma\left[\frac{1}{2}(t+2)-i+\frac{r_\eta}{p_\eta}\right]} \end{aligned} \tag{3.41}$$

$$\begin{aligned} B_{21io}^{(0)} &= \sum_{j=1}^{b}\psi\left(j-i+\frac{1}{2}\right)-\sum_{j=b+1}^{b'}b_j\left(j-i+\frac{1}{2}\right)^{-1}-\sum_{\substack{j=1\\ \neq i}}^{c'}c_j(j-i)^{-1} \\ &-c\psi\left(\frac{1}{2}(t+2)-i\right)-\sum_{\eta=1}^{h}\sum_{\substack{r_\eta=1\\ \neq p_\eta/2}}^{p_\eta-1}\psi\left(\frac{1}{2}(t+2)-i+\frac{r_\eta}{p_\eta}\right) \end{aligned} \tag{3.42}$$

$$B_{12io}^{(m)} = (-1)^{m+1}m!\Bigg[\sum_{j=1}^{b}\zeta\left(m+1,j-i+\frac{1}{2}\right)+\sum_{j=b+1}^{b'}b_j\left(j-i+\frac{1}{2}\right)^{-1-m}$$

$$+\sum_{\substack{j=1\\ \neq i}}^{c'} c_j(j-i)^{-1-m} - c\zeta\Big(m+1, \frac{1}{2}(t+2)-i\Big)$$

$$-\sum_{\eta=1}^{h}\sum_{\substack{r_\eta=1\\ \neq p_\eta/2}}^{p_\eta-1} \zeta\Big(m+1, \frac{1}{2}(t+2)-i+\frac{r_\eta}{p_\eta}\Big)\Bigg] \tag{3.43}$$

$$A_{22io}^{(0)} = \frac{\prod_{j=\beta_i'+1}^{b}\Gamma(j-i)}{\prod_{\substack{j=b+1\\ \neq i}}^{b'}(j-i)^{b_j}\prod_{j=1}^{\beta_i'-1}(j-i)^j\prod_{j=\beta_i'}^{i-1}(j-i)^{\beta_i'}\prod_{j=1}^{c'}\left(j-i-\frac{1}{2}\right)^{c_j}}$$

$$\cdot\frac{1}{\Gamma^c\left[\frac{1}{2}(t+1)-i\right]\prod_{\eta=1}^{h}\prod_{\substack{r_\eta=1\\ \neq p_\eta/2}}^{p_\eta-1}\Gamma\left[\frac{1}{2}(t+1)-i+\frac{r_\eta}{p_\eta}\right]} \tag{3.44}$$

$$B_{22io}^{(0)} = \beta_i'\psi(1) + \sum_{j=\beta_i'+1}^{b}\psi(j-i) - \sum_{\substack{j=b+1\\ \neq i}}^{b'} b_j(j-i)^{-1}$$

$$-\sum_{j=1}^{\beta_i'-1} j(j-i)^{-1} - \beta_i'\sum_{j=\beta_i'}^{i-1}(j-i)^{-1} - \sum_{j=1}^{c'} c_j\Big(j-i-\frac{1}{2}\Big)^{-1}$$

$$-c\psi\Big(\frac{1}{2}(t+1)-i\Big) - \sum_{\eta=1}^{h}\sum_{\substack{r_\eta=1\\ \neq p_\eta/2}}^{p_\eta-1}\psi\Big(\frac{1}{2}(t+1)-i+\frac{r_\eta}{p_\eta}\Big) \tag{3.45}$$

$$B_{22io}^{(m)} = (-1)^{m+1}m!\Bigg[\beta_i'\zeta(m+1,1) + \sum_{j=\beta_i'+1}^{b}\zeta(m+1,j-i)$$

$$+\sum_{\substack{j=b+1\\ \neq i}}^{b'} b_j(j-i)^{-1-m} + \sum_{j=1}^{\beta_i'-1} j(j-i)^{-1-m} + \beta_i'\sum_{j=\beta_i'}^{i-1}(j-i)^{-1-m}$$

$$+\sum_{j=1}^{c'} c_j\Big(j-i-\frac{1}{2}\Big)^{-1-m} - c\zeta\Big(m+1, \frac{1}{2}(t+1)-i\Big)$$

$$-\sum_{\eta=1}^{h}\sum_{\substack{r_\eta=1\\ \neq p_\eta/2}}^{p_\eta-1}\zeta\Big(m+1,\frac{1}{2}(t+1)-i+\frac{r_\eta}{p_\eta}\Big)\Big]. \tag{3.46}$$

Also evaluating R_{1i} and R_{2i}, using (3.9) and (3.15) substituting them in (3.16), we get the density for the Case 3 given in the following theorem.

Theorem 3.4. The p.d.f. of $V=[\lambda_1(\mathrm{mvc})]^{2/N}$, where $\lambda_1(\mathrm{mvc})$ is the likelihood ratio criterion for testing $H_1(\mathrm{mvc})$, for Case 3, is given by

$$\begin{aligned} f(v) &= K(p_1,\ldots,p_h;t)v^{(N-t-3)/2} \\ &\cdot\Bigg[\sum_{i=1}^{\infty}\frac{v^{i-1/2}}{(\alpha_i-1)!}\sum_{k=0}^{\alpha_i-1}\binom{\alpha_i-1}{k}(-\ell nv)^{\alpha_i-1-k}A_{31io}^{(k)} \\ &+\sum_{i=1}^{c'}\frac{v^{i}}{(\beta_i-1)!}\sum_{k=0}^{\infty}\binom{\beta_i-1}{k}(-\ell nv)^{\beta_i-1-k}A_{32io}^{(k)}\Bigg],\ 0<v<1, \end{aligned} \tag{3.47}$$

where α_i,β_i and $K(p_1,\ldots,p_h;t)$ are given by (3.15) and (2.2) respectively, and $A_{31io}^{(k)}$ and $A_{32io}^{(k)}$ are given by (3.48)–(3.54) below.

$$A_{3\varepsilon io}^{(k)} = \sum_{m=0}^{k-1}\binom{k-1}{m}A_{3\varepsilon io}^{(k-1-m)}B_{3\varepsilon io}^{(m)},\quad \varepsilon=1,2 \tag{3.48}$$

$$\begin{aligned} A_{31io}^{(0)} &= \frac{\prod_{j=\alpha_i'+1}^{a}\Gamma(j-i)}{\prod_{\substack{j=a+1\\ \neq i}}^{a'}(j-i)^{a_j}\prod_{j=1}^{\alpha_i'-1}(j-i)^{j}\prod_{j=\alpha_i'}^{i-1}(j-i)^{\alpha_i'}\prod_{j=1}^{c'}\left(j-i+\frac{1}{2}\right)^{c_j}} \\ &\cdot\frac{1}{\Gamma^{c}\left[\frac{1}{2}(t+2)-i\right]\prod_{\eta=1}^{h}\prod_{\substack{r_\eta=1\\ \neq p_\eta/2}}^{p_\eta-1}\Gamma\left[\frac{1}{2}(t+2)-i+\frac{r_\eta}{p_\eta}\right]} \end{aligned} \tag{3.49}$$

$$B_{31io}^{(0)} = \alpha_i'\psi(1)+\sum_{j=\alpha_i'+1}^{a}\psi(j-i)-\sum_{\substack{j=a+1\\ \neq i}}^{a'}a_j(j-i)^{-1}$$

$$- \sum_{j=1}^{\alpha_i'-1} j(j-i)^{-1} - \alpha_i' \sum_{j=\alpha_i'}^{i-1} (j-i)^{-1} - \sum_{j=1}^{c'} c_j \Big(j-i+\frac{1}{2}\Big)^{-1}$$

$$- c\psi\Big(\frac{1}{2}t+1-i\Big) - \sum_{\eta=1}^{h} \sum_{\substack{r_\eta=1 \\ \neq p_\eta/2}}^{p_\eta-1} \psi\Big(\frac{1}{2}t+1-i+\frac{r_\eta}{p_\eta}\Big) \tag{3.50}$$

$$B_{31io}^{(m)} = (-1)^{m+1} m! \Big[\alpha_i' \zeta(m+1,1) + \sum_{j=\alpha_i'+1}^{a} \zeta(m+1, j-i)$$

$$+ \sum_{\substack{j=a+1 \\ \neq i}}^{a'} a_j (j-i)^{-1} + \sum_{j=1}^{\alpha_i'-1} j(j-i)^{-1-m} + \sum_{j=\alpha_i'}^{i-1} (j-i)^{-1-m}$$

$$+ \sum_{j=1}^{c'} c_j \Big(j-i+\frac{1}{2}\Big)^{-1-m} - c\zeta\Big(m+1, \frac{1}{2}t+1-i\Big)$$

$$- \sum_{\eta=1}^{h} \sum_{\substack{r_\eta=1 \\ \neq p_\eta/2}}^{p_\eta-1} \zeta\Big(m+1, \frac{1}{2}t+1-i+\frac{r_\eta}{p_\eta}\Big)\Big] \tag{3.51}$$

$$A_{32io}^{(0)} = \frac{\prod_{j=1}^{a} \Gamma\Big(j-i-\frac{1}{2}\Big)}{\prod_{j=a+1}^{a'} \Big(j-i-\frac{1}{2}\Big)^{a_j} \prod_{\substack{j=1 \\ \neq i}}^{c'} (j-i)^{c_j}}$$

$$\cdot \frac{1}{\Gamma^c\Big[\frac{1}{2}(t+1)-i\Big] \prod_{\eta=1}^{h} \prod_{\substack{r_\eta=1 \\ \neq p_\eta/2}}^{p_\eta-1} \Gamma\Big[\frac{1}{2}(t+1)-i+\frac{r_\eta}{p_\eta}\Big]} \tag{3.52}$$

$$B_{32io}^{(0)} = \sum_{j=1}^{a} \psi\Big(j-i-\frac{1}{2}\Big) - \sum_{j=a+1}^{a'} a_j \Big(j-i-\frac{1}{2}\Big)^{-1} - \sum_{\substack{j=1 \\ \neq i}}^{c'} c_j (j-i)^{-1}$$

$$- c\psi\Big(\frac{1}{2}(t+1)-i\Big) - \sum_{\eta=1}^{h} \sum_{\substack{r_\eta=1 \\ \neq p_\eta/2}}^{p_\eta-1} \psi\Big(\frac{1}{2}(t+1)-i+\frac{r_\eta}{p_\eta}\Big) \tag{3.53}$$

$$B_{32io}^{(m)} = (-1)^{m+1} m! \Bigg[\sum_{j=1}^{a} \zeta\Big(m+1, j-i-\frac{1}{2}\Big) + \sum_{j=a+1}^{a'} a_j \Big(j-i-\frac{1}{2}\Big)^{-1-m}$$

$$+ \sum_{\substack{j=1 \\ \neq i}}^{c'} c_j (j-i)^{-1-m} - c\zeta\Big(m+1, \frac{1}{2}(t+1)-i\Big)$$

$$- \sum_{\eta=1}^{h} \sum_{\substack{r_\eta=1 \\ \neq p_\eta/2}}^{p_\eta-1} \zeta\Big(m+1, \frac{1}{2}(t+1)-i+\frac{r_\eta}{p_\eta}\Big)\Bigg]. \tag{3.54}$$

Specializing $p_1, \ldots, p_h$ and t in the expression for the density derived in this section and simplifying the resulting expression using results on gamma-, psi-, and generalized Riemann zeta functions one can easily derive several special cases of the density.

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