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# On Some Weak Positive Dependence Notions[†]

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**ABSTRACT**

A random vector $\underline{X} = (X_1, \ldots, X_n)$ is weakly associated if and only if for every pair of partitions $\underline{X}_1 = (X_{\pi(1)}, \ldots, X_{\pi(k)})$, $\underline{X}_2 = (X_{\pi(k+1)}, \ldots, X_{\pi(n)})$ of $\underline{X}$, $P(\underline{X}_1 \in A, \underline{X}_2 \in B) \geq P(\underline{X}_1 \in A)\underline{P}(\underline{X}_2 \in B)$ whenever $A$ and $B$ are open upper sets and $\pi$ is a permutation of $\{1, \ldots, n\}$. In this paper, we develop notions of weak positive dependence , which are weaker than a positive version of negative association(weak association) but stronger than positive orthant dependence by arguments similar to those of Shaked. We also illustrate some concepts of a particular interest. Various properties and interrelationships are derived.

**KEYWORDS:** Positive dependence, Association, Orthant dependence, Upper sets.

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# 1. INTRODUCTION

Among various notions of positive dependence, that of association has proved to be quite useful. Esary, Proschan, and Walkup([1],[4]) introduced this concept to obtain bounds related to coherent(coordinatewise increasing) functions in the theory of reliability. The defining property for a random vector $\underline{X}$ to be associated is that for every coordinatewise nondecreasing functions $f$ and $g$

$$\mathrm{Cov}(f(\underline{X}), g(\underline{X})) \geq 0. \tag{1.1}$$

Since association represents a strong positive dependence, weaker concepts have been considered in the literature. As pointed out by Shaked([8]) many of these can be viewed as variations of the classes from which $f$, $g$ are chosed and then (1.1) imposed. We often divide a $n$-components system into two disjoint subsystems consisting of $k$ and $(n-k)$ components, respectively, and desire to investigate dependence between them : one of examples is 'weak association' introduced by Burton, Dabrowski and Dehling([2])(See Definition 2.1). They also suggested that each random vector may have negative dependence but every two partitions are to be positive.

In this paper we will obtain various new positive dependence notions by splitting the random vector into two partitions and considering dependence between them with arguments similar to those of Shaked([8]). If for every real vector $\underline{x} = (x_1, \ldots, x_n)$, $P(\underline{X} > \underline{x}) \geq \prod_{i=1}^{n} P(X_i > x_i)$ then we say that $\underline{X} = (X_1, \ldots, X_n)$ is positively upper orthant dependent(PUOD) and if for every real vector $\underline{x} = (x_1, \ldots, x_n), P(\underline{X} \leq \underline{x}) \geq \prod_{i=1}^{n} P(X_i \leq x_i)$ then we say that $\underline{X}$ is positively lower orthant dependent(PLOD). When $n = 2$ the $\underline{X} = (X_1, X_2)$ is PUOD if and only if $\underline{X}$ is PLOD(see Lehman[6]); we say then that $\underline{X}$ is positive quadrant dependent(PQD). It is well known that $(X_1, X_2)$ is PQD if and only if $\mathrm{Cov}(h_1(X_1), h_2(X_2)) \geq 0$, whenever $h_1, h_2$ are univariate nonincreasing functions, that is, $(X_1, X_2)$ is weakly associated(Definition 2.1) and that $\underline{X}'$s are weakly associated then they are PUOD and PLOD similar to the fact that if the $\underline{X}'$s are associated then they are PUOD and PLOD(Dykstra et al. [3]).

Various results in probability and statistics have been derived under the assumption that some underlying random variables are weakly associated. In some case one may inspect that results are valid even if one relaxes the assumption of weak association, however, the validity of these results may be violated if instead of assumption of weak association one merely assumes positive upper orthant dependence or positive lower orthant dependence. Thus, we will derive some new weak positive dependence between weak association and orthant dependence.

The general propositions and some observations are given in Section 2. In Section 3 the illustrative special cases are introduced. Based on the general propositions some properties are proven in Section 4. Some counterimplications and an example of applications are given in Section 5.

## 2. GENERAL PROPOSITION AND DEFINITIONS

**Definition 2.1.** (Burton et al.(1986)). A random vector $\underline{X} = (X_1, \ldots, X_n)$ is said to be weakly associated(WA) if for every pair of partitions $\underline{X}_1 = (X_{\pi(1)}, \ldots, X_{\pi(k)})$ and $\underline{X}_2 = (X_{\pi(k+1)}, \ldots, X_{\pi(n)})$ of $\underline{X}$, and every pair of increasing functions $f$ on $R^k$ and $g$ on $R^{(n-k)}$

$$\mathrm{Cov}(f(\underline{X}_1), g(\underline{X}_2)) \geq 0 \tag{2.1}$$

whenever $\pi$ is any permutation of $\{1, 2, \ldots, n\}$ and $1 \leq k \leq n-1$.

Weak association defines a strictly larger class of random variables than does association(e.g. A-Compound processes of Burton et al.[2]).

**Proposition 2.2.** A random vector $\underline{X} = (X_1, \ldots, X_n)$ is weakly associated if and only if for every pair of partitions $\underline{X}_1 = (X_{\pi(1)}, \ldots, X_{\pi(k)})$ and $\underline{X}_2 = (X_{\pi(k+1)}, \ldots, X_{\pi(n)})$ of $\underline{X}$

$$P(\underline{X}_1 \in A, \underline{X}_2 \in B) \geq P(\underline{X}_1 \in A)P(\underline{X}_2 \in B) \tag{2.2}$$

whenever $A$ and $B$ are open upper sets, $1 \le k \le n-1$, and $\pi$ is any permutation of $\{1,\ldots,n\}$.

**Proof.** We only show the converse : Let $\pi$ denote any permutation of $\{1,2,\ldots,n\}$, $\underline{X}_1 = (X_{\pi(1)},\ldots,X_{\pi(k)})$, $\underline{X}_2 = (X_{\pi(k+1)},\ldots,X_{\pi(n)})$ be arbitrary partitions of $\underline{X}$, and $f, g$ be arbitrary increasing functions on $R^{(k)}, R^{(n-k)}$ respectively. Then for every real $s$ and $t$ , $A = \{f(\underline{X}_1) > s\}$ and $B = \{g(\underline{X}_2) > t\}$ are open upper sets. Thus

$$\begin{aligned} P(f(\underline{X}_1) > s, g(\underline{X}_2) > t) &= P(\underline{X}_1 \in A, \underline{X}_2 \in B) \\ &\ge P(\underline{X}_1 \in A)P(\underline{X}_2 \in B) \\ &= P(f(\underline{X}_1) > s)P(g(\underline{X}_2) > t). \end{aligned}$$

Define

$$X_f(s) = \begin{cases} 1, & \text{if } f(\underline{X}_1) > s, \\ 0, & \text{otherwise,} \end{cases}$$

$$X_g(t) = \begin{cases} 1, & \text{if } g(\underline{X}_2) > t, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} &\mathrm{Cov}(f(\underline{X}_1), g(\underline{X}_2)) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{Cov}(X_f(s), X_g(t))\, ds\, dt \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Big(E(X_f(s)X_g(t)) - EX_f(s)EX_g(t)\Big)\, ds\, dt \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Big\{P\big(f(\underline{X}_1) > s, g(\underline{X}_2) > t\big) - P\big(f(\underline{X}_1) > s\big)P\big(g(\underline{X}_2) > t\big)\Big\}\, ds\, dt \\ &\ge 0. \end{aligned}$$

So by (2.1), $\underline{X}$ is weakly associated.

A possible way relaxing the condition of weak association is to require that (2.2) holds for all $A$ and $B$ which belong to subcollections of the collections of all upper sets in $R^{(k)}$ and $R^{(n-k)}$, respectively. This will be the approach in this paper. Let $\mathcal{A}^{(k)}$ and $\mathcal{A}^{(n-k)}$ be collections of sets in $R^{(k)}$ and $R^{(n-k)}$, respectively.

**Definition 2.3.** A random vector $\underline{X} = (X_1, \ldots, X_n)$ is weakly positive dependent relative to $\mathcal{A}^{(k)}$ and $\mathcal{A}^{(n-k)}$(denoted WPD($\mathcal{A}^{(k)}, \mathcal{A}^{(n-k)}$) if for a pair of partitions
$\underline{X}_1 = (X_{\pi(1)}, \ldots, X_{\pi(k)})$, $\underline{X}_2 = (X_{\pi(k+1)}, \ldots, X_{\pi(n)})$ of $\underline{X}$

$$P(\underline{X}_1 \in A, \underline{X}_2 \in B) \geq P(\underline{X}_1 \in A)P(\underline{X}_2 \in B) \tag{2.3}$$

whenever $A \in \mathcal{A}^{(k)}$ and $B \in \mathcal{A}^{(n-k)}$ and $\pi$ is any permutation of $\{1, 2, \ldots, n\}$ and a random vector $\underline{X}$ is weakly positive dependent relative to $\mathcal{A}^{(n)}$(denoted by WPD($\mathcal{A}^{(n)}$) if (2.3) holds for every $k(1 \leq k \leq n-1)$.

The following general propositions similar to those of Shaked([8]) are obtained by considering every pair of partitions $\underline{X}_1$ and $\underline{X}_2$ of $\underline{X}$.

**Proposition 2.4.** If, for fixed $k$, $\mathcal{A}^{(k)} \subset \tilde{\mathcal{A}}^{(k)}$ and $\mathcal{A}^{(n-k)} \subset \tilde{\mathcal{A}}^{(n-k)}$ then WPD($\tilde{\mathcal{A}}^{(k)}, \tilde{\mathcal{A}}^{(n-k)}$) $\Longrightarrow$ WPD($\mathcal{A}^{(k)}, \mathcal{A}^{(n-k)}$) and if for every $k(1 \leq k \leq n)$, $\mathcal{A}^{(k)} \subset \tilde{\mathcal{A}}^{(k)}$ then WPD($\tilde{\mathcal{A}}^{(n)}$) $\Longrightarrow$ WPD($\mathcal{A}^{(n)}$).
Put $\overline{\mathcal{A}}^{(k)} = \{\overline{A} : A \in \mathcal{A}^{(k)}\}$ ($\overline{A}$ denotes the commplement of $A$ in $R^{(k)}$) and $-\mathcal{A}^{(k)} = \{A : -A \in \mathcal{A}^{(k)}\}$ ($-A$ denotes $\{\underline{X} : -\underline{X} \in A\}$).

**Proposition 2.5.** The random vector $\underline{X}$ is WPD($\mathcal{A}^{(k)}, \mathcal{A}^{(n-k)}$) if and only if $\underline{X}$ is WPD($\overline{\mathcal{A}}^{(k)}, \overline{\mathcal{A}}^{(n-k)}$) and the random vector $\underline{X}$ is WPD ($\mathcal{A}^{(n)}$) if and only if $\underline{X}$ is WPD ($\overline{\mathcal{A}}^{(n)}$).

**Proof.** For fixed $k$, assume that $\underline{X}$ is WPD($\mathcal{A}^{(k)}, \mathcal{A}^{(n-k)}$). Let $\pi$ be any permutation of $\{1, 2, \ldots, n\}$ and $\underline{X}_1 = (X_{\pi(1)}, \ldots, X_{\pi(k)})$, $\underline{X}_2 = (X_{\pi(k+1)}, \ldots, X_{\pi(n)})$ be any pair of partitions of $\underline{X}$. Then for every $A \in \mathcal{A}^{(k)}$ and $B \in \mathcal{A}^{(n-k)}$

$$\begin{aligned} & P(\underline{X}_1 \in A, \underline{X}_2 \in B) \\ \geq \; & P(\underline{X}_1 \in A)P(\underline{X}_2 \in B) \end{aligned}$$

$$\begin{aligned} &= (1 - P(\underline{X}_1 \in \overline{A}))(1 - P(\underline{X}_2 \in \overline{B})) \qquad (2.4) \\ &= 1 - P(\underline{X}_1 \in \overline{A}) - P(\underline{X}_2 \in \overline{B}) + \underline{P}(\underline{X}_1 \in \overline{A})P(\underline{X}_2 \in \overline{B}). \end{aligned}$$

Since, in general,

$$P(\underline{X}_1 \in A, \underline{X}_2 \in B) = 1 - P(\underline{X}_1 \in \overline{A}) - P(\underline{X}_2 \in \overline{B}) + P(\underline{X}_1 \in \overline{A}, \underline{X}_2 \in \overline{B})$$

it follows from (2.4) that

$$P(\underline{X}_1 \in \overline{A}, \underline{X}_2 \in \overline{B}) \geq P(\underline{X}_1 \in \overline{A})P(\underline{X}_2 \in \overline{B}).$$

The converse is proved in the same way as above.

Next, assume that $\underline{X}$ is WPD($\mathcal{A}^{(n)}$). Then (2.4) holds for every positive integer $k(1 \leq k \leq n-1)$ that is, if $\underline{X}$ is WPD($\mathcal{A}^{(n)}$) then $\underline{X}$ is WPD($\overline{\mathcal{A}}^{(n)}$).

**Proposition 2.6.** The random vector $\underline{X}$ is WPD($\mathcal{A}^{(k)}, \mathcal{A}^{(n-k)}$) if and only if $-\underline{X}$ is WPD($-\mathcal{A}^{(k)}, -\mathcal{A}^{(n-k)}$) and $\underline{X}$ is WPD($\mathcal{A}^{(n)}$) if and only if $-\underline{X}$ is WPD($-\mathcal{A}^{(n)}$).

## 3. CONCEPTS OF WEAK POSITIVE DEPENDENCE

Various weaker concepts than weak association will be considered in this section. Many of these can be viewed by as variations of the class from which upper sets $A, B$ are chosed and (2.2) imposed.

**Theorem 3.1.**

$$\begin{array}{ccccc} & & \text{WA} = \text{WPD}(\mathcal{A}_5^{(n)}) & & \\ & \swarrow & & \searrow & \\ & \text{WPD}(\mathcal{A}_4^{(n)}) & & \text{WPD}(\mathcal{A}_3^{(n)}) & \\ \swarrow & & \searrow \quad \swarrow & & \\ \text{WPD}(\mathcal{A}_2^{(n)}) & & \text{WPD}(\mathcal{A}_1^{(n)}) \Longrightarrow \text{PUOD} & & \end{array}$$

(See Section 3 of Shaked[8] for the notions of $\mathcal{A}_j^{(n)}$ $j = 1, 2, 3, 4, 5$, collections of upper sets in $R^n$.)

**Remark 3.2.** (1) From Proposition 2.2 it follows that random vector $\underline{X}$ is WA if and only if $\underline{X}$ is WPD($\mathcal{A}_5^{(n)}$).
(2) From Proposition 2.4 it follows that the relationships of WPD($\mathcal{A}_j^{(n)}$) for $j = 1, \ldots, 5$ are obtained.
(3) Note that it will be seen later(Theorem 3.4) that WPD($\mathcal{A}_1^{(n)}$) $\Longrightarrow$ PUOD, thus, for $j = 1, 3, 4, 5$, the WPD($\mathcal{A}_j^{(n)}$) are weaker than, weak association and stronger than the orthant dependence.

Some of the results of Section 2 can be specialized now to the notions of this section as follows. Since for every positive integer $k$, $\overline{\mathcal{A}}_j^{(k)} = -\mathcal{A}_j^{(k)}, j = 2, 3, 5$, we obtain the following theorem from the fact that if $\overline{\mathcal{A}}^{(k)} = -\mathcal{A}^{(k)}$ for every $k(1 \leq k \leq n)$ then $\underline{X}$ is WPD($\mathcal{A}^{(n)}$) if and only if $-\underline{X}$ is WPD($\mathcal{A}^{(n)}$).

**Theorem 3.3.** For $j = 2, 3, 5$, $\underline{X}$ is WPD ($\mathcal{A}_j^{(n)}$) if and only if $-\underline{X}$ is WPD ($\mathcal{A}_j^{(n)}$).

**Theorem 3.4.** (a) If $\underline{X}$ is WPD($\mathcal{A}_j^{(n)}$) then $\underline{X}$ is PUOD, $j = 1, 3, 4, 5$.
(b) If $\underline{X}$ is WPD($-\mathcal{A}_j^{(n)}$) then $\underline{X}$ is PLOD, $j = 1, 3, 4, 5$.

**Proof.** By Theorem 3.1 it is enough to prove (a) for $j = 1$. Let $\underline{X} = (X_1, \ldots, X_n)$. If $\underline{X}$ is WPD($\mathcal{A}_1^{(n)}$) then $\underline{X}$ is WPD($\mathcal{A}_1^{(k)}, \mathcal{A}_1^{(n-k)}$) for every $k(1 \leq k \leq n-1)$. When $k = 1$ take $\underline{X}_1 = (X_1)$ and $\underline{X}_2 = (X_2, \ldots, X_n)$ as a partition of $\underline{X}$ and take $\underline{\mathbf{a}}_1 = (a_1)$ and $\underline{\mathbf{a}}_2 = (a_2, \ldots, a_n)$ as a partition of $\underline{\mathbf{a}}$, respectively, then

$$\begin{aligned} P(X_1 > a_1, \ldots, X_n > a_n) &= P(\underline{X} > \underline{\mathbf{a}}) \\ &= P(\underline{X}_1 > \underline{\mathbf{a}}_1, \underline{X}_2 > \underline{\mathbf{a}}_2) \\ &\geq P(\underline{X}_1 > \underline{\mathbf{a}}_1) P(\underline{X}_2 > \underline{\mathbf{a}}_2) \\ &= P(X_1 > a_1) P(X_2 > a_2, \ldots, X_n > a_n). \end{aligned}$$

When $k = 2$ take $\underline{X}_1 = (X_1, X_2)$ and $\underline{X}_2 = (X_3, \ldots, X_n)$ as a partition of $\underline{X}_1$

and take $\underline{\mathbf{a}}_1 = (a_1, a_2)$ and $\underline{\mathbf{a}}_2 = (a_3, \ldots, a_n)$ as a partition of $\underline{\mathbf{a}}$ then

$$\begin{aligned} P(X_1 > a_1, \ldots, X_n > a_n) &= P(\underline{X}_1 > \underline{\mathbf{a}}_1, \underline{X}_2 > \underline{\mathbf{a}}_2) \\ &\geq P(\underline{X}_1 > \underline{\mathbf{a}}_1) P(\underline{X}_2 > \underline{\mathbf{a}}_2) \qquad (3.1) \\ &= P(X_1 > a_1, X_2 > a_2) P(X_3 > a_3, \ldots, X_n > a_n). \end{aligned}$$

By choosing $a_1 = -\infty$ in (3.1) we obtain

$$P(X_2 > a_2, \ldots, X_n > a_n) \leq P(X_2 > a_2) P(X_3 > a_3, \ldots, X_n > a_n).$$

We proceed by induction and finally, when $k = n-1$ take $\underline{X}_1 = (X_1, \ldots, X_{n-1})$ and $\underline{X}_2 = (X_n)$ as a partition of $\underline{X}$ and take $\underline{\mathbf{a}}_1 = (a_1, \ldots, a_{n-1})$ and $\underline{\mathbf{a}}_2 = (a_n)$ as a partition of $\underline{\mathbf{a}}$, then

$$\begin{aligned} P(X_1 > a_1, \ldots, X_n > a_n) &= P(\underline{X}_1 > \underline{\mathbf{a}}_1, \underline{X}_2 > \underline{\mathbf{a}}_2) \\ &\geq P(\underline{X}_1 > \underline{\mathbf{a}}_1) P(\underline{X}_2 > \underline{\mathbf{a}}_2) \qquad (3.2) \\ &= P(X_1 > a_1, \ldots, X_{n-1} > a_{n-1}) P(X_n > a_n). \end{aligned}$$

By choosing $a_1 = -\infty, \ldots, a_{n-2} = -\infty$ in (3.2) we obtain

$$P(X_{n-1} > a_{n-1}, X_n > a_n) \geq P(X_{n-1} > a_{n-1}) P(X_n > a_n).$$

Finally

$$P(X_1 > a_1, \ldots, X_n > a_n) \geq \prod_{i=1}^{n} P(X_i > a_i).$$

Since for every $k, -\mathcal{A}_j^{(k)} \subset -\mathcal{A}_1^{(k)}$, $j = 3, 4, 5$, it is enough to prove (b) for $j = 1$:

$$\begin{aligned} \underline{X} \textit{ is } WPD(-\mathcal{A}_1^{(n)}) &\Longrightarrow -\underline{X} \text{ is WPD}(\mathcal{A}_1^{(n)}) \\ &\Longrightarrow -\underline{X} \text{ is PUOD} \\ &\Longrightarrow \underline{X} \text{ is PLOD.} \end{aligned}$$

**Theorem 3.5.** For $j = 3, 4, 5$, (a) If $\underline{X}$ is WPD($\mathcal{A}_j^{(n)}$) then $\underline{X}$ is PLOD. (b) If $\underline{X}$ is WPD($-\mathcal{A}_j^{(n)}$) then $\underline{X}$ is PUOD.

**Proof.** (a). By Proposition 2.5, if $\underline{X}$ is WPD($\mathcal{A}_j^{(n)}$) then $\underline{X}$ is WPD($\overline{\mathcal{A}}_j^{(n)}$), $j = 3, 4, 5$. It is easy to see that for every $k$, $\overline{\mathcal{A}}_j^{(k)} \supset -\mathcal{A}_1^{(k)}$, for $j = 3, 4, 5$, thus, if $\underline{X}$ is WPD($\mathcal{A}_j^{(n)}$) then $\underline{X}$ is WPD($-\mathcal{A}_1^{(n)}$). Hence, by part (b) of Theorem 3.4, $\underline{X}$ is PLOD.
(b). The proof is similar to that of (a).

# 4. CONCEPTS OF FUNCTIONAL WEAK POSITIVE DEPENDENCE

In many instances, if $\underline{X}$ is WPD($\mathcal{A}^{(k)}, \mathcal{A}^{(n-k)}$) then there exist a family of real $k$-variate functions $\mathcal{F}^{(k)}$ and a family of real $(n-k)$ variate functions $\mathcal{F}^{(n-k)}$ such that for any partitions $\underline{X}_1 = (X_{\pi(1)}, \ldots, X_{\pi(k)})$ and $\underline{X}_2 = (X_{\pi(k+1)}, \ldots, X_{\pi(n)})$ of $\underline{X} = (X_1, \ldots, X_n)$

$$\mathrm{Cov}\Big(f(\underline{X}_1), g(\underline{X}_2)\Big) \geq 0 \tag{4.1}$$

whenever $f \in \mathcal{F}^{(k)}$, $g \in \mathcal{F}^{(n-k)}$ and $\pi$ is a permutation of $\{1, \ldots, n\}$, provided the expectations exist.

**Definition 4.1.** If the random vector $\underline{X}$ satisfies (4.1) we say that $\underline{X}$ is functionally weak positive dependent relative to $\mathcal{F}^{(k)}$ and $\mathcal{F}^{(n-k)}$ (denoted by FWPD($\mathcal{F}^{(k)}, \mathcal{F}^{(n-k)}$)) and if $\underline{X}$ satisfies (4.1) for every $k(1 \leq k \leq n-1)$ then $\underline{X}$ is functionally weak positive dependent relative to $\mathcal{F}^{(n)}$ (denoted by FWPD($\mathcal{F}^{(n)}$)).

The following concepts of functional weak positive dependence are obtained by using the arguments similar to those of Shaked [8].

**Proposition 4.2.** If $\mathcal{F}^{(k)} \subset \tilde{\mathcal{F}}^{(k)}$ and $\mathcal{F}^{(n-k)} \subset \tilde{\mathcal{F}}^{(n-k)}$ then FWPD($\mathcal{F}^{(k)}$, $\mathcal{F}^{(n-k)}$) $\Longrightarrow$ FWPD ($\tilde{\mathcal{F}}^{(k)}, \tilde{\mathcal{F}}^{(n-k)}$) and if for every $k(1 \leq k \leq n)$ $\tilde{\mathcal{F}}^{(k)} \subset \tilde{\mathcal{F}}^{(k)}$ then FWPD($\mathcal{F}^{(n)}$) $\Longrightarrow$ FWPD($\tilde{\mathcal{F}}^{(n)}$).

From Proposition 4.2 and Definition 4.1 we obtain the following theorem.

**Theorem 4.3.**

$$\mathrm{FWPD}(\mathcal{F}^{(n)}) = \mathrm{WPD}(\mathcal{A}_5^{(n)})$$

$$\swarrow \qquad \searrow$$

$$\mathrm{FWPD}(\mathcal{F}_4^{(n)}) \quad \mathrm{FWPD}(\mathcal{F}_3^{(n)})$$

$$\swarrow \qquad \searrow \qquad \swarrow$$

$$\mathrm{FWPD}(\mathcal{F}_2^{(n)}) \quad \mathrm{FWPD}(\mathcal{F}_1^{(n)})$$

(See Section 4 of Shaked[8] for the notion of $\mathcal{F}_j^{(n)}$, $j = 1, \ldots, 5$, collections of increasing functions in $R^n$.)

We are going to show now that for $j = 1, \ldots, 5$, the notion of $\mathrm{WPD}(\mathcal{A}_j^{(n)})$ essentially implies the notion of $\mathrm{FWPD}(\mathcal{F}_j^{(n)})$. First the following lemma which characterizes $\mathrm{WPD}(\mathcal{A}_j^{(n)})$ is proven.

**Lemma 4.4.** For $j = 1, \ldots, 5$, $\underline{X}$ is $\mathrm{WPD}(\mathcal{A}_j^{(n)})$ if and only if for every partitions $\underline{X}_1 = (X_{\pi(1)}, \ldots, X_{\pi(k)})$ and $\underline{X}_2 = (X_{\pi(k+1)}, \ldots, X_{\pi(n)})$ of $\underline{X}$, and every $k (1 \le k \le n-1)$,

$$\big(f(\underline{X}_1), g(\underline{X}_2)\big) \quad \text{is PQD whenever } f \in \mathcal{F}_j^{(k)}, g \in \mathcal{F}_j^{(n-k)}, \tag{4.2}$$

provided $\underline{X}$ is nonnegative. If $\underline{X}$ is not nonnegative then the above equivalence is true for $j = 2, 4, 5$.

**Proof.** When $j = 1$ and $P(\underline{X} \ge \underline{0}) = 1$ then the $\underline{X}'s$ in $A \in \mathcal{A}_1^{(n)}$ are nonnegative. It is easily seen that for every $k (1 \le k \le n-1)$ $A \in \mathcal{A}_1^{(k)}$, $B \in \mathcal{A}_1^{(n-k)}$ if and only if $A = \{\underline{x}_1 : \min_{1 \le j \le k} b_j x_{\pi(j)} > s\}$ for some $s \in [-\infty, \infty]$ and $b_j \ge 0$, $j = 1, \ldots, k$, and $B = \{\underline{x}_2 : \min_{k+1 \le j \le n} c_j x_{\pi(j)} > t\}$ for some $t \in [-\infty, \infty]$ and $c_j \ge 0$, $j = k+1, \ldots, n$. Then the result follows from the definition of $\mathrm{WPD}(\mathcal{A}_1^{(n)})$. When $j = 2$ then the result follows directly from the definition of $\mathrm{WPD}(\mathcal{A}_2^{(n)})$; see (3.2) of Shaked([8]).
When $j = 3$ and $P(\underline{X} \ge \underline{0}) = 1$ then, to construct sets in $\mathcal{A}_3^{(k)}$ and $\mathcal{A}_3^{(n-k)}$, we

can consider (3.3) or (3.4) in Shaked([8]) only the sets for every $k(1 \le k \le n-1)$

$$\left\{\underline{x}_1 : b_j x_{\pi(j)} > 1\right\} \quad \text{for some } j \in \{1, \dots, k\} \quad \text{and } b_j \in [0, \infty] \tag{4.3}$$

and

$$\left\{\underline{x}_2 : c_j x_{\pi(j)} > 1\right\} \quad \text{for some } j \in \{k+1, \dots, n\} \quad \text{and } c_j \in [0, \infty]. \tag{4.4}$$

It is easy to see that by taking unions and intersections of the form (4.3) and (4.4), respectively, we obtain sets of the form

$$\left\{\underline{x}_1 : f(\underline{x}_1) > 1\right\} \quad \text{for some } f \tag{4.5}$$

and

$$\left\{\underline{x}_2 : g(\underline{x}_2) > 1\right\} \quad \text{for some } g \tag{4.6}$$

of the form (4.1) or (4.2) in Shaked([8]) that is, for every $k(1 \le k \le n-1)$ $A \in \mathcal{A}_3^{(k)}, B \in \mathcal{A}_3^{(n-k)}$ if and only if $A$ is of the form (4.5) and B is of the (4.6). Using the homogeneity and nonegativity of (4.5) and (4.6) finally we observe for every $k(1 \le k \le n-1)$ $A \in \mathcal{A}_3^{(k)}$ and $B \in \mathcal{A}_3^{(n-k)}$ if and only if $A = \{\underline{x}_1 : f(\underline{x}_1) > b\}$ for some $f \in \mathcal{F}_3^{(k)}$ and some $b \in [0, \infty]$ and $B = \{\underline{x}_2 : g(\underline{x}_2) > c\}$ for some $g \in \mathcal{F}_3^{(n-k)}$ and $c \in [0, \infty]$. The result then follows from the definition of WPD($\mathcal{A}_3^{(n)}$).

When $j = 5$ we deal with weakly associated random variables. It is easy to see that (4.2) and (2.1) are equivalent and this proves the result. Finally let $j = 4$. First assume $\underline{X}$ satisfies (4.2). Let $\pi$ be any permutation of $\{1, 2, \dots, n\}$, $\underline{X}_1 = (X_{\pi(1)}, \dots, X_{\pi(k)})$ and $\underline{X}_2 = (X_{\pi(k+1)}, \dots, X_{\pi(n)})$ be arbitrary partitions of $\underline{X} = (X_1, \dots, X_n)$ and $A$ and $B$ be in $\mathcal{A}_4^{(k)}, \mathcal{A}_4^{(n-k)}$ respectively. Since $A$ and $B$ are open, convex, and upper sets they can be approximated by intersections of sets of the form

$$\left\{\underline{x}_1 : \sum_{i=1}^{k} a_i x_{\pi(i)} > 1\right\}, \quad \text{where } a_i \ge 0, \quad i = 1, \dots, k,$$

and

$$\left\{\underline{x}_2 : \sum_{i=k+1}^{n} b_i x_{\pi(i)} > 1\right\}, \quad \text{where } b_i \ge 0, \quad i = k+1, \dots, n,$$

respectively. Explicitly, for every $\epsilon > 0$ there exist $K_1$ and $K_2$ such that

$$\left|P\Big(\underline{X}_1 \in A\Big) - P\Big(\min_{1\leq l\leq K_1} \sum_{i=1}^{k} a_i^{(l)} X_{\pi(i)} > 1\Big)\right| < \epsilon,$$

where $a_i^{(l)} \geq 0$, $i = 1, \ldots, k$; $l = 1, \ldots, K_1$ and

$$\left|P\Big(\underline{X}_2 \in B\Big) - P\Big(\min_{1\leq r\leq K_2} \sum_{i=k+1}^{n} b_i^{(r)} X_{\pi(i)} > 1\Big)\right| < \epsilon,$$

where $b_i^{(r)} \geq 0$, $j = k+1, \ldots, n$; $r = 1, \ldots, K_2$.
Denoting $f_{K_1}(\underline{x}_1) = \min_{1\leq l\leq K_1} \sum_{i=1}^{k} a_i^{(l)} x_{\pi(i)}$ and
$g_{K_2}(\underline{x}_2) = \min_{1\leq r\leq K_2} \sum_{i=k+1}^{n} b_i^{(r)} x_{\pi(i)}$ we can also assume that

$$\Big|P(\underline{X}_1 \in A, \underline{X}_2 \in B) - P(f_{K_1}(\underline{X}_1) > 1, g_{K_2}(\underline{X}_2) > 1)\Big| < \epsilon.$$

It is easy to check that for every $k (1 \leq k \leq n-1)$ $f_{K_1} \in \mathcal{F}_4^{(k)}$ and $g_{K_2} \in \mathcal{F}_4^{(n-k)}$, thus, it follows from (4.2) that

$$\begin{aligned} P(\underline{X}_1 \in A, \underline{X}_2 \in B) + \epsilon &\geq P(f_{K_1}(\underline{X}_1) > 1, g_{K_2}(\underline{X}_2) > 1) \\ &\geq P(f_{K_1}(\underline{X}_1) > 1) P(g_{K_2}(\underline{X}_2) > 1) \\ &\geq \Big[P(\underline{X} \in A) - \epsilon\Big]\Big[P(\underline{X}_2 \in B) - \epsilon\Big] \end{aligned}$$

Letting $\epsilon \to 0$ we obtain

$$P(\underline{X}_1 \in A, \underline{X}_2 \in B) \geq P(\underline{X}_1 \in A) P(\underline{X}_2 \in B),$$

that is, $\underline{X}$ is WPD($\mathcal{A}^{(n)}$).
To show the converse assume that $X$ is WPD($\mathcal{A}_4^{(n)}$). Let $f \in \mathcal{F}_4^{(k)}$ and $g \in \mathcal{F}_4^{(n-k)}$ for every $k (1 \leq k \leq n-1)$. Then for every $a$ and $b$ the sets $A = \{\underline{x}_1 : f(\underline{x}_1) > a\}$ and $B = \{\underline{x}_2 : g(\underline{x}_2) > b\}$ are in $\mathcal{A}_4^{(k)}, \mathcal{A}_4^{(n-k)}$, respectively.
Thus, since $\underline{X}$ is WPD($\mathcal{A}_4^{(n)}$).

$$\begin{aligned} P(f(\underline{X}_1) > a, g(\underline{X}_2) > b) &= P(\underline{X}_1 \in A, \underline{X}_2 \in B) \\ &\geq P(\underline{X}_1 \in A) P(\underline{X}_2 \in B) \\ &= P(f(\underline{X}_1) > a) P(g(\underline{X}_2) > b), \end{aligned}$$

that is, (4.2) holds.

**Theorem 4.5.** For $j = 1, \ldots, 5$, if the random vector $\underline{X}$ is WPD($\mathcal{A}_j^{(n)}$) and nonnegative then $\underline{X}$ is FWPD($\mathcal{F}_j^{(n)}$). If it is not assumed that $\underline{X}$ is nonnegative the above is true for $j = 2, 4, 5$.

**Proof.** If $\underline{X}$ is WPD($\mathcal{A}_j^{(n)}$) then, by (4.2), $\mathrm{Cov}(f(\underline{X}_1), g(\underline{X}_2)) \geq 0$ whenever the expectations exist and $f \in \mathcal{F}_4^{(k)}, g \in \mathcal{F}_4^{(n-k)}$ for every $k (1 \leq k \leq n-1)$, that is, $\underline{X}$ is FWPD($\mathcal{F}_j^{(n)}$).

**Theorem 4.6.** For $j = 1, 2, 5$, if $\underline{X} = (X_1, \ldots, X_n)$ and $\underline{Y} = (Y_1, \ldots, Y_n)$ are nonnegative independent random vectors which are WPD($\mathcal{A}_j^{(n)}$) and WPD ($\mathcal{A}_j^{(m)}$), respectively, then $(\underline{X}, \underline{Y})$ is WPD($\mathcal{A}_j^{(n+m)}$). Without the nonnegativity assumption, the above is true for $j = 2, 5$.

**Proof.** First note that since $\underline{X}$ and $\underline{Y}$ are independent $\mathrm{Cov}(f(\underline{X}), g(\underline{Y})) \geq 0$ for $f \in \mathcal{F}_j^{(n)}, g \in \mathcal{F}_j^{(n)}$, $j = 1, 2, 3, 4, 5$. Let $(\underline{X}_1, \underline{X}_2)$ and $(\underline{Y}_1, \underline{Y}_2)$ denote arbitrary partitions of $\underline{X}$ and $\underline{Y}$, respectively. Put $\underline{X}_1 = (X_{\alpha(1)}, \ldots, X_{\alpha(k)}), \underline{X}_2 = (X_{\alpha(k+1)}, \ldots, X_{\alpha(n)})$ $\underline{Y}_1 = (Y_{\beta(1)}, \ldots, Y_{\beta(r)})$ and $\underline{Y}_2 = (Y_{\beta(r+1)}, \ldots, Y_{\beta(m)})$, where $\alpha$ and $\beta$ are arbitrary permutations of $\{1, 2, \ldots, n\}$ and $\{1, \ldots, m\}$, respectively. Then $f(\underline{X}_1, \underline{y}_1)$ and $g(\underline{X}_2, \underline{y}_2)$ are increasing functions in $\mathcal{F}_j^{(k)}, \mathcal{F}_j^{(n-k)}$ respectively. Note that $E\{f(\underline{X}_1, \underline{Y}_1) \mid \underline{Y}_1\}$ is a measurable function, so that

$$E\Big\{f(\underline{X}_1, \underline{Y}_1) \mid \underline{Y}_1, \underline{Y}_2\Big\} = E\Big\{f(\underline{X}_1, \underline{Y}_1) \mid \underline{Y}_1\Big\} \tag{4.7}$$

almost surely. A similar result holds for $E\{g(\underline{X}_2, \underline{Y}_2) \mid \underline{Y}_2\}$.
Thus by (4.2) and (4.7),

$$\begin{aligned} E\Big\{f(\underline{X}_1, \underline{Y}_1) g(\underline{X}_2, \underline{Y}_2)\Big\} &= E\Big[E\Big\{f(\underline{X}_1, \underline{Y}_1) g(\underline{X}_2, \underline{Y}_2) \mid \underline{Y}_1, \underline{Y}_2\Big\}\Big] \\ &\geq E\Big[E\Big\{f(\underline{X}_1, \underline{Y}_1) \mid \underline{Y}_1\Big\} E\Big\{g(\underline{X}_2, \underline{Y}_2) \mid \underline{Y}_2\Big\}\Big]. \end{aligned}$$

Denote the conditional expectations $E\{f(\underline{X}_1, \underline{Y}_1) \mid \underline{Y}_1\}$, $E\{g(\underline{X}_2, \underline{Y}_2) \mid \underline{Y}_2\}$ by $h_1(\underline{Y}_1)$ and $h_2(\underline{Y}_2)$ respectively, and note that $h_1, h_2$ are increasing functions in $\mathcal{F}_j^{(r)}$ and $\mathcal{F}_j^{(m-r)}$ respectively. Thus again by (4.2),

$$E\Big[h_1(\underline{Y}_1) h_2(\underline{Y}_2)\Big] \geq E\Big\{h_1(\underline{Y}_1)\Big\} E\Big\{h_2(\underline{Y}_2)\Big\}$$

$$= E\{f(\underline{X}_1, \underline{Y}_1)\}E\{g(\underline{X}_2, \underline{Y}_2)\}.$$

So, for every $k(1 \le k \le n-1)$ and $r(1 \le r \le m-1)$,

$$\mathrm{Cov}(f(\underline{X}_1, \underline{Y}_1), g(\underline{X}_2, \underline{Y}_2) \ge 0$$

whenever $f \in \mathcal{F}_j^{(k+r)}$, $g \in \mathcal{F}_j^{(n+m-k-r)}$, hence, by Lemma 4.4. $(\underline{X}, \underline{Y})$ is WPD$(\mathcal{A}_j^{(n+m)})$

## 5. EXAMPLES AND APPLICATION

### 5.1 Counterexamples.

First we present examples that when $n \ge 3$ neither WPD$(\mathcal{A}_1^{(3)})$ nor WPD$(\mathcal{A}_2^{(3)})$ implies the other.

**Example 5.1.** [WPD$(\mathcal{A}_1^{(3)}) \not\Longrightarrow$ WPD$(\mathcal{A}_2^{(3)})$ ]. Let $\underline{X} = (X_1, X_2, X_3)$ be the discrete random variables with joint probability $p(x_1, x_2, x_3) \equiv P[X_1 = x_1, X_2 = x_2, X_3 = x_3]$; $p(0,1,0) = p(0,2,0) = p(1,0,1) = p(1,1,0) = 1/14$, $p(0,2,1) = p(1,0,0) = 2/14$ and $p(0,0,0) = p(1,2,1) = 3/14$, hence $(X_1, X_2, X_3)$ is not WPD$(\mathcal{A}_2^{(3)})$ since $3/14 = P[X_1 > 0, X_2 + X_3 > 1] < P[X_1 > 0]P[X_2 + X_3 > 1] = (7/14)^2$ while a lengthy verification shows that it is in fact WPD$(\mathcal{A}_1^{(3)})$.

**Example 5.2.** [WPD$(\mathcal{A}_2^{(3)}) \not\Longrightarrow$ WPD$(\mathcal{A}_1^{(3)})$ ]. Let $\underline{X} = (X_1, X_2, X_3)$ have the following probability; $P(\underline{X} = (2,2,1)) = P(\underline{X} = (3,2,1)) = P(\underline{X} = (2,3,1)) = P(\underline{X} = (3,3,1)) = P(\underline{X} = (1,1,2)) = P(\underline{X} = (2,1,2)) = P(\underline{X} = (3,1,2)) = P(\underline{X} = (1,2,2)) = P(\underline{X} = (1,3,2)) = 1/17$, and $P(\underline{X} = (1,1,1)) = P(\underline{X} = (3,3,2)) = 4/17$ ; hence $X = (X_1, X_2, X_3)$ is not WPD$(\mathcal{A}_1^{(3)})$, $P[X_1 > 1, X_2 > 1, X_3 > 1] = 4/17 < P[X_1 > 1, X_2 > 1]P[X_3 > 1] = (8/17)(9/17)$ while a lengthy verification shows that it is in fact WPD$(\mathcal{A}_2^{(3)})$.

The following example shows that when $n \ge 3$ then PUOD $\not\Longrightarrow$ WPD$(\mathcal{A}_1^{(3)})$.

**Example 5.3.** [PUOD $\not\Longrightarrow$ WPD$(\mathcal{A}_1^{(3)})$ ]. Let $\underline{X} = (X_1, X_2, X_3)$ have the following probability ; $P(\underline{X} = (0,0,0)) = 0.16$ $P(\underline{X} = (1,1,0)) = P(\underline{X} =$

$(1,0,1)) = P(\underline{X} = (0,1,1)) = 0.18$ and $P(\underline{X} = (1,1,1)) = 0.30$. Lengthy verification shows that $\underline{X}$ it is PUOD. Let $\underline{X}_1 = (X_1)$ and $\underline{X}_2 = (X_2, X_3)$, $A = \{\underline{X}_1 : X_1 > 0.5\}$ and $B = \{\underline{X}_2 : X_2 > 0.5, X_3 > 0.5\}$. Then

$$P(\underline{X}_1 \in A, \underline{X}_2 \in B) = 0.3 < 0.168 = P(\underline{X}_1 \in A)P(\underline{X}_2 \in B),$$

Thus $\underline{X}$ is not WPD$(\mathcal{A}_1^{(3)})$.

### 5.2 Convergence in Distribution.

By combining Lemma 4.4 and the proof of Theorem 10 of Newman [7] it can be shown that $X_1, \ldots, X_n$ are WPD$(\mathcal{A}_2^{(n)})$ finite variance random variables with joint and marginal characteristic functions, $\phi$ and $\phi_i$, $i = 1, \ldots, n$; then

$$\left|\phi(r_1, \ldots, r_n) - \prod_{i=1}^{n} \phi_i(r_i)\right| \leq \frac{1}{2} \sum \sum_{1 \leq i \neq k \leq n} |r_i||r_k||\mathrm{Cov}(X_i, X_k)|,$$

and by using Theorem 3.1, we obtain that for $j = 2, 4, 5$, if $\{X_1, \ldots, X_n\}$ is WPD$(\mathcal{A}_j^{(n)})$ and if the $X_j'$s are uncorrelated then $X_1, \ldots, X_n$ are jointly independent. Moreover, if $\{X_1, X_2, \ldots, X_n\}$ is a strictly stationary sequence of WPD$(\mathcal{A}_2^{(n)})$, finite variance random variables and satisfies

$$\sigma^2 = \mathrm{Var}(X_1) + 2\sum_{i=2}^{\infty} \mathrm{Cov}(X_1, X_i) < \infty,$$

then $\{X_j : j \geq 1\}$ satisfies a central limit theorem.

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