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Nonparametric Selection Procedures and Their Efficiency Comparisons

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Abstract

We consider nonparametric procedures for the selection and ranking problems. Tukey's generalized lambda distribution is considered as the distribution for the score function because the distribution can approximate many well-known contionuous distributions. Also we compare these procedures in terms of efficiency,defined by the ratio of a probability of a correct selection divided by the expected selected subset size.

1. Introduction

Since the selection and ranking problems were introduced and formulated, many papers have been concernd with nonparametric selection procedures. In practice, there are many situations in which one cannot observe the complete samples because of lack of resources, such as time, budget, unexpected accidents, but one can at least observe ranks. Also this kind of difficulity occurs in lifetesting very frequently. Thus, to avoid some difficulties and also sensitivity to the assumptions on the underlying distributions of the parametric approaches, nonparametric approaches are frequently used. These can provide robustness against deviations from the assumptions about the underlying distributions.

Some nonparametric selection procedures in terms of quantiles were considered by Rizvi and Sobel (1967), Balow and Gupta (1969), among others. Also nonparametric subset selection procedures based on ranks were studied by Nagel (1970), McDonald (1969, 1972, 1973, 1975), Gupta and McDonald (1970), Hsu (1978, 1981), Gupta, Huang and Nagel (1979), Huang and Panchapakesan (1982), Gupta and Leu (1983a), Gupta and Liang (1984), Matsui (1984) among others.

Nagel (1970) and Gupta and McDonald (1970) proposed and studied some nonparametric subset selection procedures for the location and scale models which choose a subset including the best population among k populations. The latter authors considered locally optimal selection procedures based on some functions. But the optimal choice of the score function for these procedures has not been studied. Since the rank sum statistic is easy to deal with, many proposed nonparametric subset selection procedures are based on this statistic.

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In this paper we consider the problem of choosing the optimal score function for different procedures proposed by Nagel (1970) and Gupta and McDonald (1970). The Tukey's lambda family of distributions is considered as the distribution for the score function because this family of distributions can be used to approximate many theoretical (unimodal) continuous distributions.

In Section 2, the problem of selection and ranking with nonparametric subset selection procedures is formulated and notations and definitions including proposed procedures are given.

In Section 3, we evaluate those procedures and compute constants which are necessary to carry out the procedures. Also the score function which leads the procedures to be locally optimal in the neighborhood of some points is introduced and evaluated.

A Monte Carlo study for the optimal choice of the score function is carried out in Section 4. Some tables containing the results of simulations are provided.

2. Formulation of the problem

Let $\pi_1, \ldots, \pi_k$ be $k(\geq 2)$ independent populations and let X_i be an observable characteristic of $\pi_i, i = 1, 2, \ldots, k$, respectively. Assume that a random variable X_i follows a continuous distribution $F(\cdot|\theta_i)$, and that the family $\{F(\cdot|\theta)\}$ is stochastically increasing in θ. Here we assume that the θ_i are unknown location parameters.

Let $X_{ij}, j = 1, \ldots, n$ be n independent random observations from $\pi_i, i = 1, 2, \ldots, k$,. Let R_{ij} denote the rank of the observation X_{ij} in the pooled sample of kn observations. Define

$$nH_i = \sum_{j=1}^{n} a(R_{ij}), \quad i = 1,2, \ldots, k, \tag{2.1}$$

where $a(r)$ is a score function defined by

$$-\infty < a(r) = E(T(r)|G) < \infty,$$

where $T(1) \leq T(2) \leq \ldots \leq T(N)$ is an ordered sample of size $N=nk$ from a continuous distribution G. Let $\theta_{[1]} \leq \theta_{[2]} \leq \ldots \leq \theta_{[k]}$ be the ordered θ_i's . The population associated with $\theta_{[k]}$, *i.e.* $F(x \mid \theta_{[k]})$, is called the best. When several populations have the same largest value $\theta_{[k]}$, randomly one of them is tagged as the best.

Our goal is to select a subset which contains the best with the usual requirement on the probability of a correct selection (PCS), *i.e.*, for any procedure R,

$$\inf_{\underline{\theta} \in \Omega} P_{\underline{\theta}}(CS|R) \geq P^*, \tag{2.2}$$

where $\Omega = \{\underline{\theta} | \underline{\theta} = (\theta_1, \ldots, \theta_k), \ \underline{\theta} \in R^k\}$ is the parameter space. Gupta and McDonald (1970) proposed procedures $R_1(G)$ and $R_2(G)$, which choose a subset containing the best, and

which depend on the choice of G as follows:

$R_1(G)$: Select π_i if and only if $H_I \geq \max_j H_J - d$, $i = 1, 2, \ldots, k$,

and

$R_2(G)$: Select π_i if and only if $H_i \geq D$, $i = 1, 2, \ldots, k$,

where $d(\geq 0)$ and $D(-\infty < D < \infty)$ are chosen so as to meet the P^*-condition. Note that rules $R_1(G)$ and $R_2(G)$ are equivalent if $k=2$. Also the rule $R_2(G)$ may select an empty set. A usual choice of G is a uniform distribution which is appealing because of simplicity. Let $\pi_{(i)}$ be the population associated with $\theta_{[i]}$. It is easy to see that, for rules $R_1(G)$ and $R_2(G)$.

$$\Pr(CS|R_1(G)) = \Pr(H_{(k)} \geq \max_j H_{(j)} - d, \quad j = 1, \ldots, k-1) \tag{2.3}$$

and

$$\Pr(CS|R_2(G)) = \Pr(H_{(k)} \geq D), \tag{2.4}$$

where $H_{(i)}$ is the H_i associated with $\pi_{(i)}$, $i = 1, 2, \ldots, k$, respectively.

3. Comparison of the Procedures $R_1(G)$ and $R_2(G)$

In order to compare $R_1(G)$ and $R_2(G)$ for various choices of G, we need first the results relating to the infimum of the PCS and evaluation of necessary constants.

3.1 PCS for $R_1(G)$ and $R_2(G)$ and Evaluation of Associated Constants

We state below (without proof) the results reguarding the infimum of PCS for rules $R_1(G)$ and $R_2(G)$ obtained by Gupta and McDonald (1970).

Theorem 3.1 For procedures $R_1(G)$ and $R_2(G)$

$$\inf_{\underline{\theta} \in \Omega} P_{\underline{\theta}}(CS|R_j(G)) = \inf_{\underline{\theta} \in \Omega_k} P_{\underline{\theta}}(CS|R_j(G)), \quad j = 1, 2, \tag{3.1}$$

and further, for the procedures $R_2(G)$,

$$\inf_{\underline{\theta} \in \Omega} P_{\underline{\theta}}(CS|R_2(G)) = \inf_{\underline{\theta} \in \Omega_0} P_{\underline{\theta}}(CS|R_2(G)), \tag{3.2}$$

where $\Omega_k = \{\underline{\theta} \in \Omega \mid \theta_{[k-1]} = \theta_{[k]}\}$ and $\Omega_0 = \{\underline{\theta} \in \Omega \mid \theta_{[1]} = \ldots = \theta_{[k]}\}$

Remark : When $\underline{\theta} \in \Omega_0$, procedures $R_1(G)$ and $R_2(G)$ are distribution free in the sense that the distributions of the statistics $\max_{1 \le j \le k} H_J - H_i$ and H_i do not depend upon the underlying distribution $F(\cdot|\theta)$.

In general, the least favoriable configuration(LFC) of the rule $R_1(G)$ is unknown except for $k=2$; however, it is known (see Rizvi and Woodworth (1970)) that the LFC need not occur in Ω_0. In order to compare rules $R_1(G)$ and $R_2(G)$, for various choices of G, the constants d and D are chosen to yield approximately the same P^* when $\underline{\theta} \in \Omega_0$.

The ratio EFF(R) $\equiv P(CS|R)/E(S|R)$ is used to compare the rules, where $E(S|R)$ is the expected size of the subset selected. Now, taking G to be a symmetric lambda distribution with location parameter α, scale parameter β and shape parameter γ, for $\underline{\theta} \in \Omega_0$, we have the following:

$$\begin{aligned} a(r) &= E(T(r)|G) \\ &= \alpha + \frac{\Gamma(N+1)}{\beta\Gamma(r)\Gamma(N-r+1)} \left\{ \frac{\Gamma(r+\gamma)\Gamma(N-r+1) - \Gamma(r)\Gamma(N+\gamma-r+1)}{\Gamma(N+\gamma+1)} \right\} \end{aligned} \tag{3.3}$$

$$\sum_{r=1}^{N} a(r) = \alpha N, \tag{3.4}$$

and

$$\sum_{r=1}^{N} H_i = \alpha K. \tag{3.5}$$

Now, let $a(r) = \alpha + \xi_r$. When $N = 2m+1$, $m \ge 0$, we have from (3.3)

$$\xi_{2m+1} = -\xi_1, \ldots, \xi_{m+2} = -\xi_m, \xi_{m+1} = 0.$$

In this case, we obtain

$$E(H_i) = \alpha, \tag{3.6}$$

$$n^2 Var(H_i) = \frac{2N(k-1)}{k^2(N-1)} \sum_{j=m+2}^{N} \xi_j^2 \tag{3.7}$$

$$n^2 Cov(H_i, H_j) = -\frac{-2 \sum_{j=m+2}^{N} \xi_j^2}{k(N-1)} - \frac{\alpha^2 N(n-1)}{k}, \tag{3.8}$$

$$-\frac{1}{k-1} \le Cov(H_i, H_j) < 0. \tag{3.9}$$

On the other hand, when $N = 2m$, $m > 0$, we get

$$\xi_{2m} = -\xi_1, \ldots, \xi_{m+1} = -\xi_m.$$

Consequently, in this case also we obtain results (3.6) through (3.9) except that the summations in (3.7) and (3.8) will be from $m+1$ to N instead of $m+2$ to N. Gupta and

McDonald (1970) derived the exact distribution of $\max_{1\le j\le k} H_j - H_i$ for the case of $a(R_{ij}) = R_{ij}$ for $k = 3$ and $n = 2(1)5$. Also, for $a(R_{ij}) = R_{ij}$, H_i is the well-known Mann-Whitney U-statistic. But in general the distribution of $\max_{1\le j\le k} H_j - H_i$ is not known since it depends on G. However, with $a(r)$ definewd as in (3.3), for $k = 3$ and $d \ge 0$,

$$\Pr\{\max_{1\le j\le 3} H_j - H_i \le d\} = \Pr\{H_2 - H_1 \le d,\ H_3 - H_1 \le d\}$$

can be evaluated on the computer.

Without loss generality, one can assume that $\alpha = 0$. The values of $a(r)$, d-values for the procedure $R_1(G)$, and D-values for the rules $R_2(G)$, respectively, for $k = 3$, $n = 3,5$ and (β, γ) = (0.57735, 1.00000),(0.19745, 0.13491),(-0.0006589, -0.0003630),(-0.16857, -0.080199) can be evaluated easily and hence are omitted. These are available based upon request. The four choices of (β, γ) specified above correspond to the cases where the lambda distribution can be used to approximate uniform, normal, logistic and double exponential distributions, respectively, each with mean 0 and variance 1. Accordingly, these choices are denoted by U, N, L, and D, respectively.

Finally, we briefly discuss how approximate values of d and D can be obtained by using asymptotic theory.

Theorem 3.2 For $\underline{\theta} \in \Omega_0$ and for the rule $R_1(G)$,

$$P(CS|R_1(G)) = \int_{-\infty}^{\infty} \Phi^{k-1}\left(x + \frac{nd}{v}\right) d\Phi(x),$$

where $v^2 = Var(H_i) - C_v$, C_v is common covariance between H_i and H_j for $i \ne j$, and $\Phi(x)$ is the cdf of a standard normal distribution.

Proof. By checking Lindeberg's condition, one can show that $nH_i\sqrt{Var(H_i) - C_v}$ is asymptotically normally distributed. Hence the result follows.

The value of d satisfying

$$\int_{-\infty}^{\infty} \Phi^{k-1}\left(x + \frac{nd}{v}\right) d\Phi(x) = P^*$$

can be obtained from the tables of Gupta (1963), Gupta, Nagel and Panchapakesan (1969) or Gupta, Panchapakesan and Sohn (1985), who have tabulated $h = nd/\sqrt{2}v$. Similarly the following theorem holds for the rule $R_2(G)$, whose proof is analogous to that of Theorem 3.2 and hence is omitted.

Theorem 3.3 For $\underline{\theta} \in \Omega_0$ and $N = 2m+1$,

$$P(CS|R_2(G)) \simeq \Phi^k\left(\frac{D}{nw}\right),$$

where $w^2 = \frac{2(k-1)}{nk(kn-1)} \sum_{j=n+2}^{kn} \xi_i^2$.

From the above theorem, we have $D \simeq \Phi^{-1}(nwP^{*i/k})$.

3.2 Evaluation of Constants for $R_1(G)$ and $R_2(G)$ using scores $a_0^*(r)$

In this section, we use a score function $a_0^*(r)$ (to be defined later) in the rules $R_1(G)$ and $R_2(G)$ and evaluated the associated constants d and D. In order to define the scores $a_0^*(r)$, consider the density $d(x,\theta)$, $\theta \in \Theta$, on an interval containing the origin, satisfying the following regularity conditions.

(i) $d(x,\theta)$ is absolutely continuous in θ for almost every x:

(ii) the limit

$$\dot{d}(x,0) = \lim_{\theta \to 0} \frac{1}{\theta} [d(x,\theta) - d(x,0)]$$

exists for almost every x:

(iii) $\lim_{\theta \to 0} \int_{-\infty}^{\infty} |\dot{d}(x0)| dx = \int_{-\infty}^{\infty} |\dot{d}(x,0)| dx < \infty$ holds, with $\dot{d}(x,\theta)$ denoting the partial derivative with respect to θ.

Note that the existence of $\dot{d}(x,\theta)$ for almost every θ is insured at every point x such that $d(x,\theta)$ is absolutely continuous in θ. This, however, does not make the condition (ii) superfluous.

In deriving locally most powerful tests for equality of location Gupta, Huang and Nagel (1979) used the score function $a_0^*(r)$ defined by

$$a_0^*(r) = E\left\{ \frac{\dot{d}(X_N^{(r)}, 0)}{d(X_N^{(r)}, 0)} \right\}, \tag{3.10}$$

where $X_N^{(r)}$ denotes the r-th order statistic in a sample of size N from the distribution with density $d(x,0)$. For the location parameter case, $a_0^*(r)$ can be written as

$$a_0^*(r) = E\left\{ \frac{\dot{f}(F^{-1}(U_{(r)},0),0)}{f(F^{-1}(U_{(r)},0),0)} \right\}, \tag{3.11}$$

where $U_{(r)}$ denotes the r-th order statistic in a sample of size N from the uniform distribution.

Now, specifying $d(x,\theta)$ to be the symmetric lambda density with parameters α(location), β (scale) and γ(scale), we obtain

$$a_0^*(r) = \begin{cases} \int_0^1 N\binom{N-1}{r-1} \frac{\beta(\gamma-1)u^{\gamma-1}(1-u)^{N-r}(u^{\gamma}-(1-u)^{\gamma-2})}{\gamma^2(u^{\gamma-1}+(1-u)^{\gamma-1})^2} du, & \beta \geq 0 \\ \int_0^1 N\binom{N-1}{r-1} \frac{-\beta(\gamma-1)u^{\gamma-1}(1-u)^{N-r}(u^{\gamma-1}-(1-u)^{\gamma-2})}{\gamma^2(u^{\gamma-1}+(1-u)^{\gamma-1})^2} du, & \beta > 0 \end{cases}$$

For the same values of k, n and (β, γ), the value of $a_0^*(r)$, the constants d and D are computed and are available based upon request.

Remark: Nagel (1970) and Gupta, Huang and Nagel (1979) have derived locally optimal subset selection procedures. It follows from their results that the rules $R_2(G)$ is locally optimal in sense that the rule maximizes the PCS in a neighborhood of any $\underline{\theta} \in \Omega_0$ among all rules which satisfy $\inf_{\underline{\theta} \in \Omega_0} P(CS|R) = P^*$.

3.3 Comparisons of Procedures $R_1(G)$ and $R_2(G)$

As we have stated in Section 3.1, the procedures $R_1(G)$ and $R_2(G)$ are compared in terms of EFF(R), which is used as a measure of efficiency. A large value indicates high efficiency.

For a proper comparison of the two procedures, we should have the constants d and D such that the two procedures will have the PCS aprroximately equal to P^* for $\underline{\theta} \in \Omega_0$. In our Monte Carlo studies with k=3, this led to the choice of P^* =0.90, 0.95, 0.975, for n=3, and P^* =0.75, 0.90, 0.95, 0.975, for n=5.

Further, we considered normal, logistic, and double exponential distributions all with variance 1, as three possible choices of the underlying distributions. Let $\theta_1, \theta_2, \theta_3$ be the means of the tree populations π_1, π_2, π_3. We considered four different configurations of $\underline{\theta} = (\theta_1, \theta_2, \theta_3)$, namely,

I: $\underline{\theta} = (0, 0, 0.1)$ II: $\underline{\theta} = (0, 0, 0.5)$

III: $\underline{\theta} = (0, 0, 1)$, IV: $\underline{\theta} = (0, 0.5, 1.0)$.

For comparisons using the score function $a(r)$, we choice the four choices of the parameter (β, γ) of the lambda distribution,referred to by U, N, L, and D in Section 3.1. For comparisons using $a_0^*(r)$ the choice of (β, γ), denoted by UD, is made so that the lambda distribution can be used to approximate the underlying distributions with variance 1.

For each choice of the underlying distribution, random samples were generated by using the random number generator RVP, developed by Professor Rubin at Purdue University. Our results are based on 1000 simulations in the case of n=3 and 500 simulations in the case of n=5. Table I is reproduced for the cases where the underlying distributions are normal and

logistic distributions with the mean configuration II for $(n, P^*) = (3, 0.90)$; the patterns in the other case are similar. Besides comparing the efficiencies of the rules $R_1(G)$ and $R_2(G)$ under each choice of G, we are also interested in comparing the different choices of G for each rule. Based on the Monte Carlo study, our conclusions are summarized below.

(1) When the means are close to each other, no rule performs uniformly better than the other when the underlying distributions are normal or double exponential; however, as $P^* \to 1$, the rules $R_2(G)$ performs slightly better than the rules $R_1(G)$. With means close to each other, the situation changes when the underlying distributions are uniformly better than the rule $R_1(G)$.

(2) When the largest mean is sufficiently away from the next largest, the rule $R_1(G)$ generally performs better than the rule $R_2(G)$ no matter what the choice of G is. This behavior becomes more clear as n increases. Also, when P^* is close to 1, the difference in the performances of the two rules narrows down, even through $R_1(G)$ still is better.

(3) Generally, the rule $R_1(G)$ performs better than the rule $R_2(G)$ when the choices of G are the lambda distribution to be the uniform and the underlying distribution F (i.e., G is U or UD) both with variance 1.

(4) Considering the efficiency of the procedure $R_1(G)$, the best choice of G is the lambda distribution which approximates the uniform distribution with unit variance (*i.e.*, G is U).

(5) For the rules $R_2(G)$, the best choice of G is the lambda distribution approximating the underlying distribution with unit variance. This is all the more clear when the underlying distributions are normal or double exponential with their means close to each other.

This study indicates that the score function based on uniform distribution is optimal and robust against possible deviations from the underlying distributions. Also the score function which is a weighted sum of ranks turn out to be optimal for some procedures. Furthermore, it shows that the Gupta-type procedure is almost uniformly better than another available procedure. This is not the same conclusion as that in Gupta and McDonald (1970). The reason why these results are different is due to the lack of number of simulations in Gupta and McDonald (1970) for various underlying populations. Also it is due to the fact that they only use the rank sum statistics.

Considering all the findings of the study, the overall recommendations will be:

(1) When the means of the underlying distributions are expected to be close to each other, use either the rule $R_1(G)$ with U as the choice for G or the rule $R_2(G)$ with UD as the choice for G.

(2) When the largest mean is expected to be sufficiently away from the next largest, use the $R_1(G)$ with U as the choice for G.

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Table I. Comparisons of the Procedures $R_1(G)$ and $R_3(G)$ under the configuration $\underline{\theta}$ = (0,0,0.5) and P^*= 0.90

(a) n=3

Underlying Distribution	G	P(CS|R_1(G))	P(CS|R_2(G))	E(S|R_1(G))	E(S|R_2(G))	EFF(R_1(G))	EFF(R_2(G))
Normal	U	0.969 (0.005)	0.985 (0.004)	2.583 (0.019)	2.712 (0.014)	0.400 (0.005)	0.374 (0.003)
	N	0.971 (0.005)	0.975 (0.005)	2.607 (0.018)	2.658 (0.015)	0.394 (0.005)	0.378 (0.003)
	L	0.971 (0.005)	0.975 (0.005)	2.604 (0.018)	2.658 (0.015)	0.393 (0.005)	0.378 (0.003)
	D	0.971 (0.005)	0.975 (0.005)	2.604 (0.018)	2.658 (0.015)	0.393 (0.005)	0.378 (0.003)
	UD	0.971 (0.005)	0.973 (0.005)	2.607 (0.018)	2.627 (0.015)	0.394 (0.005)	0.382 (0.003)
Logistic	U	0.927 (0.008)	0.947 (0.007)	2.668 (0.018)	2.753 (0.014)	0.357 (0.005)	0.348 (0.003)
	N	0.937 (0.008)	0.947 (0.007)	2.704 (0.017)	2.726 (0.014)	0.355 (0.004)	0.353 (0.003)
	L	0.940 (0.008)	0.947 (0.007)	2.669 (0.017)	2.726 (0.014)	0.355 (0.004)	0.353 (0.003)
	D	0.940 (0.008)	0.947 (0.008)	2.669 (0.017)	2.726 (0.014)	0.355 (0.004)	0.753 (0.003)
	UD	0.927 (0.008)	0.937 (0.008)	2.668 (0.018)	2.719 (0.014)	0.357 (0.005)	0.349 (0.004)

U : Uniform distribution
N : Normal distribution
L : Logistic distribuiton
D : Double exponential distribution
UD :Tukey's generalized lambda

Table I (continued)

(b) n=5

Underlying Distribution	G	$P(CS\|R_1(G))$	$P(CS\|R_2(G))$	$E(S\|R_1(G))$	$E(S\|R_2(G))$	$EFF(R_1(G))$	$EFF(R_2(G))$
Normal	U	0.988 (0.005)	0.990 (0.004)	2.528 (0.029)	2.590 (0.022)	0.431 (0.008)	0.397 (0.004)
	N	0.984 (0.006)	0.986 (0.005)	2.534 (0.022)	2.584 (0.022)	0.427 (0.008)	0.396 (0.004)
	L	0.986 (0.005)	0.986 (0.005)	2.542 (0.022)	2.604 (0.022)	0.426 (0.008)	0.392 (0.004)
	D	0.986 (0.005)	0.988 (0.005)	2.546 (0.022)	2.612 (0.022)	0.426 (0.008)	0.392 (0.004)
	UD	0.984 (0.006)	0.986 (0.005)	2.532 (0.022)	2.586 (0.022)	0.425 (0.008)	0.396 (0.004)
Logistic	U	0.952 (0.010)	0.954 (0.009)	2.732 (0.023)	2.726 (0.020)	0.428 (0.008)	0.356 (0.005)
	N	0.952 (0.010)	0.948 (0.010)	2.726 (0.023)	2.716 (0.020)	0.359 (0.006)	0.355 (0.005)
	L	0.948 (0.010)	0.948 (0.010)	2.726 (0.023)	2.710 (0.020)	0.361 (0.006)	0.355 (0.005)
	D	0.948 (0.010)	0.950 (0.010)	2.734 (0.022)	2.710 (0.020)	0.357 (0.006)	0.356 (0.005)
	UD	0.956 (0.010)	0.952 (0.010)	2.732 (0.023)	2.712 (0.020)	0.359 (0.006)	0.357 (0.005)