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3차원 덕트유동에서의 변형 척도

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Deformation Measure in Three-Dimensional Duct Flows

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요 약

덕트유동에서, 재료입자의 변형은 그 입자의 초기위치와 방향에 의존한다. 이러한 변형과정을 이해하기 위하여, 변형구배텐서(deformation gradient tensor)를 덕트 전단면에 걸쳐 초기위치와 시간의 함수로 계산할 수 있어야 한다. 따라서 본 논문은 입자궤적 궤도(particle trajectory orbit)에 접하는 직교좌표계를 선택하여 3차원 덕트 유동에서의 변형구배텐서를 효과적으로 계산할 수 있는 간단한 방법을 제안한다. 이러한 특수 좌표계로부터 구해진 변형구배텐서가 덕트유동에서의 변형 척도를 이해하는데에 매우 중요함을 알 수 있었다.

Abstract－Deformation of materials in the duct flow depends on the initial position and the orientation of the material element. To understand the details of the deformation process, one has to evaluate the deformation gradient tensor as a function of time and the initial position over the entire cross section in the three-dimensional duct flow. Therefore, the present paper proposes a simple method to effectively calculate the deformation gradient tensor over the entire section in the three-dimensional duct flow in a cartesian coordinate system aligned with the particle trajectory orbit. Components of the deformation gradient tensor in such a special coordinate system are found to play an important role in understanding a deformation measure in duct flows.

Keywords: Duct flow, deformation measure, particle trajectory, screw extrusion, FEM analysis

1. Introduction

It is of great interest to enhance our understanding of the kinematics of deformation in the three-dimensional duct flow, which are steady and incompressible velocity fields composed of a recirculating two-dimensional cross-sectional flow and a unidirectional axial flow, both flows remaining unchanged along the axial direction [1].

One of typical industrial examples of duct flows [1] is the pressure and drag driven flow in a channel formed by a barrel and a screw in a single-screw extruder. Recently, we have studied the extrusion process in terms of the flow characterisitcs [2], the residence time distribution [3] and deformation characteristics [4, 5] including the effect of 3-D circulatory flow in a single-screw extruder. It may be noted that the stretching or deformation of materials in the duct flow depends on the initial position and the orientation of the material element. To understand the details of the deformation process, one has to evaluate the deformation gradient tensor as a function of time and the initial position over the entire cross section, which requires a tremendous computational time.

With this in mind, the present paper proposes a simple and convenient method to calculate the deformation gradient tensor over the entire cross section in the three-dimensional duct flow and finally discusses a representative deformation measure based on components of the deformation gradient tensor in a cartesian coordinate system along the particle trajectory.

2. Kinematic Considerations

Fig. 1 schematically represents a particle motion in a duct flow with a coordinate system defined, x_1 and x_2 coordinates lying in the cross-sectioned plane of the duct with x_3 coordiate chosen in the axial direction. Since the velocity field does not vary with the axial direction, a trajectory of a fluid particle, after one full circulation in the x_1-x_2 plane, will form a closed orbit which is actually the constant line of the stream function in the x_1-x_2 plane as shown in Fig. 1, where T_{cir}, the period of circulation, denotes the time it takes a fluid particle to circulate once the trajectory contour in the x_1-x_2 plane.

The motion of a fluid particle might be represented as the following autonomous continuous-time dynamical system:

$$\frac{dx}{dt}=\mathbf{v}(\mathbf{x}) \text{ with } \mathbf{x}(0)=\mathbf{X} \tag{1}$$

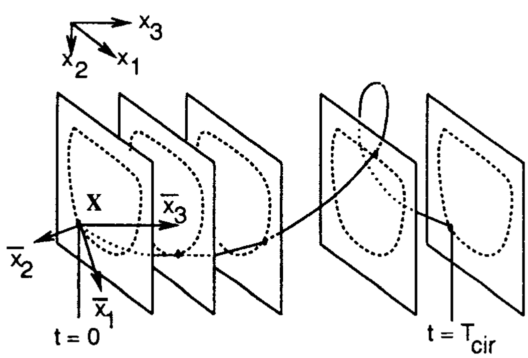

Fig. 1. Particle trajectory during one circulation in a duct flow with coordinate systems.

where x denotes a position vector in space occupied at time t by a particle which occupies $\mathbf{X}$ at time t=0. Then, the solution of this system can be represented as:

$$\mathbf{x}(t)=\phi(\mathbf{X},\ t), \text{ with } \mathbf{X}=\phi(\mathbf{X},\ 0) \tag{2}$$

The deformation gradient tensor, $\mathbf{F}(\mathbf{X}, t)$, can represent the deformation of a material initially located at $\mathbf{X}$ by relating the initial material vector, $\delta\mathbf{X}$, with the deformed material vector, $\delta\mathbf{x}(t)$, after time t as follows [6-8].

$$\delta\mathbf{x}(t)=\mathbf{F}(\mathbf{X},\ t)\cdot\delta\mathbf{X} \tag{3}$$

The deformation gradient tensor is governed by the following evolution equation:

$$\frac{d\mathbf{F}(\mathbf{X},\ t)}{dt}=\mathbf{L}(\phi,\ t))\cdot\mathbf{F}(\mathbf{X},\ t) \text{ with } \mathbf{F}(\mathbf{X},\ 0)=\mathbf{I} \tag{4}$$

where $\mathbf{L}$ is the velocity gradient tensor [4, 6]. By differentiating Eq. (3) with respect to time and applying Eqs. (3) and (4) into this result, the rate of change of $\delta\mathbf{x}$ can be obtained as follows:

$$\frac{d(\delta\mathbf{x})}{dt}=\mathbf{L}(\phi(\mathbf{X},\ t))\cdot\delta\mathbf{x} \tag{5}$$

To analyze the deformation process in duct flows, one has to first obtain the velocity field, which could be determined numerically, for instance via a finite element method [3]. Once the velocity field is known in duct flows, the particle trajectory, i.e. $\mathbf{x}(t)$, can be determined by numerically integrating Eq. (1) by means of the fourth-order Runge-Kutta method in general. During the Runge-Kutta integration of Eq. (1), one can also determine the period of circulation, T_{cir} [3]. Similarly, one may also integrate Eq. (4) to obtain the history of deformation gradient tensor by the Runge-Kutta method [4, 9].

Unfortunately, however, making use of Eq. (4) requires a numerical evaluation of the velocity gradient tensor from the velocity field. Thus numerically obtained velocity gradient field is worse in accuracy than the velocity field, especially when singular corners exist, for instance, in the extruder channel. Therefore, a numerical integration of Eq. (4) may lead to an inaccurate deformation gradient tensor.

In this regard, we derived new formulae to determine the deformation gradient tensor without having to numerically evaluate the velocity gradient so that the difficulty encountering in making use of Eq. (4) is eliminated. The derivation of the new formulae is described below.

3. Simple Method for Determining a Deformation Measure

We will adopt the following notations. In a cartesian coordinate system depicted in Fig. 1, the spatial velocity field is denoted by $\mathbf{v}(\mathbf{x})=(v_1, v_2, v_3)$. A particle initially located at $\mathbf{X}=(X_1, X_2, X_3)$ has a velocity denoted by $\mathbf{v}(\mathbf{X})=(V_1, V_2, V_3)$ at the initial position at time t equal to zero and after one circulation time $T_{cir}(\mathbf{X})$.

An integration of Eq. (4), following a fluid particle $\mathbf{X}$, from $t=0$ to a certain time t results in the following equation:

$$F_{ij}(\mathbf{X}, t)-F_{ij}(\mathbf{X}, 0)=\int_0^t \frac{\partial v_i}{\partial x_k}\frac{\partial x_k}{\partial X_j}dt=\int_0^t \frac{\partial v_i}{\partial X_j}dt \tag{6}$$

where the integrand should be evaluated following the particle. In particular, after one circulation, the above equation can be rewritten as;

$$F_{ij}(\mathbf{X}, T_{cir})-F_{ij}(\mathbf{X}, 0)=\int_0^{T_{cir}(\mathbf{X})} \frac{\partial v_i}{\partial X_j}dt \tag{7}$$

Noting that T_{cir} is a function of the intial position, $\mathbf{X}$, one can have the following equation from Leibnitz's rule:

$$\frac{\partial}{\partial X_j}\left(\oint v_i dt\right)=\int_0^{T_{cir}(\mathbf{X})} \frac{\partial v_i}{\partial X_j}dt+V_i\frac{\partial T_{cir}}{\partial X_j} \tag{8}$$

where $\oint \cdots dt$ denotes the integral over the closed trajectory orbit in the x_1-x_2 plane. Applying the above equation to Eq. (7) results in the following interesting formula for the deformation gradient tensor:

$$F_{ij}(\mathbf{X}, T_{cir})-F_{ij}(\mathbf{X}, 0)=\frac{\partial}{\partial X_j}\left(\oint v_i dt\right)-V_i\frac{\partial T_{cir}}{\partial X_j} \tag{9}$$

The above equation can be expressed explicitly for i=1, 2, 3, as follows:

$$F_{1j}(\mathbf{X}, T_{cir})-F_{1j}(\mathbf{X}, 0)=\frac{\partial}{\partial X_j}\left(\oint v_1 dt\right)-V_1\frac{\partial T_{cir}}{\partial X_j}=-V_1\frac{\partial T_{cir}}{\partial X_j} \tag{10}$$

$$F_{2j}(\mathbf{X}, T_{cir})-F_{2j}(\mathbf{X}, 0)=\frac{\partial}{\partial X_j}\left(\oint v_2 dt\right)-V_2\frac{\partial T_{cir}}{\partial X_j}=-V_2\frac{\partial T_{cir}}{\partial X_j} \tag{11}$$

$$F_{3j}(\mathbf{X}, T_{cir})-F_{3j}(\mathbf{X}, 0)=\frac{\partial}{\partial X_j}\left(\oint v_3 dt\right)-V_3\frac{\partial T_{cir}}{\partial X_j} \tag{12}$$

where $\oint v_1 dt=\oint v_2 dt=0$ are utilized. Since $\oint v_3 dt$ and T_{cir} are independent of x_3, the final matrix form for the deformation gradient tensor after one circulation can be expressed as:

$$\mathbf{F}(\mathbf{X}, T_{cir})=\begin{bmatrix} 1-V_1\frac{\partial T_{cir}}{\partial X_1} & -V_1\frac{\partial T_{cir}}{\partial X_2} & 0 \\ -V_2\frac{\partial T_{cir}}{\partial X_1} & 1-V_2\frac{\partial T_{cir}}{\partial X_2} & 0 \\ \frac{\partial}{\partial X_1}\left(\oint v_3 dt\right)-V_3\frac{\partial T_{cir}}{\partial X_1} & \frac{\partial}{\partial X_2}\left(\oint v_3 dt\right)-V_3\frac{\partial T_{cir}}{\partial X_2} & 1 \end{bmatrix} \tag{13}$$

One may introduce a deformation tensor, $\mathbf{M}$, defined by $\mathbf{F}(X, T_{cir})=(\mathbf{I}-\mathbf{M})$ [1]. Then, the tensor $\mathbf{M}$ can be expressed as:

$$\mathbf{M}=\begin{bmatrix} V_1\frac{\partial T_{cir}}{\partial X_1} & V_1\frac{\partial T_{cir}}{\partial X_2} & 0 \\ V_2\frac{\partial T_{cir}}{\partial X_1} & V_2\frac{\partial T_{cir}}{\partial X_2} & 0 \\ -\frac{\partial}{\partial X_1}\left(\oint v_3 dt\right)+V_3\frac{\partial T_{cir}}{\partial X_1} & -\frac{\partial}{\partial X_2}\left(\oint v_3 dt\right)+V_3\frac{\partial T_{cir}}{\partial X_2} & 0 \end{bmatrix} \tag{14}$$

After k-times the period of recirculation, k being any integer, the deformed material vector $\delta\mathbf{x}$ can be expressed with the help of Eq. (3) as:

$$\delta\mathbf{x}(kT_{cir})=(\mathbf{I}-\mathbf{M})^k\cdot\delta\mathbf{X} \tag{15}$$

It might be mentioned that Franjione and Ottino

[1], following a different argument from the present one, derived an equation similar to Eq. (14) but with some mistakes yielding a wrong expression for **M** (see Appendix A). Even with such mistakes, their conclusion that **M** is nilpotent order 2 remains unchanged; in other words, $\mathbf{M}^n=0$ for $n\geq 2$ [1]. $\mathbf{M}^2=0$ can be easily shown because $\mathbf{v}(\mathbf{X})\cdot\nabla\mathbf{x}T_{cir}=0$ and $\mathbf{v}(\mathbf{X})\cdot\nabla\mathbf{x}(\oint v_3dt)=0$ (see [1] for the proof), $\nabla\mathbf{x}$ denoting gradients with respect to the initial position **X**.
Therefore, Eq. (15) reduces to:

$$\delta\mathbf{x}(kT_{cir})=(\mathbf{I}-k\mathbf{M})\cdot\delta\mathbf{X} \tag{16}$$

which explains that the lineal stretch in duct flows increases linearly with time, or number of recirculations [1].

As far as the deformation measure over the entire cross-section of the duct is concerned, Eq. (13) can be utilized to determine components of the deformation gradient tensor in the global cartesian coordinate system (x_1, x_2, x_3) for a material initially located at **X** after one circulation in the x_1-x_2 plane. In using Eq. (13), one has to evaluate T_{cir} and $\oint v_3dt$ as a function of X_1 and X_2. Those deformation gradient tensor components thus obtained are not in a convenient form to represent the deformation measure. In this regard, we want to introduce a special cartesian coordiante system in which physically more meaningful components of the deformation gradient tensor can be defined, as explained below.

In order to consider the deformation gradient tensor for a material initially located at **X** after one period of recirculation, let us introduce a cartesian coordiate system, ($\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$) such that $\bar{x}_1$ and $\bar{x}_2$ are tangential and normal to the closed orbit at **X**, respectively, and $\bar{x}_3$ is along the axial direction, as depiced in Fig. 1. From now on, the bar system represents components of tensors in this coordinate system for convenience. In the bar coordinate system, the velocity components at the initial location **X** can be expressed as $\mathbf{v}(\mathbf{X})=(\bar{V}_1, 0, \bar{V}_3)$. It may be noted that T_{cir} and $\oint\bar{v}_3dt$ vary with the closed trajectory orbits. Therefore, it is obvious that $\frac{\partial T_{cir}}{\partial\bar{X}_1}=0$ and $\frac{\partial}{\partial\bar{X}_1}(\oint\bar{v}_3dt)=0$ at the initial **X**. Then it follows that $\bar{F}_{11}=\bar{F}_{22}=1$, $\bar{F}_{21}=\bar{F}_{31}=0$ according to Eq. (13) (see Appendix B for an alternative derivation of this result). Therefore, in the bar coordinate system, the deformation gradient tensor **F** after one circulation, expressed in Eq. (13), reduces to

$$\mathbf{F}(\mathbf{X}, T_{cir})=\mathbf{I}-\mathbf{M}=\mathbf{I}+\begin{bmatrix} 0 & \bar{F}_{12} & 0 \\ 0 & 0 & 0 \\ 0 & \bar{F}_{32} & 0 \end{bmatrix} \tag{17}$$

where $\bar{F}_{12}$ and $\bar{F}_{32}$ can be obtained from

$$\bar{F}_{12}(\mathbf{X}, T_{cir})=-\bar{V}_1\frac{\partial T_{cir}}{\partial\bar{X}_2},$$

$$\bar{F}_{32}(\mathbf{X}, T_{cir})=-\bar{V}_3\frac{\partial T_{cir}}{\partial\bar{X}_2}+\frac{\partial}{\partial\bar{X}_2}\left(\oint\bar{v}_3dt\right) \tag{18}$$

From Eq. (16), $\bar{F}_{12}$ and $\bar{F}_{32}$ after k-times the period of recirculation can be expressed as:

$$\bar{F}_{12}(\mathbf{X}, kT_{cir})=k\bar{F}_{12}(\mathbf{X}, T_{cir}),$$
$$\bar{F}_{32}(\mathbf{X}, kT_{cir})=k\bar{F}_{32}(\mathbf{X}, T_{cir}) \tag{19}$$

It may be noted that the physical significance of $\bar{F}_{12}$ and $\bar{F}_{32}$ is obviously the shear strain component in the normal direction to the closed orbit and that along the axial direction, respectively. Thus, we propose $\sqrt{\bar{F}_{12}^2+\bar{F}_{32}^2}$ and a deformation measure to characterize the deformation process at the particular position **X** [4].

Now, it becomes of our interest to efficiently evaluate the deformation gradient tensor **F** in the bar coordinate system as a function of the initial starting point, **X**. Fig. 2 schematically shows a particle trajectory in the x_1-x_2 plane with several positions indicated as $\mathbf{x}_i$. Let's consider one of those points, say, $\mathbf{x}_i$, as the starting position, **X**. Then, the deformation gradient tensor $\bar{\mathbf{F}}$ for the starting position, $\mathbf{x}_i$, can be expressed by Eq. (17) in the bar coordinate system defined at the location $\mathbf{x}_i$ as indicated in Fig. 2. Therefore, $\bar{F}_{12}(\mathbf{x}_i, T_{cir})$ and $\bar{F}_{32}(\mathbf{x}_i, T_{cir})$ can be rewritten, instead of Eq. (18), as:

$$\bar{F}_{12}(\mathbf{x}_i, T_{cir})=-\bar{V}_1\Big|_{X=x_i}\frac{\partial T_{cir}}{\partial\bar{X}_2}\Big|_{X=x_i} \tag{20}$$

$$\bar{F}_{32}(\mathbf{x}_i, T_{cir})=-\bar{V}_3\Big|_{X=x_i}\frac{\partial T_{cir}}{\partial\bar{X}_2}\Big|_{X=x_i}+\frac{\partial}{\partial\bar{X}_2}\left(\oint v_3dt\right)\Big|_{X=x_i} \tag{21}$$

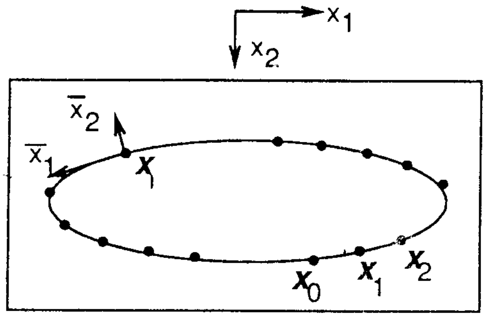

Fig. 2. A particle trajectory orbit in the x_1-x_2 plane with a bar coordinate system defined at an initial positon $\mathbf{x}_i$.

It might be noted that any point, $\mathbf{x}_i$, in Fig. 2 lies on the constant streamline so that the following mass conservation equation hold:

$$\bar{V}_1\Big|_{X=x_0} d\bar{X}_2\Big|_{X=x_0} = \bar{V}_1\Big|_{X=x_i} d\bar{X}_2\Big|_{X=x_i} \tag{22}$$

where $d\bar{X}_2$ represents the normal distance one. With the help of Eq. (22), $\left.\frac{\partial T_{cir}}{\partial \bar{X}_2}\right|_{X=x_i}$ and $\left.\frac{\partial(\oint \bar{v}_3 dt)}{\partial \bar{X}_2}\right|_{X=x_i}$ in Eqs. (20) and (21) can be easily determined for each initial position $\mathbf{x}_i$ once these terms are evaluated at $\mathbf{X}=\mathbf{x}_0$ as a reference position. That is, applying Eq. (22) into Eqs. (20) and (21) finally results in the most convenient formulae for $\bar{F}_{12}$ and $\bar{F}_{32}$ as follows:

$$\bar{F}_{12}(\mathbf{x}_i, T_{cir}) = -\frac{\bar{V}_1^2|_{X=x_i}}{\bar{V}_1|_{X=x_0}} \left.\frac{\partial T_{cir}}{\partial \bar{X}_2}\right|_{X=x_0} \tag{23}$$

$$\bar{F}_{32}(\mathbf{x}_i, T_{cir}) = -\left.\frac{\bar{V}_3|_{X=x_i}\bar{V}_1|_{X=x_i}}{\bar{V}_1|_{X=x_0}} \frac{\partial T_{cir}}{\partial \bar{X}_2}\right|_{X=x_0} + \frac{\bar{V}_1|_{X=x_i}}{\bar{V}_1|_{X=x_0}} \left.\frac{\partial}{\partial \bar{X}_2}\left(\oint \bar{v}_3 dt\right)\right|_{X=x_0} \tag{24}$$

In might be mentioned that, in using Eq. (23) and (24), one has to determine the period of recirculation, T_{cir}, or each closed orbit in the cross-sectioned plane, the tangential velocity component, $\bar{V}_1$, and the axial velocity component, $\bar{V}_3$ along the closed orbit during the Runge-Kutta integration of Eq. (1). Then, a deformation measure, defined by $\sqrt{\bar{F}_{12}^2+\bar{F}_{32}^2}$, can be easily determined over the entire cross-sectioned plane.

4. Conclusion

Concerned about the kinematicks of deformation in duct flows, the present paper proposes a simple method to calculate a deformation measure over the entire cross section of a duct. The simple method requires determining only the velocity field, particle trajectories, and recirculating time for each trajectory, thus obviating difficulties encountering in the numerical evaulation of the velocity gradient tensor, especially when there exist singlular corners like in an extruder channel. With the help of the simple formulae, Eqs. (23) and (24), a deformation measure, defined by $\sqrt{\bar{F}_{12}^2+\bar{F}_{32}^2}$ in a cartesian coordinate system aligned with the particle trajectory orbit, can be easily determined over the entire cross-sectioned plane.

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Appendix A

Instead of Eq. (14), Franjione and Ottino [1] has mistakenly obtained the following expression for $\mathbf{M}$:

$$\mathbf{M} = \begin{bmatrix} V_1\dfrac{\partial T_{cir}}{\partial X_1} & V_1\dfrac{\partial T_{cir}}{\partial X_2} & 0 \\ V_2\dfrac{\partial T_{cir}}{\partial X_1} & V_2\dfrac{\partial T_{cir}}{\partial X_2} & 0 \\ -\dfrac{\partial}{\partial X_1}\left(\oint v_3 dt\right) & -\dfrac{\partial}{\partial X_2}\left(\oint v_3 dt\right) & 0 \end{bmatrix} \tag{A.1}$$

which should be compared with Eq. (14). The derivation in [1] shoud be corrected as follows:

An equation (which appeared just before Eq. (10) in [1])

$$x_3+\delta x_3=X_3+\delta X_3+f[\mathbf{X}_c+\delta\mathbf{X}_c,\ T(\mathbf{X}_c+\delta\mathbf{X}_c)] \quad (A.2)$$

should be changed to

$$x_3+\delta x_3=X_3+\delta X_3+f[\mathbf{X}_c+\delta\mathbf{X}_c,\ T(\mathbf{X}_c)] \quad (A.3)$$

where T indicates T_{cir} in the present paper, the subscript "c" denotes the cross-sectional portion for the flow and f is defined as $f[\mathbf{X}_c,\ T(\mathbf{X}_c)]=\int_0^{T(\mathbf{X}_c)} v_3[\phi_c(\mathbf{X}_c,\ t)]dt$.

$f[\mathbf{X}_c+\delta\mathbf{X}_c,\ T(\mathbf{X}_c)]$ can be expanded about $\mathbf{X}_c$ in a Taylor series ignoring the second- and higher-order terms as follows:

$$\begin{aligned} f[\mathbf{X}_c+\delta\mathbf{X}_c,\ T(\mathbf{X}_c)] &= f[\mathbf{X}_c+\delta\mathbf{X}_c,\ T(\mathbf{X}_c+\delta\mathbf{X}_c)] \\ &\quad - v_3[\phi_c(\mathbf{X}_c,\ T(\mathbf{X}_c)]\nabla_{\mathbf{X}_c}T\delta\mathbf{X}_c \\ &= f[\mathbf{X}_c,\ T(\mathbf{X}_c)]+\nabla_{\mathbf{X}_c}f\delta\mathbf{X}_c-v_3(\mathbf{X}_c)\nabla_{\mathbf{X}_c}T\delta\mathbf{X}_c \end{aligned} \quad (A.4)$$

Applying Eq. (A.4) into Eq. (A.3), with the definition $\delta\mathbf{x}[T(\mathbf{X}_c)]=(\mathbf{I}-\mathbf{M})\cdot\delta\mathbf{X}_c$, results in the third row in Eq. (14), instead of in Eq. (A.1).

Appendix B

Applying Eq. (2) into Eq. (1) results in:

$$\frac{d\phi(\mathbf{X},\ t)}{dt}=\mathbf{v}(\phi(\mathbf{X},\ t)) \text{ with } \phi(\mathbf{X},\ 0)=\mathbf{X} \quad (B.1)$$

Differentiating the above equation with respect to time yields:

$$\frac{d^2\phi(\mathbf{X},\ t)}{dt^2}=\mathbf{L}(\phi(\mathbf{X},\ t))\cdot\frac{d\phi(\mathbf{X},\ t)}{dt} \quad (B.2)$$

Eq. (B.2) shows that $\frac{d\phi(\mathbf{X},\ t)}{dt}$ satisfies Eq. (5), which indicates that $\frac{d\phi(\mathbf{X},\ t)}{dt}$ evolves in the same way as $\delta\mathbf{x}$. Therefore, Eq. (3) can be rewritten in terms of $\frac{d\phi(\mathbf{X},\ t)}{dt}$ instead of δx as follws:

$$\frac{d\phi(\mathbf{X},\ t)}{dt}=\mathbf{F}(\mathbf{X},\ t)\cdot\frac{d\phi(\mathbf{X},\ t)}{dt} \quad (B.3)$$

Since a fluid particle returns to its original position in the x_1-x_2 plane after one full circulation, one may find that

$$\frac{d\phi(\mathbf{X},\ T_{cir})}{dt}=\frac{d\phi(\mathbf{X},\ 0)}{dt}=(\bar{V}_1,\ 0,\ \bar{V}_3) \quad (B.4)$$

in the bar coordinate system.

From Eqs. (B.3) and (B.4) with $\det[\mathbf{F}]=1$ for incompressibility, one can easily find that $\bar{F}_{11}=\bar{F}_{22}=1$, $\bar{F}_{21}=\bar{F}_{31}=0$.

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