오토마타와 그들의 입력 반군들에 대한 어떤 성질들

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요 약

본 논문의 주된 목적은 입력 반군들에 대해서 환에서 얻어진 것들에 비슷한 래디칼 및 원시성과 이행성의 개념들을 발전시킨다. 더구나 우리는 그들의 어떤 성질들을 조사한다.

Some Properties on Automata and Their Input Semigroups

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ABSTRACT

The purpose of this paper is to develope for input semigroups the notions of radical and primitiveness similar to those which have been developed for rings and of transitiveness. Moreover, some of thier properties anr investigated.

1. Introduction and Preliminaries

We will start with the definition of an automaton.

DEFINITION 1.1.

(1) An automaton, $A=(M, S, \delta)$, is a triple where M is a nonempty set(the set of states), S is a nonempty semigroup (the set of inputs), δ is a function(called the state transition function) mapping $M \times S$ into M. Also, we shall assume the useful property that $\delta(m, st)=\delta(\delta(m, s), t)$ for all $s t, \in S$ and $m \in M$.

NOTE. An automaton A means a triple(M, S, δ) and M does not mean an automaton. But the attribute "*automaton*" will be sometimes used for M.

NOTATION. For convenience we will denote δ (m, s) as ms.

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(2) A ***subautomaton*** of M is a non-void subset H of M such that $HS \subset H$.

(3) An automaton, $A=(M, S, \delta)$, is called irreducible if $MS \not\subset F(M)$ and M has no non-trivial subautomata where $F(M)=\{m \in M: ms=m \text{ for all } s \in S\}$.

(4) A right congruence of a semigroup S is called ***modular*** if there is an element e of S such that $(es, s) \in \alpha$ for all $s \in S$. The element e is called a ***left identity*** for α.

DEFINITION 1.2

Let $A=(M, S, \delta_A)$ and $B=(N, S, \delta_B)$ be automata.

(1) A mapping $f: A \to B$(or $M \to N$) is an S-***homomorphism***(or S-map or S-operation preserving) if $f(ms)=f(m)s$ for all $m \in M$ and $s \in S$. f is called an S-***isomorphism*** if it is bijective and an S-homomorphism. f is called an S-homomorphism.

(2) An automaton A is *cyclic* if $M=mS$ for some $m \in M$. Also, m is called a generator.

(3) An automaton A is ***strongly connected*** if every element of M is a generator.

(4) An automaton A is ***abelian*** if $m(st)=m(ts)$ for all $m \in M$ and $s, t \in S$.

(5) S is ***M-abelian*** if $m(ab)=m(ba)$ for all $m \in M$ and a, $b \in S$.

(6) An automaton A is ***perfect*** iff A is strongly connected and S is M-abelian.

(7) $T_a: M \to M$ is called ***a right translation*** if $T_a(m)=ma$ for all $m \in M$ where $a \in S$.

(8) We define a congruence $\mu_M \subset S \times S$ on S through $(a, b) \in \mu_M \Leftrightarrow T_a = T_b$.

(9) M is faithful iff $\mu_M = O_c$(the identity relation).

(10) We define a right congruence μ_M on S by $(a,b) \in \mu_M \Leftrightarrow T_a(m) = T_b(m)$ for $a, b \in S$ where $m \in M$.

(11) Let α be a right congruence on S. For any $s \in S$ we define a relation αs on S by $(a, b) \in \alpha s$ if and only if $(sa, sb) \in \alpha$

LEMMA 1.3.

(1) $\mu_{s/\alpha} = \cap_{t \in s} \alpha t$ if α is a right congruence on S. We note that if α is a modular right congruence on S, $\mu_{s/\alpha} = \cap_{a \in s} a\alpha \leq \alpha$.

(2) $[\alpha]_{\cap M \in IS} \mu_M = \cap_{M \in IS} [\alpha]\mu_M$ for $\alpha \in S$.

(3) Let M and N be two automata. *If* f:M $\to$N is an S-isomorphism, then $\mu_M = \mu_N$.

Proof.

For(1), let M be an automaton and $m \in M$. Then for $a, b \in S$ $(a, b) \in \mu_m \Leftrightarrow ma = mb$ define a right congruence on S.

$(a, b) \in \mu_{S/\alpha} = \cap_{[t]_\alpha \in S/\alpha} \mu_{[t]_\alpha} \Leftrightarrow$(a, b)$\in \mu_{[t]_\alpha}$ for all $[t]_\alpha \in S/\alpha \Leftrightarrow [t]_\alpha a = [t]_\alpha b \Leftrightarrow (ta, tb) \in \alpha$ for all $t \in S \Leftrightarrow (a, b) \in \alpha t$ for all $t \in S \Leftrightarrow (a, b) \in \cap_{t \in S} \alpha t$. For(2) $x \in [\alpha]_{\cap M \in IS \mu_M} \Leftrightarrow (x, a) \in \cap_{M \in IS \mu_M} \Leftrightarrow (x, a) \in \mu_M$ for all $M \in IS \Leftrightarrow x \in \cap_{M \in IS} [a]_{\mu M}$. For (3), $(a, b) \in \mu_M \Leftrightarrow ma = mb$ for all $m \in M \Leftrightarrow f(ma) = f(mb) \Rightarrow f(m)a = f(m)b$ for all $m \in M$.

Now for any $n \in N$, $n = f(m)$ for some m$\in$M. Hence na$=$nb implies $(a, b) \in \mu_N$. Also it is easy for us to check the converse.

The following proposition is a generalization for a new right congruence induced by right congruences on S and right ideals of S. This follows from Oehmke [6].

PROPOSITION 1.4.

Let A be an indexes set. Let τ_α be a right congruence on S and let I_α be a right ideal of S for each $\alpha \in A$. We define a relation $\omega = \omega(\cap_{\alpha \in A} \tau_\alpha, \cap_{\alpha \in A} I_\alpha$ as follow:

$(a, b) \in \omega \Leftrightarrow (a, b) \in \cap_{\alpha \in A} \tau_\alpha$ or $(a, b) \in \cap_{\alpha \in A} I_\alpha$.

Then ω is a right congruence on S with $\cap_{\alpha \in A} I_c \leq \omega$.

PROPOSITION 1.5.

Let A be an indexes set. Let Iα be an ideal of S and ω(Iα) be a congruence induced by Iα for each $\alpha \in$A. If $\cap_{\alpha \in A} I_a = \{0\}$, then $\cap_{\alpha \in A} \omega(I_\alpha) = O_r$ where O_r means the identity relation.

Proof.

For each $(a, b) \in \cap_{\alpha \in A} \omega(I_\alpha)$ it its enough to show that a$=$b. Now, $(a, b) \in \omega$(Iα) for all $\alpha \in$A. This means that a$=$b or a, b$\in$Iα for all $\alpha \in$A. The latter case inplies that $a, b \in \cap_{\alpha \in A} I_\alpha = \{0\}$. *Hence* $a=b$.

PROPOSITION 1.6.

Let G be a non-trivial group and let H be a proper subgroup of G. We define the relation α on G by

$$(a,b) \in \alpha \Leftrightarrow ab^{-1} \in H.$$

Let L(α)=\{all left identities of α\}. Then

(1) $H \subset L(\alpha)$;

(2) α is a modular right congruence on G with $\alpha \neq 1_C$ where 1_C means the universal relation;

(3) $\tau(L(\alpha)) \subset \alpha$ where $\tau(L(\alpha)) = Sup_{\mu \in L(\alpha)} \tau(\mu)$ and $\tau(\mu)$ = the intersection of all modular right congruences with respect to u;

Proof.

(1) and (2) are clear. (3) comes from Seidel [4].

LEMMA 1.7.

Let M be an automaton and let H and K be subsets of M. Let A be a subset of S. if $KA = H$, then $A \subset K^{-1}H = \{s \in S \mid Ks \subset H\}$

DEFINITION 1.8.

(1) A right congruence τ on S is said to be ***modular*** with respect to $a \in S$ if and only if $(s, as) \in \tau$ for all $s \in S$.

(2) An element $a \in S$ is a ***right q-element***(or a ***right quasi-regular element***) if $\tau(a) = 1_C$ or equivalent to $a^m s = a^m t$ for some m, n≥ 0 and all s, $t \in S$ where $\tau(a)$ = the intersection of all modular right congruences with respect to a.

(3) An element $a \in S$ is an $O(S)$-potent if $a^n \in O(S)$ for some $n \geq 1$ where $O(S)$ = {all left zero elements in S}.

(4) An element $a \in S$ is an ***eigentlich*** $O(S)$-***potent*** if $(as)^n \in O(S)$ for some $n \geq 1$ and for all $s \in S$.

(5) A set M is called an ***automaton with null*** (*0*) if M is an automaton and $F(M) = \{0\}$ where $0 \in S$.

NOTE :

with each automaton M we associate the set $M_0 = \{s \in S \mid mss_1 = m$ implies $m \in F(M)$ where $m \in M$, $s_1 \in S^1\}$ where $S^1 = S \cup \{1\}$.

(6) We define $rad_0 S = \bigcap_{M \in IS_0} K_{\sigma}, S_M$ or $rad_0 S = \bigcap_{M \in IS_0} M_0$ where S_M is the representation of S generated by an automaton M and IS_0 = {all irreducible automata with null(0) where $0 \in S$}.

NOTATION.

$Q(S)$ = {all q-elements in S}.

$E(S)$ = {all idempotent elements in S}.

$O(S)$ = {all left zero elements in S}.

$R(S) = \{a \in S \mid (as, s) \in \alpha$ for all s in S implies $\alpha = 1_r$ where α is a right congruence on S}.

$O(S)p$ = {all O(S)-potent elements in S}.

The following proposition gives us a relationship among these.

PROPOSITION 1.9.

(1) $Q(S) \cap E(S) \subset O(S) \subset O(S)_P \subset Q(S)$.

(2) $Q(S) \cap E(S) = O(S)$ if $R(S) = S$.

Proof.

For(1), this is clear from Seidel[4]. For(2), We have $O(S) = E(S)$ if $R(S) = S$ and also, we have $O(S) \subset Q(S)$ from Seidel[4].

PROPOSITION 1.10.

Let $N(S)$ = {all eigentlich $O(S)$-potent elements in S}. Then we have

(1) $F\left[\dfrac{S}{N(S)}\right] = \{N(S)\}$ and $F\left[\dfrac{S}{M_0}\right] = \{M_0\}$;

(2) $F\left[\dfrac{S}{rad_0(S)}\right] = \{rad_0 S\}$.

Proof.

We note that $F(M/H) = \{H\}$ if $F(M) \subset H \neq \phi$ where H is a noon-void subautomaton of an automation M and also we know that $rad_0 S = N($

S). Since $F(S)=O(S)$, we have $F(S)=O(S)\subset rad_0 S=N(S)\subset M_0$.

PROPOSITION 1.11.

Let $0\in S$ and $S^1=S\cup\{1\}$.

Let $\Omega=\{$all right ideals I_α in S, $\alpha\in A\}$. Let $\Pi=\{$all right congruences on S$\}$. We define the relation ω_{I_α} (σ_α) on S by $(a, b)\in\omega_{I_\alpha}$ $(\sigma_\alpha)\Leftrightarrow(as_1, bs_1)\in\sigma_\alpha$ for all $s_1\in S^1$ where $\sigma_\alpha=I_\alpha\times I_\alpha\cup(S-I_\alpha)\times(S-I_\alpha)$. then

(1) there exists a function $F:\Omega\to\Pi$ given by $F(I_\alpha)=\omega_{I_\alpha}$ (σ_α);

(2) F is 1−1.

Proof.

For(1), from proposition 15 of Seidel [4] there exists a unique maximal right congruence ω_{I_α} (σ_α) for each right ideal I_α in S. Hence it holds. For(2), To prove that F is 1−1, it is enough to show that $F(I_\alpha)=F(I_\beta)$ inplies $I_\alpha=I_\beta$. *Also, we note that by proposition 15 if Seidel[4] we have* $\omega_{I_\alpha}(\sigma_\alpha)=Sup_\Pi\{\gamma\in\Pi \mid I_\alpha=[0]_\gamma\}$.

Since ω_{I_α} $(\sigma_\alpha)=\omega_{I_\beta}$ $(\sigma_{\alpha\beta})$ and these are in Π, we have ω_{I_α} $(\sigma_{\alpha\alpha})=\gamma_0$ with $I_\alpha=[0]\gamma_0$ for some γ_0 in Π and ω_{I_β} $(\sigma_{\alpha\beta})=\gamma_1$ with $I_\beta=[0]\gamma_1$ for some γ_1 in Π. Hence $I_\alpha=I_\beta$.

COROLLARY 1.11.1.

$|\Omega|\le|\Pi|$ where $|\ |$ means the cardinality.

PROPOSITION 1.12.

Let $0\in S$ and let α be a right congruence on S. Let $\beta([0]_\alpha)$ on S and let $\omega_{[0]_\alpha}$ (σ_α) be the right congruence induced by $[0]_\alpha$ on S and let $\omega_{[0]_\alpha}$ (σ_α) be the unique maximal right congruence induced by σ_α with respect to $[0]_\alpha$. Then we have

$$\beta([0]_\alpha)\subset\alpha\subset\omega_{[0]_\alpha}(\sigma_\alpha).$$

Proof.

For the first part it is clear since (x, y) belongs to $\beta([0]_\alpha)$ if and only if $x=y$ or $x, y\in[0]_\alpha$.

For the second part we note that $(a, b)\in\omega_{[0]_\alpha}(\sigma_\alpha)\Leftrightarrow(as_1, bs_1)\in\sigma_\alpha$ for all s_1 in S^1. So it enought to show that for each $(x, y)\in\alpha$ for all s_1 in S^1.

(i) suppose $(x, 0)\in\alpha$. Then $(x, y)\in\sigma_\alpha$. For $(y, 0)\in\alpha$ implies $x, y\in[0]_\alpha$. Since α is a right congruence on S, we have $(xs, 0)\in\alpha$ for all s in S. This implies that $(xs, ys)\in\alpha_\alpha$ for all s in S. For $(ys, 0)\in\alpha$ implies $xs, ys\in[0]_\alpha$. Hence $(x, y)\in\alpha_\alpha$ and $(xs, ys)\in\alpha_\alpha$ for all s in S.
i.e., we have $(xs_1, ys_1)\in\sigma_\alpha$ for all s_1 in S^1.

(ii) suppose $(x, 0)\notin\alpha$. Then $(x, y)\notin\sigma_\alpha$ For $(y, 0)\notin\alpha$ implies $x, y\in[0]_\alpha$ and $x, y\in S-[0]_\alpha$. *Hence we have* $(x, y)\in(S-[0]_\alpha)\times(S-[0]_\alpha)\subset\sigma_\alpha$. Next, if $(xs, 0)\in\alpha$ for all s in S, $(xs, ys)\in\sigma_\alpha$ from (i). So, we have done. Now, suppose $(xs, 0)\notin\alpha$ for all s in S. Then $(xs, ys)\in\sigma_\alpha$ For $(ys, 0)\notin\alpha$ implies $xs, ys\notin[0]_\alpha$ and $xs, ys\in S-[0]_\alpha$. Hence (x, y) and (xs, ys) are contained in σ_α for all s in S. This means that $(xs_1\ ys_1)\in\sigma_\alpha$ for all s_1 in S^1.

2. Radical and Maximality

DEFINITION 2.1.

(1) An automaton M is called ***totally irreducible*** if $MS\not\subset F(M)$ and M has no non-trivial homomorphism.

(2) An automaton M is called ***strictly cyclic*** if $M=m_0S$ for some $m_0\in$M.

LEMMA 2.2.

Let α and β be modular right congruences on S. if S/α and S/β are isomorphic and α is maximal, then β is maximal.

Proof.

We note that S/α has a non-trivial homomorphism if and only if there exists a right congruence ω on S such that $\alpha<\omega<1$. Suppose that β is not maximal. Then there exists a modular right congruence μ on S with $\beta<\mu<1$. This means that S/β has a non-trivial homomorphism and also this means that S/α has a non-trivial homomrphism. Therefore there exists a right congruence ω_1 on S with $\alpha<\omega<1$. Also, ω_1 is a modular since α is a modular. It is impossible since α is maximal.

PROPOSITION 2.3.

Let M be an automaton with $|M|\geq 2$ and no homomorphism except for isomorphism.

Suppose $O(S)\neq\phi$. Then M is a totally irreducible if and only if M is strictly cyclic.

Proof.

($\Rightarrow$) It is clear. ($\Leftarrow$) Suppose M is strictly cyclic. Then M and S/α are isomorphic where α is a modular right congruence on S. Now, from Hoehnke[5] there exists a maximal modular right congruence β with $\alpha\subset\beta$. Also, from $\alpha\subset\beta$ there is an onto-homomorphism from S/α to S/β. But by the assumption S/α and S/β are isomorphic. Hence α is maximal from Lemma 2.2. This means that M is totally irreducible.

NOTATION. I_r means the identical relation and 1_r means the unversal relation. $TA=\{$all totally irreductble automata$\}$.

DEFINITION 2.4.

(1) We define $rad\ S=\cap_{M\in IS}\ \mu_M$.

(2) S is called *radical-frec* if rad $S=Ir$.

The following statement is a similar one of a solvable group G. Let N be a normal subgroup of G. We note that F is solvable if and only if N and G/N are solvable.

PROPOSITION 2.5.

Let I be an ideal of S with $I\subset KerS_M$. *Then* S *is radical-free if and only if* I *and* S/N are radical-free.

Proof.

($\Rightarrow$) We note that rad $I=I\times I\cap$ *rad* S from Seidel[4]. Since *rad* $S=Ir$, we have *rad* $I=Ir$

Also, from Seidel[4] we note that for every ideal $I\subset KerS_M$ M *is an* S*-automaton if and only if* M *is an* S/I automaton. Let ISI be the set of all irreducible S/I-automata. Then we have *rad* S/I $=\cap_{M\in ISI}\ \mu_M(S/I)=\cap_{M\in IS}\ \mu_M(S)=rad\ S=Ir$.

($\Leftarrow$) It is clear from Seidel[4].

DEFINITION 2.6.

(1) We define $Rad\ S=\cap_{M\in TA}\ \mu_M$.

(2) S is O-radical free if $rad_0S=\{0\}$.

PROPOSITION 2.7.

(1) $rad_0S\times rad_0S\subset rad\ S\subset Rad\ S$;

(2) Let $0\in S$. If S is radical-free, then S is O -radical free.

Proof.

For(1), since rad_0S is a congruence ideal with respect to rad S $rad_0S=[a_0]_{rad\ S}$ for some $a_0\in rad_0$ S. Now, for each $(x, y)\in rad_0S\times rad_0S$ we have $x, y\in rad_0S$. So, $(x, a_0)\in rad\ S$ and $(y, a_0)\in$ *rad* S.

Hence we have (x, y) rad S. For the second part it comes from Hoehnke [5]. For (2) we note that we can replace the condition $0\in S$ by

rad_0 $S \neq \phi$ since rad $S = Ir$ and rad_0 $S \neq \phi$ implies that there exist a null element 0 in S from Seidel [4]. Hence it is clear.

We note that any finitely generated semigroup S contains at least one maximal subsemigroup.

The following proposition is easy to check it using Zorn's Lemma.

PROPOSITION 2.8.

Let A be a non-empty subset of S. Suppose there exist an ideal I of S such that $I \cap A = \phi$.

Let $W = \{J \mid J$ is ideal of S with $J \cap A = \phi\}$. Then there exists a maximal subautomaton K containing H.

PROPOSITION 2.9.

Let M be an automaton with $MS = M$. If H is an subautomaton of M with $H \neq M$, there exists a maximal subautomaton K containing H.

Proof.

Let $W = \{L \mid L$ is a proper subautomaton of M with $H \subset L < M\}$ and partially order W by set inclusion(i.e., $L_1 \leq L_2$ if and only if $L_1 \subset L_2$). We claim that W is a non-empty induxtively ordered set. To prove this,(1) $W \neq \phi$: It is clear since $H \in W$. (2)W is inductively ordered : Let T be any non-empty totally ordered subset of W. To show that T has an upper bound in W, let $U = \cup_{L \in T} L$. *Then* $(a)U$ is a subautomaton of M(i.e., $US \subset U$). To show this, choose any $x \in US$. Then $x = ys$ for some $y \in U$ and $s \in S$. $y \in U$ implies $y \in L$ for some $L \in T$. Hence $x = ys \in LS \subset L \subset U$. (b) $H \subset U \subset M(U \neq M)$(i.e., $U \subset W$). Now, $H \subset U$ is clear. To show that $U \neq M$, suppose $U = M$. Then $US = MS = M$. For each $m \in M$, $m \in US$. This implies $m \in LS$ for some $L \in T$. So, $m \in LS \subset L$. Hence $M \subset L$ and $M = L$. It is impossible. (c) U has an upper bound for T(it is clear). Hence By Zorn's lemma W has a maximal element K in W.

COROLLARY 2.9.1.

Let M be a strictly cyclic automaton. If H is a subautomaton of M with $H \neq M$, then there exists a maximal subautomaton K containing H.

COROLLARY 2.9.2.

Let M be an automaton with $F(M) = M$. If H is a subautomaton of M with $H \neq M$, then there exists a maximal subautomaton K containing H.

Proof.

$F(M) = M$ implies $MS = M$.

COROLLARY 2.9.3.

Let M be an irreducible (or totally irreducible) automaton. If H is a subautomaton of M with $H \neq M$, then there exists a maximal subautomaton K containing H.

Proof.

The fact that M is totally irreducible implies that M is irreducible and also this impliees $MS = M$.

PROPOSOTION 2.10.

Let $0 \in S$ and let α be α maximal right congruence on S. Then $S/[0]_\alpha$ is an irreducible S-automaton$\Leftrightarrow S \neq Ker\ S_{s/\alpha}$.

Proof.

Let I be a maximal right ideal of S. Then from Hoehnke[5] S/I is an irreducible S-automaton if and only if $S \neq S^{-1}I$. ($\Rightarrow$):$[0]_\alpha$ is a maximal right ideal since α is a maximal right congruence on S. This implies that $S \neq S^{-1}[0]_\alpha = (S/\alpha)^{-1}\{[0]_\alpha\} = Ker\ S_{s/\alpha}$.

($\Leftarrow$) : It is clear since $S \neq S^{-1}[0]_\alpha$ and $[0]_\alpha$ is a maximal right ideal of S.

DEFINITION 2.11.

Let M be an S-automaton. M is cyclic if $M = mS \cup \{m\}$ for some in M.

NOTATION. $M^* = \{$all non-generators in $M\}$ where M is cyclic.

PROPOSITION 2.12.

Let be α right congruence on S. Let S/α be an S-automaton with $F(S/\alpha) = \phi$. Then S/α is an irreducible S-automaton if and only if α is a modular right congruence.

Proof.

S/α is a strictly cyclic S-automaton since S/α is irreducible. This means that α is modular.

Conversely, we assume that α is a modular right congruence on S. Then S/α is strictly cyclic. This means that $(S/\alpha)^* = \phi$ since $F(S/\alpha) = \phi$. Hence it holds.

3. Faithfulness, Primitveness and transitiveness.

DEFINITION 3.1.

Let $0 \in S$ and let M be an S-automaton.

(1) M is faithful if $\mu_M = I_r$ where $0 \notin S$.

(2) M is 0-faithful if $Ker\ S_M = M^{-1}[0] = \{0\}$.

(3) S is 0-primitive if S has an 0-faithful irreducible S-automaton.

(4) Let P be an ideal of S. P is 0-primitive if S/P is 0-primitive semigroup and $S \neq P$.

PROPOSITION 3.2.

Let $0 \in S$. If α is a maximal modular right congruence, then $Ker\ S_{S/\alpha}$ is an 0-primitive ideal of S.

Proof.

By the asumption $[0]_\alpha$ is a maximal modualr right ideal of S. This means that $S^{-1}[0]_\alpha$ is an 0-primitive ideal by Hoehnke[3]. Hence we have $S^{-1}[0]_\alpha = (S/\alpha)^{-1}\{[0]_\alpha\} = Ker\ S_{s/\alpha}$.

DEFINITION 3.3.

Let M be an S-automaton.

(1) M is 2-minimal if $|M| \geq 2$ and M has the only trivial S-auotmaton.

(2) M is 2-null if $|M| = 2$ and $|MS| = 1$.

(3) M is 0-transitive if M is strictly cyclic with $|M| \geq 2$ and $|F(M)| = 1$.

LEMMA 3.4.

Let M be an S-automaton. If M is 2-minimal reducible, then M is either 2-null or $|M| = |MS| = |F(M)| = 2$.

PROPOSITION 3.5.

Let M be an 2-minimal S-automaton with $|F(M)| = 1$. Then

(1) If $MS \not\subset F(M)$, then M is 0-transitive.

(2) If M is reducible, then M is 2-null.

Proof.

(1) $MS \not\subset F(M)$ means that M is irreducible. This means that M is strictly cyclic. Hence it holds.

(2) It is clear from lemma 3.4 and $|F(M)| = 1$.

COROLLARY 3.5.1.

Let M be an 2-minimal S-automaton with $|F(M)| = 1$. If $MS \not\subset F(M)$ or M is reducible, then M, $F(M)$ and ϕ are the only invariant subsets of M.

Proof.

It comes from Tully[1] and proposition 3.5.

DEFINITION 3.6.

(1) S is transitive if S is strictly cyclic with $O(S)=\phi$.
(2) S is 0-transitive if S is strictly cyclic with $|S| \geq 2$ and $O(S)=\phi$.
(3) S is h-primitive if S has a faithful irreducible S-automaton M.
(4) S is t-primitive if I_r and 1_r are the only right conguence on S.

We have the following lemma a from Tully[1].

LEMMA 3.7.

(1) If S is 0-transitive, then $|O(S)|=1$ and S, $O(S)$ and ϕ are the only invariant subsets of S.
(2) If S is t-primitive with $|S| \geq 3$, then S is either 0-transitive or transitive.

PROPOSITION 3.8.

If S is t-primitive with $|S| \geq 3$ and $E(S) \not\subset 0(S)$, then S is an h-primitive.

Proof.

By lemma 3.7, S is either 0-transitive or transitive. To prove that S is a faithful irreducible S-automaton, (1) assume that S is 0-transitive. Then $\mu_s = I_r$ from the fact that for each a in S, μ_a is a right congruence on S and $\mu_a = I_r$ or 1_r. We will show that S is irreducible. (i) $SS \not\subset F(S)=O(S)$ since $|O(S)|=1$ by lemma 3.7. (ii) S has the only trivial S-automaton since S, $O(S)$ and ϕ are the only invariant subsets of S by lemma 3.7. (2) assume that S is transitive.

Then $\mu_s = I_r$ from the case (1). To show that S is irreducible, we know that if S is transitive, then S is strictly cyclic with $F(S)=O(S)=\phi$. This implies that S is irreducible since $S^*=\varphi$ from Hoehnke[5].

DEFINITION 3.9.

Let M be an S-automaton. M is strongly connected(or transitive) if M is strictly cyclic with $F(M)=\phi$(i.e., every element of M is a strict generator).

We have the following proposition by combining proposition 2.1 of Tully[1] with theorem 3 of Oehmke[2].

PROPOSITION 3.10.

Let M be a strictly cyclic S-automaton. Then the following statements are equivalent:

(1) M is strongly connected(or transitive);
(2) *S/μ_m is strongly connected for every m in M;*
(3) For every $a,b \in S$, $(ac, b) \in \mu_m$ for some c in S and each m in M;
(4) any μ_m-class $\cap$ any right ideal of $S \neq \phi$ for every m in M.

Proof.

(1)⇒(2):By the definition we have $M=mS$ for every m in M. Also, we have that mS and S/μ_m are isomorphic from theorem 3 of Oehmke[2]. Hence it holds. (1)⇐(2): $S/\mu_{m_0} \sim m_0 S = M$ for some $m_0 \in M$. The proofs that (2), (3) and (4) are equivalent come from Tully[1].

PROPOSITION 3.11.

Let $0 \in S$.

(1) If S is 0-primitive, then S is 0-radical free.
(2) If S is h-primitive, then S is radical free.
(3) If S is 0-primitive or h-primitive, then S is 0-radical free.

Proof.

(1) and (2) come from the definitions. (3) is clear from proposition 2.7.

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