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A GLOBAL EIGHTH ORDER SPLINE PROCEDURE FOR A CLASS OF BOUNDARY VALUE PROBLEMS

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1. Introduction

Boundary value problems are common in nature. Here we restrict our attention to the second order differential equations of the form

$$(1.1)\qquad \begin{cases} \dfrac{d^2y}{dx^2} &= P(x)y(x) + Q(x), \quad 0 \le x \le 1, \\ y(0) &= \alpha, \\ y(1) &= \beta, \end{cases}$$

where $P(x)$ and $Q(x)$ are continuous functions with $P(x) \ge 0, x \in [0,1]$.

In literature, Usmani and Warsi(1980) have derived a fourth order method based on quintic spline. Bhatta and Sastri[1] discussed a global seventh order method based on heptic and octic splines. We present a global eighth order procedure where an octic spline scheme coupled with a heptic spline function is used for solving the above problem. In section 2 we derive the numerical scheme with convergence of eighth order and in section 3 we present the analysis of our scheme And in section 4 we briefly remark the scheme using splines of degrees $2s$ and $2s-1$. Finally in section 5 we present the numerical experiments that verify our discussion.

2. Development of the numerical scheme

We consider a uniform partition of the interval $[0,1]$ into N subintervals: step-length $h = 1/N$ and $I_k = [x_{k-1}, x_k], k = 1, \ldots, N-1$. In the interval I_j, the j-th element may be written as

$$(2.1)\qquad S_j(x) = \sum_{k=1}^{9} a_{jk}\phi_k, \quad x_j \le x \le x_{j+1}, \quad j = 0, \ldots, N-1,$$

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where $\phi = (x - x_j)^{k-1}$. Using continuity conditions of spline function, we obtain a relation between τ and s values as

$$\begin{aligned}(2.2)\quad &\tau_{i-3} + 54\tau_{i-2} + 135\tau_{i-1} - 380\tau_i + 135\tau_{i+1} + 54\tau_{i+2} + \tau_{i+3} \\ &= \frac{h^2}{56}(s_{i-3} + 246s_{i-2} + 4047s_{i-1} + 11572s_i + 4047s_{i+1} \\ &+ 246s_{i+2} + s_{i+3}), \qquad i = 3, \cdots, N-3,\end{aligned}$$

where $\tau_i = S_j(x_i)$ and $s_i = S_j''(x_i)$.

The above relation gives $N-5$ equations in $N-1$ unknowns $\tau_i, i = 1, \ldots, N-1$ because boundary conditions determine $\tau_0 = \alpha, \tau_N = \beta$. So we need four more extra equations for complete determination of all the unknowns.

First of all, we consider the following equation.

$$(2.3)\quad a_0\tau_0 + a_1\tau_1 + a_2\tau_2 + a_3\tau_3 + a_4\tau_4 = h^2(b_0s_0 + b_1s_1 + b_2s_2 + b_3s_3 + b_4s_4).$$

By assuming temporarily τ_i and s_i as solution $y(x_i)$ and $y''(x_i)$, Taylor's expanding $y(x)$ at x_1 and setting $a_4 = 1$, then by equating like powers of h we get the linear problem:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1/2! & 0 & 1/2! & 2^2/2! & 1 & 1 & 1 & 1 & 1 \\ -1/3! & 0 & 1/3! & 2^3/3! & -1 & 0 & 1 & 2 & 3 \\ 1/4! & 0 & 1/4! & 2^4/4! & 1/2! & 0 & 1/2! & 2^2/2! & 3^2/2! \\ -1/5! & 0 & 1/5! & 2^5/5! & -1/3! & 0 & 1/3! & 2^3/3! & 3^3/3! \\ 1/6! & 0 & 1/6! & 2^6/6! & 1/4! & 0 & 1/4! & 2^4/4! & 3^4/4! \\ -1/7! & 0 & 1/7! & 2^7/7! & -1/5! & 0 & 1/5! & 2^5/5! & 3^5/5! \\ 1/8! & 0 & 1/8! & 2^8/8! & 1/6! & 0 & 1/6! & 2^6/6! & 3^6/6! \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 3^2/2! \\ 3^3/3! \\ 3^4/4! \\ 3^5/5! \\ 3^6/6! \\ 3^7/7! \\ 3^8/8! \end{pmatrix}$$

By solving the above simultaneous equations we obtain

$$(2.4)\quad \tau_0 + \frac{128}{31}\tau_1 - \frac{318}{31}\tau_2 + \frac{128}{31}\tau_3 + \tau_4 = \frac{h^2}{465}(23s_0 + 688s_1 + 2358s_2 + 688s_3 + 23s_4).$$

Similarly, we obtain by setting $a_5 = 1, a_4 = 54$ and $a_3 = 135$ and using Taylor's expansion at x_2,

(2.5)

$$\frac{9141}{221}\tau_0 + \frac{36084}{221}\tau_1 - \frac{87215}{221}\tau_2 + 135\tau_3 + 54\tau_4 + \tau_5 = \frac{h^2}{53040}(110393s_0 + 3201551s_1 + 10883686s_2 + 3834586s_3 + 234001s_4 + 743s_5\ ),$$

$$(2.6)\quad \tau_N + \frac{128}{31}\tau_{N-1} - \frac{318}{31}\tau_{N-2} + \frac{128}{31}\tau_{N-3} + \tau_{N-4} = \frac{h^2}{465}(23s_N + 688s_{N-1} + 2358s_{N-2} + 688s_{N-3} + 23s_{N-4} \quad \text{at } x_{N-1},$$

and

(2.7)

$$\frac{9141}{221}\tau_N + \frac{36084}{221}\tau_{N-1} - \frac{87215}{221}\tau_{N-2} + 135\tau_{N-3} + 54\tau_{N-4} + \tau_{N-5} = \frac{h^2}{53040}(110393s_N + 3201551s_{N-1} + 10883686s_{N-2} + 3834586s_{N-3} + 234001s_{N-4} + 743s_{N-5} \quad \text{at } x_{N-2}.$$

Now the above equations form a system of $N-1$ equations in $N-1$ unknowns since this system is linear, we can obtain the solution as the nodal approximation of true solution $y(x)$. Once this values τ_i are known, s_i values of second derivative can be easily computed using equations (2.2), (2.4), (2.5), (2.6) and (2.7). On the other hand we can obtain the off node approximation of solution up to $O(h^8)$ by using a heptic spline according to [1].

3. Convergence analysis

For the ease of presentation, let us introduce vector and matrix notations. Let $Y = (y_1, \dots, y_N)^T$, $\tau = (\tau_1, \dots, \tau_N)^T$, $s = (s_1, \dots, s_N)^T$, $Q = (q_1, \dots, q_N)^T$, $P = (p_1, \dots, p_N)$ and $R = (-\tau_0 + \frac{23}{465}h^2 s_0, -\frac{9141}{221}\tau_0 + \frac{110393}{53040}h^2 s_0, -\tau_0 + \frac{1}{56}h^2 s_0, 0, \dots, 0, -\tau_N + \frac{1}{56}h^2 s_N, -\frac{9141}{221}\tau_N + \frac{110393}{53040}h^2 s_N, -\tau_N + \frac{23}{465}h^2 s_N)^T$, where P is a diagonal matrix, $y_i = y(x_i)$, $p_i = P(x_i)$ and $q_i = Q(x)$.

Let

$$A = \begin{pmatrix}
\frac{128}{31} & -\frac{318}{31} & \frac{128}{31} & 1 & & & \\
\frac{36084}{221} & -\frac{87215}{221} & 135 & 54 & 1 & & \\
54 & 135 & -380 & 135 & 54 & 1 & \\
1 & 54 & 135 & -380 & 135 & 54 & 1 \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
1 & 54 & 135 & -380 & 135 & 54 & 1 \\
 & 1 & 54 & 135 & -380 & 135 & 54 \\
 & & 1 & 54 & 135 & -\frac{87215}{221} & \frac{36084}{221} \\
 & & & 1 & \frac{128}{31} & -\frac{318}{31} & \frac{128}{31}
\end{pmatrix}$$

$$B = \begin{pmatrix}
\frac{a}{465} & \frac{b}{465} & \frac{a}{465} & \frac{c}{465} & & & \\
\frac{d}{53040} & \frac{e}{53040} & \frac{f}{53040} & \frac{g}{53040} & \frac{h}{53040} & & \\
\frac{246}{56} & \frac{4047}{56} & \frac{11572}{56} & \frac{246}{56} & \frac{1}{56} & & \\
\frac{1}{56} & \frac{246}{56} & \frac{4047}{56} & \frac{11572}{56} & \frac{4047}{56} & \frac{246}{56} & \frac{1}{56} \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\frac{1}{56} & \frac{246}{56} & \frac{4047}{56} & \frac{11572}{56} & \frac{4047}{56} & \frac{246}{56} & \frac{1}{56} \\
 & \frac{1}{56} & \frac{246}{56} & \frac{4047}{56} & \frac{11572}{56} & \frac{4047}{56} & \frac{246}{56} \\
 & & \frac{h}{53040} & \frac{g}{53040} & \frac{f}{53040} & \frac{e}{53040} & \frac{d}{53040} \\
 & & & \frac{c}{465} & \frac{a}{465} & \frac{b}{465} & \frac{a}{465}
\end{pmatrix},$$

where $a = 688$, $b = 2358$, $c = 23$, $d = 3201551$, $e = 10883686$, $f = 3834586$, $g = 234001$ and $h = 743$.

The system of equations (2.2), (2.4), (2.5), (2.6) and (2.7) can be compactly written as

$$(3.1) \qquad \begin{cases} A\tau &= h^2 Bs + R, \\ s &= P\tau + Q. \end{cases}$$

By (3.1) we get

$$(A - h^2BP)\tau = h^2BQ + R. \tag{3.2}$$

For exact solution Y, we have

$$(A - h^2BP)Y = R + h^2BQ + T, \tag{3.3}$$

where T is the truncation error.

By Taylor's expansion, $T(h) = (t_1(h), \ldots, t_{N-1}(h))^T$ is written as

(3.4)

$$t_i(h) = \begin{cases} C_1h^9y^{(9)}(\xi_1) + O(h^{10}), & x_0 < \xi_1 < x_4,\ i = 1 \\ C_2h^9y^{(9)}(\xi_2) + O(h^{10}), & x_0 < \xi_2 < x_5,\ i = 2 \\ -\frac{1}{480}h^{10}y^{(10)}(\xi_i) + O(h^{11}), & x_{i-3} < \xi_i < x_{i+3},\ i = 3, \ldots, N-1 \\ C_2h^9y^{(9)}(\xi_{N-2}) + O(h^{10}), & x_{N-5} < \xi_{N-1} < x_N,\ i = N-2 \\ C_1h^9y^{(9)}(\xi_{N-1}) + O(h^{10}), & x_{N-4} < \xi_{N-1} < x_N,\ i = N-1, \end{cases}$$

where C_1 and C_2 are constants.

Subtracting (3.2) from (3.3), the error can be written as $\tilde{A}E = T(h)$, where $\tilde{A} = A - h^2BP$ and $E = Y - \tau$. Then $E = \tilde{A}^{-1}T(h)$ if $\tilde{A}$ is invertible.

LEMMA 3.1. *Let A be a banded symmetric n by n matrix of the form $\begin{pmatrix} E_1 & C_1^T & 0 \\ C_1 & B & C_2 \\ 0 & C_2^T & E_2 \end{pmatrix}$, where E_1, E_2, B, C_1, and C_2 are block matrices, s by s, s by s, $(n-2s)$ by $(n-2s)$, $(n-2s)$ by s and s by $(n-2s)$, respectively.*

If E_1, E_2 and $\tilde{B} = B - C_1E_1^{-1}C_1^T - C_2E_2^{-1}C_2^T$ are invertible, A is invertible.

Proof. By elementary calculation, it is easily proved.

LEMMA 3.2. *For sufficiently small h, $\tilde{A}$ is invertible. i.e.,*

$$\|\tilde{A}^{-1}\| < C \quad (C > 0) \tag{3.5}$$

Proof. It's sufficient to show that A is invertible. We find T_s in order to make A to be symmetric. i.e, $A = A_s T_s^{-1}$, A_s is symmetric.

$$A_s = \begin{pmatrix} E_1 C_1^T\, 0 \\ C_1\ B\ C_2 \\ 0\ C_2^T E_2 \end{pmatrix}, \quad T_s = \begin{pmatrix} D_1 & 0 & 0 \\ 0 & I_{n-6} & 0 \\ 0 & 0 & D_2 \end{pmatrix},$$

where

$$E_1 = \begin{pmatrix} \frac{128}{31} & \frac{36084}{221} & 54 \\ \frac{36084}{221} & \frac{5054060021}{781456} & \frac{17004627}{7072} \\ 54 & \frac{17004627}{7072} & \frac{18794635711}{2489536} \end{pmatrix},$$

$$E_2 = \begin{pmatrix} \frac{18794635711}{2489536} & \frac{17004627}{7072} & 54 \\ \frac{17004627}{7072} & \frac{5054060021}{781456} & \frac{36084}{221} \\ 54 & \frac{36084}{221} & \frac{128}{31} \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 1 & 54 & 135 \\ 0 & 1 & 54 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 54 & 1 & 0 \\ 135 & 54 & 1 \end{pmatrix},$$

$$B = \begin{pmatrix} -380 & & & \\ 135 & -380 & & \\ 54 & 135 & -380 & \\ 1 & 54 & 135 & -380 \\ \ddots & \ddots & \ddots & \ddots \\ 1 & 54 & 135 & -380 \end{pmatrix},$$

$$D_1 = \begin{pmatrix} 1 & \frac{594441}{14144} & \frac{394584973}{4979072} \\ 0 & 1 & \frac{2103441}{77798} \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and}$$

$$D_2 = \begin{pmatrix} 1 & 0 & 0 \\ \frac{2103441}{77798} & 1 & 0 \\ \frac{394584973}{4979072} & \frac{594441}{14144} & 1 \end{pmatrix}.$$

By Lemma 3.1, we only have to show $\tilde{B} = B - C_1E_1^{-1}C_1^T - C_2E_2^{-1}C_2^T$ is invertible. By algebraic calculation, we get

$$\tilde{B} = \begin{pmatrix} -a & & & & & & \\ b & -c & & & & & \\ d & e & f & & & & \\ 1 & 54 & 135 & -380 & & & \\ \ddots & \ddots & \ddots & \ddots & & & \\ 1 & 54 & 135 & -380 & & & \\ & 1 & 54 & 135 & f & & \\ & & 1 & 54 & e & -c & \\ & & & 1 & d & b & -a \end{pmatrix},$$

where $a = \frac{13411083287829760571}{220716555992441 60}$, $b = \frac{500095918664211}{3120992024780}$, $c = \frac{598409390110321}{1560496012390}$, $d = \frac{2202361839}{40116610}$, $e = \frac{2705573556}{20058305}$ and $f = -\frac{7622233698}{20058305}$.

$\tilde{B}$ is the same type as A_1 mentioned in [1] and [2]. Hence $\tilde{B}$ is invertible by the same argument as that mentioned in [1] and [2].

THEOREM 3.3. *For $y \in C^{10}[0,1]$, the discretization error E of the scheme for BVP provides a convergence of order 9, that is, $\|E\| \leq Ch^9$, $C > 0$.*

REMARK 3.4. Considering the off-node approximation that Bhatta and Sastri used as before, a global eighth order approximation of the solution is obtained, since an interpolatary heptic spline is used as the global approximation of solution $y(x)$.

REMARK 3.5. We can get the better approximation up to $O(h^{11})$ at x_2 when we use

$$(3.6)\quad \begin{aligned} &-\tau_0 - \frac{97}{31}\tau_1 + \frac{446}{31}\tau_2 - \frac{446}{31}\tau_3 + \frac{97}{31}\tau_4 + \tau_5 \\ &= \frac{h^2}{465}(-23s_0 - 665s_1 - 1670s_2 + 1670s_3 + 665s_4 + 23s_5) \end{aligned}$$

instead of (2.5), but the global order of convergence is still 8.

REMARK 3.6. We can get the better approximation up to $O(h^{13})$ at x_3 by getting the new coefficients a_i and b_i instead of (2.2) in $i = 3$.

Similarly, Remark 3.5 and 3.6 are true at x_{N-2} and x_{N-3}, respectively.

4. The numerical scheme using splines of degrees $2s$ and $2s-1$

In this section, we may consider more general scheme using a spline of degree $2s$ coupled with a spline of degree $2s-1 (s \geq 4)$. We assume the continuity conditions of the form

$$(4.1)\quad \begin{aligned} &a_1\tau_{j-s+1} + a_2\tau_{j-s+2} + \cdots + a_{2s_1}\tau_{j+s-1} \\ &= h^2(b_1 s_{j-s+1} + \cdots + b_{2s-1}s_{j+s-1}),\ j = s, \ldots, N-s \end{aligned}$$

We need $2s-2$ extra equations for the complete determination of $N-1$ unknowns. Just as section 2, we try to get the values a_i and b_i to make truncation error $t_j(h)$ up to $O(h^{\gamma_j})$ by using Taylor's expansion. The values γ_j will be of the form

$$(4.2)\qquad \gamma_j = \begin{cases} 2s + 2j - 1 & ,j = 1, \ldots, s-1, \\ 2s + 2(N+1-j) - 1 & ,j = N-s+2, \ldots, N \end{cases}$$

REMARK 4.1. According to the (4.2), the discretization error is $O(h^{2s+1})$.

REMARK 4.2. Since an interpolatary spline of degree $2s-1$ is used as the global approximation of the solution $y(x)$, the norm of the off-node approximation error is $O(h^{2s})$. So, a global $2s$-th order of convergence is expected.

5. Numerical Experiments

The theoretically estimated ninth order convergence in discretization error of the scheme is numerically verified with examples.

Example 1.

$$\begin{cases} \dfrac{d^2y}{dx^2} = y - 4x\exp(x), \quad 0 \le x \le 1, \\ y(0) = y(1) = 0, \text{ exact solution } y(x) = x(1-x)\exp(x). \end{cases}$$

Example 2.

$$\begin{cases} x^2\dfrac{d^2y}{dx^2} = 2y - x, \quad 2 \le x \le 3, \\ y(2) = 0, y(3) = 0, \text{ exact solution } y(x) = (7x - 23x^2 + \dfrac{156}{x})/14. \end{cases}$$

The approximation values of the solutions are computed at the nodes using the scheme in section 2. In Table 5.1, notation $\|E\|$ is the maximum absolute error at the nodes. All computations were computed on SUN 4 SPARC station 1+ with double precision.

Example 1			Example 2		
h	$\|E\|$	order	h	$\|E\|$	order
1/8	1.923(-9)		1/8	1.157(-9)	
1/16	5.067(-12)	8.57	1/16	4.685(-12)	7.95
1/24	1.427(-13)	8.80	1/24	1.575(-13)	8.37
1/32	1.367(-14)	8.15	1/32	1.385(-14)	8.45

Table 5.1

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