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THE INVARIANCE PRINCIPLE FOR LINEARLY POSITIVE QUADRANT DEPENDENT SEQUENCES

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1. Introduction

A sequence $\{X_j : j \geq 1\}$ of random variables is said to be pairwise positive quadrant dependent (pairwise PQD) if for any real r_i, r_j and $i \neq j$

$$P\{X_i > r_i, X_j > r_j\} \geq P\{X_i > r_i\}P\{X_j > r_j\}$$

(see [8]) and a sequence $\{X_j : j \geq 1\}$ of random variables is said to be associated if for any finite collection $\{X_{j(1)}, \dots, X_{j(n)}\}$ and any real coordinatewise nondecreasing functions f, g on R^n

$$\operatorname{Cov}(f(X_{j(1)}, \dots, X_{j(n)}), g(X_{j(1)}, \dots, X_{j(n)})) \geq 0,$$

whenever the covariance is defined (see [6]). Instead of association Cox and Grimmett's [4] original central limit theorem requires only that positively linear combination of random variables are PQD (cf. Theorem A*).

A sequence $\{X_j : j \geq 1\}$ of random variables is said to be linearly positive quadrant dependent (LPQD) if for any disjoint A, B and positive $r_j's$ $\sum_{i \in A} r_i X_i$ and $\sum_{j \in B} r_j X_j$ are PQD (see [9]). Let us remark this concept of dependence is between pairwise PQD and association and it is well known. (see, for example, [8]) that neither pairwise PQD nor LPQD nor association implies the other. Using the coefficient of maximal covariances

$$u(n) = \sup_{k \geq 1} \sum_{j : |j-k| \geq n} \operatorname{Cov}(X_j, X_k)$$

Birkel [3] proved the central limit theorem and the invariance principle for LPQD sequences:

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THEOREM A*(BIRKEL, 1988). *Let* $\{X_j : j \geq 1\}$ *be an LPQD sequence with* $EX_j = 0, EX_j^2 < \infty$. *Assume*

$$(1.1) \qquad u(n) \longrightarrow_n 0, u(1) < \infty,$$

$$(1.2) \qquad \sigma_n^{-2} \sum EX_j^2 1_{\{|X_j| \geq \varepsilon\sigma_n\}} \to_n 0 \text{ for } \varepsilon > 0,$$

$$(1.3) \qquad inf_{n\geq 1} n^{-1}\sigma_n^2 > 0.$$

Then $\{X_j : j \geq 1\}$ *fulfills the central limit theorem.*

THEOREM B*(BIRKEL, 1993). *Let* $\{X_j : j \geq 1\}$ *be an LPQD sequence with* $EX_j = 0$. *Assume that*

$$(1.4) \qquad \sigma_n^{-2}\sigma_{nk}^2 \to_n k \text{ for } k \geq 1,$$

$$(1.5) \qquad u(n) = 0(n^{-\rho}) \text{ for some } \rho > 0,$$

$$(1.6) \qquad \sup E|X_j|^{2+\delta} < \infty \text{ for some } \delta > 0$$

Then $\{X_j : j \geq 1\}$ *fulfills the invariance principle, that is,*

$$W_n(t) = \sigma_n^{-1} S_{[nt]}, \quad t \in [0, 1],$$

converges weakly to standard Brownian motion W *on the set of all functions on* $[0, 1]$ *which have left hand limits and are continuous from the right.*

It is the purpose of this note to show that Theorems 1 and 2 of Birkel [2] still holds for LPQD sequences and thus to weaken (1.5) and (1.6). All results are stated in Section 2. The proofs of our theorems as well as some lemmas are given in Section 3.

2. Results

THEOREM 2.1. *Let* $\{X_j : j \geq 1\}$ *be an LPQD sequence with* $EX_j = 0, EX_j^2 < \infty$. *Assume*

$$(2.1) \qquad \sigma_n^{-2} E(S_{nk}S_{nl}) \longrightarrow_n \min\{k, l\} \text{ for } k, l \geq 1,$$

$$(2.2) \qquad \{\sigma_n^{-2}(S_{n+m} - S_m)^2 : m \geq 0, n \geq 1\} \text{ is uniformly integrable.}$$

Then $\{X_j : j \geq 1\}$ *fulfills the invariance principle.*

THEOREM 2.2. *Let* $\{X_j : j \geq 1\}$ *be an LPQD sequence with* $EX_j = 0, EX_j^2 < \infty$. *Then the following assertion are equivalent:*
(*i*) *Condition* (2.1) *is fulfilled and*

$$\{X_j : j \geq 1\} \text{ satisfies the central limit theorem,} \tag{2.3}$$

(*ii*)$\{X_j : j \geq 1\}$ *fulfills the invariance principle.*

COROLLARY 2.3. *Let* $\{X_j : j \geq 1\}$ *be an LPQD sequence with* $EX_j = 0, EX_j^2 < \infty$. *Assume that* (1.1), (1.2), (1.3), *and* (1.4) *hold. Then* $\{X_j : j \geq 1\}$ *fulfills the invariance principle.*

Proof. According to Theorem A* of Birkel [3], $\{X_j : j \geq 1\}$ fulfills the central limit theorem. Hence, by Theorem 2.1, it suffices to prove (2.1). It is easy to see that Lemmas 1 and 2 of Birkel [2] still hold for random variables which are nonnegatively correlated. Hence since (1.4) is fulfilled, (2.1) follows from

$$\sigma_n^{-2} E((S_{nj} - S_{ni})(S_{nl} - S_{nk})) \to_n 0 \text{ for } l \geq k \geq j \geq i \geq 0. \tag{2.4}$$

according to Lemma 2 of Birkel [2]. But (2.4) is a simple consequence of the estimate

$$0 \leq \sigma_n^{-2} E((S_{nj} - S_{ni})(S_{nl} - S_{nk})) \leq \sigma_n^{-2} \sum_{v=1}^{n(j-i)} u(v),$$

and the assumptions (1.1) and (1.3).

COROLLARY 2.4. *Let* $\{X_j : j \geq 1\}$ *be an LPQD sequence with* $EX_j = 0, EX_j^2 < \infty$. *Assume that* (1.1), (1.2) *and the following condition* (2.5) *hold.*

$$n^{-1}\sigma_n^2 \to_n \sigma^2 \varepsilon(0, \infty) \tag{2.5}$$

Then $\{X_j : j \geq 1\}$ *fulfills the invariance principle.*

Proof. Since (2.5) implies (1.3) and (1.4) $\{X_j : j \geq 1\}$ fulfills the invariance principle according to Corollary 2.3.

COROLLARY 2.5. *Let $\{X_j : j \geq 1\}$ be a wide sense stationary LPQD sequence with $EX_j = 0, EX_j^2 < \infty$. Assume that (1.2) and the following condition (2.6) hold.*

$$0 < \sigma^2 = Cov(X_1, X_1) + 2\sum_{j=2}^{\infty} Cov(X_1, X_j) < \infty. \tag{2.6}$$

Then $\{X_j : j \geq 1\}$ fulfills the invariance principle.

Proof. If $\{X_j : j \geq 1\}$ is stationary in the wide sense, condition (2.6) implies (1.1) and (2.5). Hence $\{X_j : j \geq 1\}$ fulfills the invariance principle according to Corollary 2.3.

3. Proof

The following lemma is a generalization of Theorem 2 of Newman and Wright [10] and will be used to provide the tightness needed for our invariance principle.

LEMMA 3.1. *Let $\{X_j : j \geq 1\}$ be an LPQD sequence with $EX_j = 0, EX_j^2 < \infty$. Define for $n \geq 1, m \geq 0$,*

$$S_{m,n} = S_{n+m} - S_m \text{ and } M_{m,n} = \max(S_{m,1}, \ldots, S_{m,n}).$$

Then

$$E(M_{m,n}^2) \leq E(S_{m,n}^2). \tag{3.1}$$

Proof. This lemma can be proved along the lines of the proof of Theorem 2 of Newman and Wright [10].
We next define for $n \geq 1, m \geq 0$,

$$S_{m,n}^* = \max(0, S_{m,1}, S_{m,2}, \ldots, S_{m,n}), \quad s_{m,n}^2 = E(S_{m,n}^2).$$

LEMMA 3.2. *Let $\{X_j : j \geq 1\}$ be an LPQD sequence with $EX_j = 0, EX_j^2 < \infty$. Then for $\lambda_2 > \lambda_1 > 0$,*

$$P(S^*_{m,n} \leq \lambda_2) \leq (1 - \frac{s^2_{m,n}}{(\lambda_2 - \lambda_1)^2})^{-1} P(S_{m,n} \leq \lambda_1) \tag{3.2}$$

$$P(\max_{1 \leq j \leq n} |S_{m,j}| \geq \lambda s_{m,n}) \geq 2P(|S_{m,n}| \leq (\lambda - \sqrt{2}) s_{m,n}) \tag{3.3}$$

Proof. For $\lambda_1 < \lambda_2$,

(3.4)

$$\begin{aligned} &P(S^*_{m,n} \geq \lambda_2) \\ &\quad \leq P(S_{m,n} \geq \lambda_1) + P(S^*_{m,n-1} \geq \lambda_2, S^*_{m,n-1} - S_{m,n} \geq \lambda_2 - \lambda_1) \\ &\quad \leq P(S_{m,n} \geq \lambda_1) + P(S^*_{m,n-1} \geq \lambda_2) P(S^*_{m,n-1} - S_{m,n} \geq \lambda_2 - \lambda_1) \\ &\quad \leq P(S_{m,n} \geq \lambda_1) + P(S^*_{m,n} \geq \lambda_2) \frac{E((S^*_{m,n-1} - S_{m,n})^2)}{(\lambda_2 - \lambda_1)^2} \end{aligned}$$

where the second inequality follows from the fact that $S^*_{m,n-1}$ and $S_{m,n} - S^*_{m,n-1}$ are PQD random variables since the X_j's are LPQD random variables and the third inequality follows from the Chebyshev's inequality. Now Lemma 3.1 with X_{i+m} replaced by $Y_{i+m} = -X_{n-i+1+m}$ yields that

$$\begin{aligned} &E([S^*_{m,n-1} - S_{m,n}]^2) \\ &\quad = E([\max(Y_{1+m}, Y_{1+m} + Y_{2+m}, \ldots, Y_{1+m} + Y_{2+m} + \cdots + Y_{n+m})]^2) \\ &\quad \leq E(S^2_{m,n}) = s^2_{m,n}. \end{aligned}$$

which together with (3.4) yields (3.2) for $(\lambda_2 - \lambda_1)^2 \geq s^2_{m,n}$. By adding to (3.2) the analogous inequality with each X_{i+m} replaced by $-X_{i+m}$ in (3.2), and by choosing $\lambda_2 = \lambda s_{m,n}, \lambda_1 = (\lambda - \sqrt{2}) s_{m,n}$, (3.3) will be obtained.

Proof of Theorem 2.1. Condition (2.1) implies (1.4) and Lemmas 1 and 2 of Birkel [2] still hold for random variables which are nonnegatively correlated.

Hence we obtained

$$\sigma_n^{-2} \sigma^2_{[nt]} \to_n t \text{ for } t > 0, \tag{3.5}$$

$$(3.6) \qquad \sigma_n^{-2} E\{(S_{[nt]} - S_{[ns]})(S_{[nv]} - S_{[nu]})\} \to_n 0,$$

$$\text{for } 0 \le s \le t \le u \le v.$$

Let X be a limit in distribution of a subsequence of $\{W_n : n \ge 1\}$. First we show that X is distributed like W. By (2.2) and (3.5) $\{W_n : n \ge 1\}$ and $\{W_n^2(t) : n \ge 1\}$ are uniformly integrable for every $t \in [0,1]$. As

$$W_n(t) \to_n X(t), \quad W_n^2(t) \to_n X^2(t)$$

in distribution (for a subsequence), Theorem 5.4 of Billingsley [1] and (3.5) imply

$$EX(t) = 0, \quad EX^2(t) = t.$$

According to Theorem 19.1 of Billingsley[1], X is distributed like W if X has independent increments, that is,

$$(3.7) \qquad X(t_1) - X(t_0), \ldots, X(t_k) - X(t_{k-1})$$

are independent for all $k \ge 1, 0 \le t_0 \le t_1 \le \cdots \le t_k = 1$.
To show (3.7), put

$$U_{ni} = W_n(t_i) - W_n(t_{i-1}), \ 1 \le i \le k.$$

Then the U_{ni} are LPQD random variables, fulfilling

$$(U_{n1}, \ldots, U_{nk}) \to_n (X(t_1) - X(t_2), \ldots, X(t_k) - X(t_{k-1}))$$

in distribution (for subsequence). By Lemma 4 of Birkel [3] the $X(t_i) - X(t_{i-1})$ are LPQD. Using Theorem 5.4 of Billingsley[1] and (3.6), we get, for $i \ne j$,

$$\mathrm{Cov}(X(t_i) - X(t_{i-1}), X(t_j) - X(t_{j-1})) = \lim_{n \ge 1} \mathrm{Cov}(U_{ni}, U_{nj}) = 0.$$

Applying (3.3) to the random variables involved in Theorem 2.1, we have, for $\lambda > 2\sqrt{2}$,

$$(3.8) \quad P\{\max_{i \le n} |S_{i+m} - S_m| \ge \lambda s_{m,n}\} \ge 2P\{|S_{m+n} - S_m| \ge \frac{1}{2}\lambda s_{m,n}\}.$$

(2.2), (3.8) and Theorem 8.4 of Billingsley [1] yield the tightness of the sequence $\{W_n : n \geq 1\}$ and $P\{X \in C[0,1]\} = 1$ by Theorem 15.5 of Billingsley [1]. Thus the proof of Theorem 2.1 is complete.

Proof of Theorem 2.2. (*i*) ⇒ (*ii*). Like in the proof of Theorem 2.1 we obtain relations (3.5) and (3.6). From (2.3) and (3.5) it follows for $t > 0$

$$\sigma_n^{-1} S_{[nt]} \longrightarrow_n N(0,t) \text{ in distribution.} \tag{3.9}$$

We will prove for $0 < s < t$,

$$\sigma_n^{-1}(S_{[nt]} - S_{[ns]}) \longrightarrow_n N(0, t-s) \text{ in distribution.} \tag{3.10}$$

To see (3.10): Let $0 < s < t$ be given. Then the sequence

$$\{(\sigma_n^{-1} S_{[ns]}, \sigma_n^{-1} S_{[nt]}) : n \geq 1\}$$

is tight ([1, p.41 problem 6]). Let Q be a probability measure on the Borel $-\sigma$ algebra of R^2 such that for a subsequence

$$(\sigma_n^{-1} S_{[ns]}, \sigma_n^{-1} S_{[nt]}) \longrightarrow_n Q \text{ in distribution.}$$

Then we have

$$(\sigma_n^{-1} S_{[ns]}, \sigma_n^{-1}(S_{[nt]} - S_{[ns]})) \longrightarrow_n Q(\pi_1, \pi_2 - \pi_1)^{-1} \text{ in distribution,}$$

where $\pi_i : R^2 \to R, i = 1, 2$, are the natural projections. Since the random variables $W_n(s)$ and $W_n(t) - W_n(s)$ are PQD by the definition of LPQD. Lemma 4 of [3] implies that π_1 and $\pi_2 - \pi_1$ are PQD with respect to Q. According to (3.9), the sets

$$\{W_n(s) : n \geq 1\}, \{W_n(t) : n \geq 1\} \text{ and } \{W_n(s)W_n(t) : n \geq 1\}$$

are uniformly integrable. Hence, using Theorem 5.4 of Billingsley [1] and (3.6) we obtain

$$\mathrm{Cov}(\pi_1, \pi_2 - \pi_1) = \lim_{n \geq 1} \mathrm{Cov}(\sigma_n^{-1} S_{[ns]}, \sigma_n^{-1}(S_{[nt]} - S_{[ns]})) = 0.$$

As uncorrelated, LPQD random variables, π_1 and $\pi_2 - \pi_1$ are Q-independent. Since $Q\pi_1^{-1} = N(0,s), Q\pi_2^{-1} = N(0,t)$, this proves (3.10). (3.8), (3.10) and Theorem 8.4 of [1] yield by standard argument the needed tightness of the distribution of the $W_n's$ to obtain the desired convergence in distribution. (see the proof Theorem 10.1 of Billingsley [1])

$(ii) \Rightarrow (i)$. It suffices to see that condition (2.1) holds. Assume that the invariance principle is fulfilled. By Remark 2.3 of Herrndorf [7] it follows that $\sigma_n^{-2}\sigma_{nk}^2 \longrightarrow_n k$, for $k \geq 1$. Since Lemmas 1 and 2 of Birkel [2] hold for random variables which are nonnegative correlated random variables it remains to prove

(*)
$$\sigma_n^{-2}E((S_{[nt]} - S_{[ns]})(S_{[nv]} - S_{[nu]})) \to_n 0 \text{ for } 0 \leq s \leq t \leq u \leq v \leq 1.$$

To prove (*): Let $0 \leq s \leq t \leq u \leq 1$ be given. Since the invariance principle is fulfilled, $\{\sigma_n^{-2}S_n^2 : n \geq 1\}$ is uniformly integrable, according to Lemma 1 of Birkel [2]. As

$$(W_n(t) - W_n(s), W_n(v) - W_n(u)) \to_n (W(t) - W(s), W(v) - W(u)) \quad \text{in distribution,}$$

it follows that $\sigma_n^{-2}(S_{[nt]} - S_{[ns]})(S_{[nv]} - S_{[nu]}) \to_n (W(t) - W(s))(W(v) - W(u))$ in distribution. According to Theorem 5.4 of Billingsley [1],

$$\sigma_n^{-2}E((S_{[nt]} - S_{[ns]})(S_{[nv]} - S_{[nu]})) \to_n E((W(t) - W(s))(W(v) - W(u))).$$

But

$$E((W(t) - W(s))(W(v) - W(u))) = E(W(t) - W(s))E(W(v) - W(u)) = 0,$$

which proves (*).

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