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ON THE HOMOLOGY OF THE MODULI SPACE OF G_2 INSTANTONS

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Introduction

Let $\pi : P \to S^4$ be a principal G-bundle over S^4 whose the structure group G is a compact, connected, simple Lie group. Since $\pi_3(G) = \pi_4(BG) = Z$, we can classify the principal bundle P_k over S^4 by the map $S^4 \to BG$ of degree k. Atiyah and Jones [2] showed that $\mathcal{C}_k = \mathcal{A}_k/\mathcal{G}_k^b$ is homotopy equivalent to $\Omega_k^3 G \simeq \Omega_k^4 BG$ where $\mathcal{A}_k$ is the space of the all connections in P_k and $\mathcal{G}_k^b$ is the based gauge group which consists of all base point preserving automorphisms on P_k. Here, $\Omega^n X$ is the space of all base-point preserving continuous map from S^n to X. Let $\mathcal{M}_k$ be the space of based gauge equivalence classes of all connections in P_k satisfying the Yang -Mills self-duality equations, which we call the moduli space of G instantons.

Then there is a natural inclusion map $i_k : \mathcal{M}_k \to \mathcal{C}_k$. While $\mathcal{C}_k \simeq \Omega_k^3 G$ is homotopy equivalent to $\Omega_0^3 G$ for any component k, each $\mathcal{M}_k$ is not homotopy equivalent to each other generally. In fact, the dimension of the space of instantons $\mathcal{M}_k$ increases as k increases. Moreover the inclusion map i: $\mathcal{M}_\infty \to \mathcal{C}_\infty$ induces a homotopy equivalence [4] where $\mathcal{M}_\infty$ and $\mathcal{C}_\infty$ are the direct limits under the inclusions. Especially $\mathcal{M} = \coprod_{k>0} \mathcal{M}_k$ has additional structures [3], that is, it behaves like four fold loop space enough to define the homology operations Q_1, Q_2, Q_3 up to homotopy.

In this paper we first compute the homology of $\Omega_0^3 G_2$, that is, $\mathcal{M}_\infty$, the direct limit of $\mathcal{M}_k$ for G_2. After then we study its gauge group and the $\mathcal{M}_1$.

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1. The homology of $\Omega_0^3 G_2$

Let $E(x)$ be the exterior algebra on x, $P(x)$ be the polynomial algebra on x and $\Gamma(x)$ be the divided power algebra on x which is free over $\gamma_i(x)$ with the product $\gamma_i(x)\gamma_j = \binom{i+j}{j}\gamma_{i+j}$. Throughout this paper, the subscript of an element always means the degree of an element, for example the degree of a_i is i. For a $(n+1)$-fold loop space, there are homology operations,

$$Q_{i(p-1)} : H_q(\Omega^{n+1}X; F_p) \longrightarrow H_{pq+i(p-1)}(\Omega^{n+1}X; F_p)$$

defined for $0 \leq i \leq n$ when $p = 2$ and for $0 \leq i \leq n$, $i \equiv q$ for mod 2 when p is an odd prime which is natural for a $(n+1)$-fold loop space. Let Q_i^a be the iterated operation $Q_i \cdots Q_i$(a times) and β be the mod p Bockstein operation.

We recall the definition of the exceptional Lie group G_2. There are only four division algebras over R, the real numbers R, complex numbers C, the quaternions H and the Cayley numbers K. K is R^8 as a vector space and it is the non-associative algebra. Then G_2 is the group of automorphisms of K. In fact G_2 is a closed subgroup of $O(7)$. Hence it is a compact Lie group. By the Cartan-Killing classification we call G_2 the exceptional Lie group of type (3,11). First we have the following well-known facts.

THEOREM 1.1.

$$\begin{aligned} H^*(G_2; F_2) &= E(x_5) \otimes P(x_3)/(x_3^4) \\ Sq^2 x_3 &= x_5 \\ H^*(G_2; F_p) &= E(x_3, x_{11}) \quad \textit{for odd prime } p. \end{aligned}$$

The group G_2, as a subgroup of $O(7)$, acts on S^7. The action is transitive and the isotropy group is $SU(3)$[5, App A]. So we have the following fibration

$$SU(3) \longrightarrow G_2 \longrightarrow S^6.$$

First we compute $H_*(\Omega^2 G_2; F_p)$ and next $H_*(\Omega_0^3 G_2; F_p)$.

THEOREM 1.2.

$$
\begin{aligned}
H_*(\Omega^2 G_2; F_2) &= E(z_1) \otimes P(z_6) \\
&\quad \otimes P(Q_1^a z_7) : a \geq 0) \otimes P(Q_1^a z_9 : a \geq 0). \\
H_*(\Omega^2 G_2; F_P) &= E(Q_{p-1}^a z_1 : a \geq 0) \otimes P(\beta Q_{p-1}^a z_1 : a > 0) \\
&\quad \otimes E(Q_{p-1}^a z_9 : a \geq 0) \otimes P(\beta Q_{p-1}^a z_9 : a > 0) \\
&\quad \textit{for odd prime } p.
\end{aligned}
$$

Proof. Consider the following path loop fibration

$$\Omega G_2 \longrightarrow * \longrightarrow G_2.$$

By the Eilenberg–Moore spectral sequence converging to $H^*(\Omega G_2; F_2)$, as a Hopf algebra we have

$$
\begin{aligned}
E_2 &= \mathrm{Tor}_{H^*(G_2;F_2)}(F_2, F_2) \\
&= \mathrm{Tor}_{P(x_3)/(x_3^4)\otimes E(x_5)}(F_2, F_2) \\
&= \mathrm{Tor}_{P(x_3)/(x_3^4)}(F_2, F_2) \otimes \mathrm{Tor}_{E(x_5)}(F_2, F_2) \\
&= E(y_2) \otimes \Gamma(y_{10}) \otimes \Gamma(y_4).
\end{aligned}
$$

Since the E_2 term concentrates on even dimensions, the spectral sequence collapses from E_2. Hence $E_2 = E_\infty$. Since the E_∞ term is the associated bigraded algebra from the some filtrations of $H^*(\Omega G_2; F_2)$, we should determine the algebra structure on $H^*(\Omega G_2; F_2)$ hiden in E_∞ by the filtration. Moreover the Eilenberg-Moore spectral sequence preserve the steenrod actions. So from the fact that $Sq^2 x_3 = x_5$, we have $Sq^2 y_2 = y_4$, that is, $y_2^2 = y_4$. From this, we can solve the algebra extension problem and we get

$$H^*(\Omega G_2; F_2) = P(y_2)/(y_2^4) \otimes \Gamma(y_{10}) \otimes \Gamma(y_8).$$

Note that as a algebra

$$\Gamma(x) = \otimes_{i \geq 0} E(\gamma_{2^i}(x)).$$

We use the Eilenberg–Moore spectral sequence again covering to $H_*(\Omega^2 G_2; F_2)$ with

$$\begin{aligned} E_2 &= \mathrm{Ext}_{H^*(\Omega G_2;F_2)}(F_2, F_2) \\ &= \mathrm{Ext}_{P(y_2)/(y_2^4)\otimes\otimes_{i\geq 0}E(\gamma_{2^i}(y_8))\otimes_{i\geq 0}E(\gamma_{2^i}(y_{10}))}(F_2, F_2) \\ &= E(z_1)\otimes P(z_6) \\ &\qquad \otimes P(Q_1^a z_7) : a \geq 0) \otimes P(Q_1^a z_9 : a \geq 0). \end{aligned}$$

Note that the source of the first differential is an indecomposable element and the target is an primitive element. Checking the bidegrees of all generators and primitive elements, there is no differential from E_2. Hence the Eilenberg–Moore spectral sequence collapses from E_2. So we get $H_*(\Omega^2 G_2; F_2)$. Now we turn to the odd prime case. Like the mod 2 case, we use the Eilenberg–Moore spectral sequence converging to $H^*(\Omega G_2; F_p)$ with

$$\begin{aligned} E_2 &= \mathrm{Tor}_{H^*(G_2;F_p)}(F_p, F_p) \\ &= \mathrm{Tor}_{E(x_3\otimes E(x_{10})}(F_p, F_p) \\ &= \Gamma(y_2)\otimes\Gamma(y_{10}). \end{aligned}$$

Since all elements in E_2 are even dimensional, the spectral sequence collapses from E_2 and we get

$$H^*(\Omega G_2; F_p) = \Gamma(y_2)\otimes\Gamma(y_{10})$$

as a Hopf algebra. Note that as a algebra

$$\Gamma(x) = \otimes_{i\geq 0} P(\gamma_{p^i}(x))/(\gamma_{p^i}(x)^p).$$

We exploit the Eilenberg–Moore spectral sequence again converging to $H_*(\Omega^2 G_2; F_p)$ with

$$\begin{aligned} E_2 &= \mathrm{Ext}_{H^*(\Omega G_2;F_p)}(F_p, F_p) \\ &= \otimes_{i\geq 0}\mathrm{Ext}_{P(\gamma_{p^i}(y_2))/(\gamma_{p^i}(y_2)^p)\otimes_{i\geq 0}P(\gamma_{p^i}(y_{10}))/(\gamma_{p^i}(y_{10})^p)}(F_p, F_p) \\ &= E(Q_{p-1}^a z_1 : a \geq 0)\otimes P(\beta Q_{p-1}^a z_1) : a > 0) \\ &\qquad \otimes E(Q_{p-1}^a z_9 : a \geq 0)\otimes P(\beta Q_{p-1}^a z_9 : a > 0). \end{aligned}$$

By the bidegree reason as mod 2 case, the spectral sequence collapses from the E_2-term and we get the conclusions.

Now we cite the following theorem for next step. Since $\pi_3(G) = Z$ for a compact, connected, simple Lie group G, $\pi_0(\Omega^3 G) = Z$. Let $\Omega^3_0 G$ be the zero component of $\Omega^3 G$.

THEOREM 1.3 [3, THM 4.17].

$$\begin{aligned}
H_*(\Omega^3_0 SU(3); F_2) = {} & P(Q_2^a(Q_2[1] * [-2]) : a \geq 0) \\
& \otimes P(Q_1^a Q_3^b v_5 : a \geq 0, b \geq 0) \\
H_*(\Omega^3_0 SU(3); F_p) = {} & P(Q^a_{2(p-1)}(Q_{2(p-1)}[1] * [-p]) : a \geq 0) \\
& \otimes E(Q^a_{p-1} \beta Q^b_{2(p-1)}(Q_{2(p-1)}[1] * [-p]) : \\
& \qquad a \geq 0,\ b \geq 0) \\
& \otimes P(\beta Q^a_{p-1} \beta Q^b_{2(p-1)}(Q_{2(p-1)}[1] * [-p]) : \\
& \qquad a > 0,\ b \geq 0) \\
& \otimes P(Q^a_{2(p-1)} u_2 : a \geq 0) \\
& \otimes E(Q^a_{p-1} \beta Q^b_{2(p-1)}(u_2) : a \geq 0, b > 0) \\
& \otimes P(\beta Q^a_{p-1} \beta Q^b_{2(p-1)} u_2 : a > 0, b > 0) \\
& \text{for odd prime } p.
\end{aligned}$$

THEOREM 1.4.

$$\begin{aligned}
H_*(\Omega^3_0 G_2; F_2) = {} & P(Q_1^a w_5 : a \geq 0) \otimes P(Q_1^a Q_2^b w_6) : a \geq 0, b \geq 0) \\
& \otimes P(Q_1^a Q_2^b w_8 : a \geq 0, b \geq 0). \\
H_*(\Omega^3_0 G_2; F_p) = {} & P(Q^a_{2(p-1)}(Q_{2(p-1)}[1] * [-p]) : a \geq 0) \\
& \otimes E(Q^a_{p-1} \beta Q^b_{2(p-1)}(Q_{2(p-1)}[1] * [-p]) : \\
& \qquad a \geq 0,\ b \geq 0) \\
& \otimes P(\beta Q^a_{p-1} \beta Q^b_{2(p-1)}(Q_{2(p-1)}[1] * [-p]) : \\
& \qquad a > 0,\ b \geq 0) \\
& \otimes P(Q^a_{2(p-1)}(w_8) : a \geq 0)
\end{aligned}$$

$$\otimes E(Q^a_{p-1}\beta Q^b_{2(p-1)}(w_8) : a \geq 0, b > 0)$$
$$\otimes P(\beta Q^a_{p-1}\beta Q^b_{2(p-1)}(w_8) : a > 0, b > 0)$$
for odd prime p.

Proof. Consider the following fibration:

$$\Omega^3 G_2 \longrightarrow * \longrightarrow \Omega^2 G_2.$$

Consider the Eilenberg-Moore converging to $H_*(\Omega^3_0 G_2; F_P)$ with

$$\begin{aligned}
E_2 &= \mathrm{Cotor}_{H_*(\Omega^2 G_2;F_2)}(F_2, F_2) \\
&= P(Q^a_1 w_5 : a \geq 0) \otimes P(Q^a_1 Q^b_2 w_6) : a \geq 0, b \geq 0) \\
&\quad \otimes P(Q^a_1 Q^b_2 w_8 : a \geq 0, b \geq 0). \\
E_2 &= \mathrm{Cotor}_{H_*(\Omega^2 G_2;F_p)}(F_p, F_p) \\
&= P(Q^a_{2(p-1)}(Q_{2(p-1)}[1] * [-p]) : a \geq 0) \\
&\quad \otimes E(Q^a_{p-1}\beta Q^b_{2(p-1)}(Q_{2(p-1)}[1] * [-p]) : a \geq 0, b \geq 0) \\
&\quad \otimes P(\beta Q^a_{p-1}\beta Q^b_{2(p-1)}(Q_{2(p-1)}[1] * [-p]) : a > 0, b \geq 0) \\
&\quad \otimes P(Q^a_{2(p-1)}(w_8) : a \geq 0) \\
&\quad \otimes E(Q^a_{p-1}\beta Q^b_{2(p-1)}(w_8) : a \geq 0, b > 0) \\
&\quad \otimes P(\beta Q^a_{p-1}\beta Q^b_{2(p-1)}(w_8) : a > 0, b > 0) \\
&\quad \text{for odd prime } p.
\end{aligned}$$

From this stage we should compute the higher differentials for next stage until E_∞. So the size of the E_∞–term, i.e. the size of the $H_*(\Omega^3_0 G_2 : F_p)$ as a F_p module is less than equal to the size of the E_2–term. Consider another fibration:

$$\Omega^3_0 SU(3) \longrightarrow \Omega^3_0 G_2 \longrightarrow \Omega^3 S^6.$$

Recall that

$$H_*(\Omega^3 S^6; F_2) = P(Q^a_1 Q^b_2(w_3) : a \geq 0, b \geq 0).$$

$$\begin{aligned} H_*(\Omega^3 S^6; F_p) = E(Q^a_{p-1} w_3 : a \geq 0) \\ \otimes P(\beta Q^a_{p-1} w_3 : a > 0) \\ \otimes P(Q^a_{2(p-1)}(w_8) : a \geq 0) \\ \otimes E(Q^a_{p-1} \beta Q^b_{2(p-1)}(w_8) : a \geq 0, b > 0) \\ \otimes P(\beta Q^a_{p-1} \beta Q^b_{2(p-1)}(w_8) : a > 0, b > 0). \end{aligned}$$

Then we should have the first non-trivial differential from w_3 to the 2 dimensional primitive element in $H_*(\Omega^3_0 SU(3); F_p)$ because of the following reasons: If the differential from w_3 is trivial, then the Serre spectral sequence collapses from the E_2-term by the naturality of the Dyer–Lashof operation. But in that case the size of $H_*(\Omega^3_0 G_2; F_p)$ is strictly bigger than the possible maximum size of $H_*(\Omega^3_0 G_2; F_p)$ through the Eilenberg–Moore spectral sequence. This is a contradiction. So we have the non-trivial differential. For $p = 2$, we have the differential from w_3 to $Q_2[1] * [-2]$. By the naturality of the Dyer–Lashof operation, we have the differentials from $Q^b_0 Q^a_1 w_3$ to $Q^b_1 Q^a_2(Q_2[1] * [-2])$ for $a, b \geq 0$. Since $Q_1(Q_2[1] * [-2]) = 0$ and $Q_3(Q_2[1] * [-2]) = 0$ in $H_*(\Omega^3_0 SU(3); F_2)$ by the dimensional reason of the primitive element, we get that

$$\begin{aligned} E_\infty = P(Q^a_1 Q^b_3(u_5) : a, b \geq 0) \otimes P((Q^a_1 w^3)^2 : a \geq 0) \\ \otimes P(Q^a_1 Q^b_2 w_8 : a, b \geq 0). \end{aligned}$$

Note that $|(Q^a_1 w_3)^2| = |Q^a_2 w_6|$ and $|Q^a_1 Q^{b+1}_3 u_5| = |Q^{a+1}_1 Q^b_2 w_6|$. So this E_∞-term is the same size of the E_2-term of the previous Eilenberg–Moore spectral sequence. Hence the Eilenberg–Moore spectral sequence collapses from E_2 and we finish the mod 2 case. Now we turn to the odd prime case. As we already proved, we have the non-trivial differential from w_3. Since u_2 is the only 2-dimensional primitive element. The target of the differential should be u_2. Then by the naturality of the Dyer–Lashof operation, there are differentials from $Q^a_{p-1} w_3, a \geq 0$ to $Q^a_{2(p-1)} u_2, a \geq 0$, from $Q^a_0 \beta Q^b_{p-1} w_3, a \geq 0, b > 0$ to $Q^a_{p-1} \beta Q^b_{2(p-1)} u_2$, $a \geq 0$, $b > 0$ and from $(Q^a_0 \beta Q^b_{p-1} w_3)^{p-1} \otimes Q^a_{p-1} \beta Q_{2(p-1)} u_2, a \geq 0, b > 0$ to $\beta Q^{a+1}_{p-1} \beta Q^b_{2(p-1)} u_2$, $a \geq 0, b > 0$. Since there does not exist 7 dimensional primitive element in $H_*(\Omega^3_0 SU(3); F_p)$, there is no other differential. So we get the E_∞–term. Comparing the size of surviving elements in this E_∞–term with the E_2–term of previous the Eilenberg–Moore spectral sequence, we know that these are the same size and this fact means that the previous Eilenberg–Moore spectral sequence collapses from the E_2-term and we get the answer.

2. $\mathcal{M}_1$ and Gauge group

Let $\mathcal{G}_k$ be the Gauge group of the principle G_2 bundle P_k over S^4 with the instanton number k. From [1, Prop 2.4] we can get

$$B\mathcal{G}_k \simeq \mathrm{Map}_{p_k}(S^4, BG_2).$$

where the subscript P_k denotes the component of a map of M into BG_2 which induces P_k. Here $\simeq$ means the homotopy equivalence.

THEOREM 2.1.

$$H_*(B\mathcal{G}_k; F_p) = H_*(\Omega_0^3 G_2; F_p) \otimes H_*(BG_2; F_p)$$

as a algebra for odd prime p.

Proof. There is a fibration:

$$\mathrm{Map}^*(S^4, BG_2) \longrightarrow \mathrm{Map}(S^4, BG_2) \longrightarrow BG_2$$

where $*$ means the base point preserving maps. Since $\mathrm{Map}^*(S^4, BG_2) = \Omega_0^3 G_2 \times Z$, we have the following fibration:

$$\Omega_0^3 G_2 \longrightarrow \mathrm{Map}_{P_k}(S^4, BG_2) \longrightarrow BG_2.$$

For odd prime p, consider the Serre spectral sequence converging to $H_*(\mathrm{Map}_{P_k}(S^4, BG_2); F_p)$ with

$$E_2 = H_*(BG_2; F_p) \otimes H_*(\Omega_0^3 G_2; F_p).$$

Since the target of the first differential is primitive and there are no 3-dimensional, 11-dimensional primitive elements in $H_*(\Omega_0^3 G_2; F_p)$, the Serre spectral sequence collapses from E_2 and we get that

$$H_*(B\mathcal{G}_k; F_p) = H_*(\Omega_0^3 G_2; F_p) \otimes H_*(BG_2; F_p).$$

Consider the Bockstein spectral sequence with $E_1 = H_*(X; F_p)$ converging to $(H_*(X; Z)/\text{torsion}) \otimes F_p$. Here the differentials in E_1–term are interpreted in terms of the Steenrod operation β and higher differentials in terms of higher order Bockstein operators.

PROPOSITION 2.2. *$H_*(B\mathcal{G}_k; Z)$ has p-torsion of all order for odd prime p.*

Proof. We consider the Bockstein spectral sequence with $E_1 = H_*(B\mathcal{G}_k; F_p)$ converging to $(H_*(B\mathcal{G}_k; Z)/\text{torsion}) \otimes F_p$. So we have

$$\begin{aligned} E_1 = & P(Q^a_{2(p-1)}(Q_{2(p-1)}[1] * [-p]) : a \geq 0) \\ & \otimes E(Q^a_{p-1}\beta Q^b_{2(p-1)}(Q_{2(p-1)}[1] * [-p]) : a \geq 0, b \geq 0) \\ & \otimes P(\beta Q^a_{p-1}\beta Q^b_{2(p-1)}(Q_{2(p-1)}[1] * [-p]) : a > 0, b \geq 0) \\ & \otimes P(Q^a_{2(p-1)}(w_8) : a \geq 0) \\ & \otimes E(Q^a_{p-1}\beta Q^b_{2(p-1)}(w_8) : a \geq 0, b > 0) \\ & \otimes P(\beta Q^a_{p-1}\beta Q^b_{2(p-1)}(w_8) : a > 0, b > 0) \\ & \otimes P(x_4) \otimes P(x_{12}) \end{aligned}$$

Since the first differential is determined by β, a tensor product of the following the form: $E(Q^a_{p-1}\beta Q^b_{2(p-1)}(x)) \otimes P(\beta Q^a_{p-1}\beta Q^b_{2(p-1)}(x))$ disappear after E_1 stage. Hence

$$\begin{aligned} E_2 = & P(Q^a_{2(p-1)}(Q_{2(p-1)}[1] * [-p]) : a \geq 0) \\ & \otimes E(\beta Q^b_{2(p-1)}(Q_{2(p-1)}[1] * [-p]) : b \geq 0) \\ & \otimes P(Q^a_{2(p-1)}(w_8) : a \geq 0) \\ & \otimes E(\beta Q^b_{2(p-1)}(w_8) : b > 0) \\ & \otimes P(x_4) \otimes P(x_{12}) \end{aligned}$$

The $(r+1)$-th order Bockstein operation is determined by

$$\beta_{r+1} x^{p^r} = x^{p^r - 1} \beta x.$$

For example $\beta_2 x^p = x^{p-1}\beta x$. So there exist the higher differentials in every stage. This mean that $H_*(B\mathcal{G}_k; Z)$ has p-torsion of all order and finally get

$$E_\infty = P(w_8) \otimes P(x_4) \otimes P(x_{12}).$$

COROLLARY 2.3. $B\mathcal{G}_k \simeq_Q K(Z,4) \times K(Z,8) \times K(Z,12)$.

There is another way to get above Corollary.

THEOREM 2.4 [1, THM 2.6]. *Let $\pi_q(Y) = 0$ for $q \neq n$ and $\pi_n(Y) = n$. Then*

$$\mathrm{Map}(X,Y) \simeq \prod_q K(H^q(X,\pi); n-q).$$

Over the rationals $BG_2 \simeq_Q K(Z,4) \times K(Z,12)$. Hence

$$\mathrm{Map}(S^4, BG_2) \simeq_Q \mathrm{Map}(S^4, K(Z,4)) \times \mathrm{Map}(S^4, K(Z,12)).$$

Applying the above Theorem over Q, We have

$$\begin{aligned}\mathrm{Map}(S^4, BG_2) &\simeq_Q \mathrm{Map}(S^4, K(Z,4)) \times \mathrm{Map}(S^4, K(Z,12)) \\ &\simeq \prod_q K(H^q(S^4; \pi, 4-q) \times \prod_q K(H^q(S^4; \pi, 12-q) \\ &\simeq Z \times K(Z,4) \times K(Z,8) \times K(Z,12).\end{aligned}$$

Hence since $\mathrm{Map}(S^4, BG_2) \simeq \mathrm{Map}_{p_k}(S^4, BG_2) \times Z$,

$$\begin{aligned}B\mathcal{G}_k &\simeq \mathrm{Map}_{P_k}(S^4, BG_2) \\ &\simeq_Q K(Z,4) \times K(Z,8) \times K(Z,12).\end{aligned}$$

So we recover Corollary 2.3.

We denoted by $\mathcal{M}_1(G)$ the based moduli space of all G instantons with instanton number 1. Let $\mathcal{M}_1'(G)$ be the moduli space of all G instantons with instanton number 1, that is, the space of all G instantons with instanton number 1 modulo the full gauge group. Let $C_G(SU(2))$ be the centralizer of $SU(2)$ in G.

THEOREM 2.5 [3, PROP 3.1]. *Let G be a compact simple simply connected Lie group. Then the based moduli space $\mathcal{M}_1(G)$ fibers trivially over $\mathcal{M}_1'(G)$ with the fiber $G/C_G(SU(2))$ and $\mathcal{M}_1'(G)$ is homeomorphic to $\mathcal{M}_1'(SU(2))$ which is homeomorphic to the five ball B^5.*

Therefore $\mathcal{M}_1(G_2)$ is homeomorphic to $G_2/C_{G_2}(SU(2)) \times B^5$. Note that $SU(2) = S^3$. We turn now to the study of $C_{G_2}(SU(2))$.

We recall the basic informations of G_2 from [5, App A]. K is R^8 as a vector space spanned by e_0, e_1, e_2,..., e_7. For the multiplicative table consider all triples (p, q, r) which can be obtained from the following triples: (1,2,4), (2,3,5), (3,4,6), (4,5,7), (5,6,1), (6,7,2), (7,1,3). For such triple (p, q, r) the subspace spanned by e_0, e_p, e_q, e_r is a subalgebra and the linear transformation of the quaternion numbers Q to this subalgebra which sends $1, i, j, k$ to e_0, e_p, e_q, e_r is isomorphism. For example, $e_1e_2 = e_4 = -e_2e_1$, $e_2e_4 = e_1 = -e_4e_2$, $e_4e_1 = e_2 = -e_1e_4$, $e_1^2 = e_2^2 = e_4^2 = -e_0$. G_2 is the automorphism group of K and a subgroup of $O(7)$. Each element of G_2 is an orthogonal transformation leaving fixed the unit vector e_0.

We consider $H = \{\tau \in G_2 | \tau(e_1) = e_1, \tau(e_2) = e_2\}$ and the map from H into K given by $\tau \to \tau(e_7)$. Then the image of the map is the set of unit vectors which are orthogonal to e_1, e_2, e_4, that is, the unit sphere in the four dimensional space and the map from H onto the image is a homeomorphism. $H \cong S^3$. Hence

$$C_{G_2}(S^3) = \{\tau' \in G_2 | \tau(\tau'(x)) = \tau'(\tau(x)) \text{ for all } \tau \in H,\ x \in K\}.$$

For $\tau \in H$, $\tau(e_4) = \tau(e_1e_2) = \tau(e_1)\tau(e_2) = e_4$ and let $\tau(e_7) = a_1e_3 + a_2e_5 + a_3e_6 + a_4e_7$ for some constant a_1, a_2, a_3, a_4. Then from the multiple table, $\tau(e_3) = a_4e_3 + a_3e_5 - a_2e_6 - a_1e_7$, $\tau(e_5) = -a_3e_3 + a_4e_5 + a_1e_6 - a_2e_7$, $\tau(e_6) = a_2e_3 - a_1e_5 + a_4e_6 - a_3e_7$. Through the elementary calculations using the above facts, we can get that the only element in G_2 which commutes with every element in H is the identity automorphism. Hence $C_{G_2}(S^3)$ is trivial. So we get

PROPOSITION 2.6. *$\mathcal{M}_1(G_2)$ is homeomorphic to $G_2 \times B^5$.*

Then from Theorem 1.1

COROLLARY 2.7.

$$H^*(\mathcal{M}_1(G_2); F_2) = E(x_5) \otimes P(x_3)/(x_3^4),$$
$$H^*(\mathcal{M}_1(G_2); F_p) = E(x_3, x_{11}) \quad \textit{for odd prime } p.$$

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