Comm. Korean Math. Soc. 9 (1994), No. 4, pp. 927–931

$\mathbb{Z}_2$-VECTOR BUNDLES OVER S^1

SUNG SOOK KIM

1. Introduction

Let G be a cyclic group of order 2 and let S^1 denote the unit circle in $\mathbb{R}^2$ with the standard metric. We consider smooth G-vector bundles over S^1 when G acts on S^1 by reflection. Then the fixed point set of G on S^1 is two points $\{z_0, z_1\}$. Let $E|_{z_0}$ and $E|_{z_1}$ be the fiber G-representation spaces at z_0 and z_1 respectively. We associate an orthogonal G-representation $\rho_i : G \to O(n)$ to $E|_{z_i}$, $i = 0, 1$. Let $\det \rho_i(g)$, $g \neq 1$, be denoted by $\det E|z_i$ since $\det \rho_i(g)$ is independent of choice of ρ_i. We obtain the following results.

THEOREM 1. *Let $E \to S^1$ be a G-vector bundle. Its underlying vector bundle is trivial if and only if* $\det E|z_0 = \det E|z_1$.

THEOREM 2. *Let (V_0, V_1) be an ordered pair of G- representation spaces of the same dimension. Then there exist a G-vector bundle E over S^1 such that (V_0, V_1) is isomorphic to $(E|_{z_0}, E|_{z_1})$.*

THEOREM 3. *Let E and E' be G-vector bundles over S^1. They are isomorphic if and only if $(E|_{z_0}, E|_{z_1})$ is isomorphic to $(E'|z_0, E'|z_1)$.*

Acknowledgments. I would like to thank Professor Mikiya Masuda for his encouragement and invaluable advices.

2. Proof of Theorems

To prove our Theorems we need the following Proposition which we proved in [3]. In our Proposition, G is a compact Lie group.

Received March 29, 1994.

The present studies were supported by the Basic Science Research Institute program, Ministry of Education, 1994, project number BSRI-94-1428.

PROPOSITION. *A smooth G-line bundle $L \to S^1$ is equivariantly isomorphic to a product bundle $S(V) \times \delta \to S(V)$ or $S(V) \times_{\mathbb{Z}_2} \delta \to S(V)/\mathbb{Z}_2 = P(V)$ according as the G-line bundle $L \to S^1$ is trivial or not when we forget the action. Here $S(V)$ denotes the unit circle of a real 2-dimensional orthogonal G-module V, δ a real 1-dimensional G-module and $\mathbb{Z}_2$ acts on $S(V)$ and δ as scalar multiplication.*

Also we need the following Lemma.

LEMMA. *Suppose G acts on S^1 by reflection. Then any smooth G-vector bundle E over S^1 is isomorphic to the Whitney sum of smooth G-line bundles.*

Proof. Let $\{z_0, z_1\}$ be the fixed set of the G-action on S^1. Choose an eigenvector v_i of E at z_i and connect v_0 and v_1 by a smooth path to get a non-zero cross section of E on the upper half circle. Extend it to a cross section of E on S^1 by using the G-action. The resulting cross section on S^1 may not be continuous, but the lines generated by it form a smooth G-line subbundle L of E. So we can decompose $E \cong E' \oplus L$ where E' is a smooth G-vector bundle. Apply the same argument to E' and so on. Then the lemma follows.

Proof of Theorem 1. By Lemma, E is isomorphic to $E_1 \oplus E_2 \oplus \cdots \oplus E_n$, where E_i is a G-line bundle. By Proposition E_i is isomorphic to a product bundle $S^1 \times \delta \to S^1$ or $S^1 \times_{\mathbb{Z}_2} \delta \to S^1$. So E can be expressed by

$$a(S^1 \times \mathbb{R}_+) \oplus b(S^1 \times \mathbb{R}_-) \oplus c(S^1 \times_{\mathbb{Z}_2} \mathbb{R}_+) \oplus d(S^1 \times_{\mathbb{Z}_2} \mathbb{R}_-),$$

where $a+b+c+d =$ dimension of E and $\mathbb{R}_+$ (resp. $\mathbb{R}_-$) is the trivial (resp. non-trivial) one dimensional G-representation space. If the underlying vector bundle is trivial, then $c+d$ is even because the first Stiefel Whitney class should vanish.

We have

$$E|_{z_0} \underset{iso}{\cong} (a+c)\mathbb{R}_+ \oplus (b+d)\mathbb{R}_-$$

and

$$E|_{z_1} \underset{iso}{\cong} (a+d)\mathbb{R}_+ \oplus (b+c)\mathbb{R}_-.$$

Since $c+d$ is even $(-1)^{b+d} = (-1)^{b+c}$. So det $E|_{z_0} =$ det $E|_{z_1}$. Conversely, if det $E|_{z_0} =$ det $E|_{z_1}$, then $(-1)^d = (-1)^c$. So $c+d$ is even. Then the first Stiefel Whitney class vanishes. Therefore our underlying vector bundle is trivial.

Proof of Theorem 2. Let $V_0 \underset{iso}{\cong} n_0\mathbb{R}_+ \oplus m_0\mathbb{R}_-$, $V_1 \underset{iso}{\cong} n_1\mathbb{R}_+ \oplus m_1\mathbb{R}_-$ and let $n = n_0 + m_0 = n_1 + m_1$. We consider

$$E \underset{iso}{\cong} a(S^1 \times \mathbb{R}_+) \oplus b(S^1 \times \mathbb{R}_-) \oplus c(S^1 \times_{\mathbb{Z}_2} \mathbb{R}_+) \oplus d(S^1 \times_{\mathbb{Z}_2} \mathbb{R}_-),$$

where $a+b+c+d = n$.

Since

$$E|_{z_0} \underset{iso}{\cong} (a+c)\mathbb{R}_+ \oplus (b+d)\mathbb{R}_-$$

and

$$E|_{z_1} \underset{iso}{\cong} (a+d)\mathbb{R}_+ \oplus (b+c)\mathbb{R}_-$$

it suffices to find solutions for the following equation

$$a+c = n_0,$$
$$b+d = m_0,$$
$$a+d = n_1$$

and

$$b+c = m_1.$$

So if $n_0 \geq n_1$, we can choose $a = n_1, b = m_0, c = n_0 - n_1, d = 0$ and if $n_0 < n_1$, we can choose $a = n_0, b = m_1, c = 0, d = n_1 - n_0$.

Proof of Theorem 3. The necessity is trivial. Let E and E' be G-vector bundles over S^1 and $(E|_{z_0}, E|_{z_1}) \underset{iso}{\cong} (E'|_{z_0}, E'|_{z_1})$. Then by Lemma and Proposition E and E' can be expressed as follows:

$$E \underset{iso}{\cong} a(S^1 \times \mathbb{R}_+) \oplus b(S^1 \times \mathbb{R}_-) \oplus c(S^1 \times_{\mathbb{Z}_2} \mathbb{R}_+) \oplus d(S^1 \times_{\mathbb{Z}_2} \mathbb{R}_-),$$

where $a+b+c+d =$ dimension of E and

$$E' \underset{iso}{\cong} a'(S^1 \times \mathbb{R}_+) \oplus b'(S^1 \times \mathbb{R}_-) \oplus c'(S^1 \times_{\mathbb{Z}_2} \mathbb{R}_+) \oplus d'(S^1 \times_{\mathbb{Z}_2} \mathbb{R}_-),$$

where $a' + b' + c' + d' =$ dimension of E'. But since $S^1 \times_{\mathbb{Z}_2} \mathbb{R}_+ \oplus S^1 \times_{\mathbb{Z}_2} \mathbb{R}_- \underset{iso}{\cong} S^1 \times \mathbb{R}_+ \oplus S^1 \times \mathbb{R}_-$, we may assume $c = 0$ or $d = 0$ for E and $c' = 0$ or $d' = 0$ for E'. First, let's assume $d = 0$ and $d' = 0$. Then we can write E and E' as follows:

$$E \underset{iso}{\cong} h(S^1 \times \mathbb{R}_+) \oplus k(S^1 \times \mathbb{R}_-) \oplus l(S^1 \times_{\mathbb{Z}_2} \mathbb{R}_+)$$

and

$$E' \underset{iso}{\cong} h'(S^1 \times \mathbb{R}_+) \oplus k'(S^1 \times \mathbb{R}_-) \oplus l'(S^1 \times_{\mathbb{Z}_2} \mathbb{R}_+).$$

Then we have the following equations.

$$E|_{z_0} \underset{iso}{\cong} (h + l)\mathbb{R}_+ \oplus k\mathbb{R}_-,$$
$$E|_{z_1} \underset{iso}{\cong} h\mathbb{R}_+ \oplus (k + l)\mathbb{R}_-,$$
$$E'|_{z_0} \underset{iso}{\cong} (h' + l')\mathbb{R}_+ \oplus k'\mathbb{R}_-$$

and

$$E'|_{z_1} \underset{iso}{\cong} h'\mathbb{R}_+ \oplus (k' + l')\mathbb{R}_-.$$

Since $E|_{z_0} \underset{iso}{\cong} E'|_{z_0}$ and $E|_{z_0} \underset{iso}{\cong} E'|_{z_0}$ it follows that

$$h + l = h' + l',$$
$$k = k',$$
$$h = h'$$

and

$$k + l = k' + l'.$$

We have similar results when $c = 0$ and $c' = 0$. Now let's assume $d = 0$ and $c' = 0$. Then we can write E and E' as follows:

$$E \underset{iso}{\cong} h(S^1 \times \mathbb{R}_+) \oplus k(S^1 \times \mathbb{R}_-) \oplus l(S^1 \times_{\mathbb{Z}_2} \mathbb{R}_+)$$

and

$$E' \underset{iso}{\cong} h'(S^1 \times \mathbb{R}_+) \oplus k'(S^1 \times \mathbb{R}_-) \oplus l'(S^1 \times_{\mathbb{Z}_2} \mathbb{R}_-).$$

Then we have the following equations.

$$E|_{z_0} \underset{iso}{\cong} (h+l)\mathbb{R}_+ \oplus k\mathbb{R}_-,$$

$$E|_{z_1} \underset{iso}{\cong} h\mathbb{R}_+ \oplus (k+l)\mathbb{R}_-,$$

$$E'|_{z_0} \underset{iso}{\cong} h'\mathbb{R}_+ \oplus (k'+l')\mathbb{R}_-$$

and

$$E'|_{z_1} \underset{iso}{\cong} (h'+l')\mathbb{R}_+ \oplus k'\mathbb{R}_-.$$

Since $E|_{z_0} \underset{iso}{\cong} E'|_{z_0}$ and $E|_{z_0} \underset{iso}{\cong} E'|_{z_0}$ it follows that

$$h+l=h',$$
$$k=k'+l',$$
$$h=h'+l'$$

and

$$k+l=k'.$$

So $l+l'=0$. But l and l' are a nonnegative integer. Therefore $l=l'=0$. We have similar results when $c=0$ and $d'=0$. So E is isomorphic to E'.

References

[1] G. E. Bredon, *Introduction to Compact Transformation Groups*, Academic Press, New York, 1972.

[2] K. Kawakubo, *The Theory of Transformation Group*, Oxford University Press, Oxford, 1991.

[3] S. S. Kim and M. Masuda, *Topological characterization of non-singular real algebraic G-surfaces*, Topology Appl. (to appear in 1994).

Department of Applied Mathematics
Paichai University
Taejon 302-735, Korea