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STABLE REDUCTIONS OF SINGULAR PLANE QUARTICS

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1. Introduction

Let $\mathcal{M}_g$ be the moduli space of isomorphism classes of genus g smooth curves. It is a quasi-projective variety of dimension $3g-3$, when $g > 2$. It is known that a complete subvariety of $\mathcal{M}_g$ has dimension $< g-1$ [D]. In general it is not known whether this bound is rigid. For example, it is not known whether $\mathcal{M}_4$ has a complete surface in it. But one knows that there is a complete curve through any given finite points [H]. Recently, an explicit example of a complete curve in moduli space is given in [G-H]. In [G-H] they constructed a complete curve of $\mathcal{M}_3$ as an intersection of five hypersurfaces of the Satake compactification of $\mathcal{M}_3$. One way to get a complete curve of $\mathcal{M}_3$ is to find a complete one dimensional family $p : X \longrightarrow B$ of plane quartics which gives a nontrivial morphism from the base space B to the moduli space $\mathcal{M}_3$. This is because every non-hyperelliptic smooth curve of genus three can be realized as a nonsingular plane quartic and vice versa. This paper has come out from the effort to find such a complete family of plane quartics. Since nonsingular quartics form an affine space some fibers of p must be singular ones. In this paper, due to the semistable reduction theorem [M], we search singular plane quartics which can occur as singular fibers of the family above. We first list all distinct plane quartics in terms of singularities.

PROPOSITION 1. *There are 55 equisingular strata in the projective space $\mathbb{P}^{14}$ of all plane quartics. The twenty irreducible quartics besides nonsingular ones are ones with one node* (1)**; with two nodes* (2)*; with three nodes* (3)*; with one cusp* (2)*; with one cusp and one node* (3)*; with*

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one cusp and two nodes (4); *with a tacnode* (3); *with a tacnode and a node* (4); *with a triple point* (4); *with two cusps* (4); *with two cusps and a node* (5); *with a double cusp* (4); *with a double cusp and a node* (5); *with a tacnode and a cusp* (5); *with a osnode* (5); *with a cusp with a smooth branch* (5); *with three cusps* (6); *with a cusp and a double cusp* (6); *with a triple cusp* (6); *with an ordinary cusp of multiplicity three* (6). *The thirty four reducible quartics are a cubic and a line* (11)**; *a cubic and a tangent line* (10); *a cubic and a flex line* (9); *a nodal cubic and a line* (10); *a nodal cubic and a tangent line* (9); *a nodal cubic and a flex line* (8); *a nodal cubic and a line through a node* (9); *a nodal cubic and a tangent line at a node* (8); *a cuspidal cubic and a line* (9); *a cuspidal cubic and a tangent line* (8); *a cuspidal cubic and a flex line* (7); *a cuspidal cubic and a line through a cusp* (8); *a cuspidal cubic and a tangent line at a cusp* (7); *two conics* (10); *two conics meeting tangentially at one point* (9); *two conics meeting tangentially at two points* (8); *two conics with an intersection multiplicity* 3 *at one point* (7); *two conics meeting only one point* (6); *double conics* (5); *a conic and two lines* (9); *a conic, a line and a tangent line* (8); *a conic and two lines which intersect on the conic* (8); *a conic and two tangent lines* (7); *a conic, a line and a tangent line through an intersection point* (7); *a conic and a double line* (7); *a conic and a double tangent line* (6); *four distinct lines* (8); *a line and three concurrent lines* (7); *four concurrent lines* (6); *two lines and a double line* (6); *three concurrent lines one of which is a double line* (5); *two double lines* (4); *a triple line and a line* (4); *a quadruple line* (2).

In Proposition 1, the number with $*$ is the codimension in $\mathbb{P}^{14}$ and the number with $**$ is the dimension of each equisingular stratum.

A k-tuple cusp of a plane curve is a double point with a unique tangent line which becomes $(k-1)$-tuple cusp after a blowup. *An ordinary cusp of multiplicity three* is a triple point with a unique tangent line which becomes a smooth point by taking a blow up once, i.e., a point whose local equation is $y^3 = x^4$.

Proof. For irreducible quartics, it is classical, or see Namba [N]. For reducible ones, combine all possible plane curves of degree < 4 with Bezout theorem.

2. Semistable reduction, preliminary

A connected and reduced curve with at worst nodes as singularities is called *semistable* (*stable*, resp.), if it has no smooth rational components meeting the rest of the curve at one point (one or two points, resp.). From now on a curve is connected. In this section we apply the semistable reduction theorem ([M], [B]) to singular curves with a cusp, a tacnode or a triple point. We start with the following known facts for the completion of this paper.

LEMMA 2. *Let X be a surface given by $x^{qn}y^r = z^q$ in $\mathbb{C}^3$. Assume q is a prime number. If $r = qs$, then the normalization of X is q disjoint union of surfaces of equation $x^n y^s = z$. If $(r,q) = 1$, it is an irreducible surface given by $x^n y^r = z$.*

Proof. If $r = qs$, then $x^{qn}y^{qs} - z^q = \prod_\xi (x^n y^s - \xi z)$ where ξ is all distinct $q-th$ root of unity. As a normalization, we take all q components. If $r = 1$, put $y = (z/x^n)^q = w^q$. We now look at the intersection of two threefolds $x^{qn}y - z^q = 0$ and $y = w^q$ in x, y, z, w space. Since $y = w^q$ is nonsingular we substitute it to $x^{qn}y - z^q = 0$: $x^{qn}w^q - z^q = \prod_\xi (x^n w - \xi z)$. Now each component $x^n w = \xi z$ in x, z, w space maps generically one to one and onto the surface $x^{qn}y = z^q$ and $y = w^q$ by mapping (x, z, w) to $(x, w^q, x^n w, w)$. So, we may take one of them as a normalization. If $(r, q) = 1$ and $r > 1$, put $y^r = (z/x^n)^q = w^q$. Since $y^r = w^q$ is singular, we map (a, b, c) space to (x, y, z, w) space sending $(a, b, c) \mapsto (a, b^q, c, b^r)$ the image of which is the threefold $y^r = w^q$ in (x, y, z, w) space. Pulling back X, we get a surface of equation $a^{qn}b^{qr} - c^q = \prod_\xi (a^n b^r - \xi c) = 0$. Again each component is smooth and maps generically one to one and onto X, so we take one of them as a normalization.

PROPOSITION 3. *Let X be a smooth surface, Δ be a unit disk of $\mathbb{C}^3$ and $p : X \longrightarrow \Delta$ a flat family of smooth curves except $p^{-1}(0) = C$. Suppose that*

(a) *C is a curve with an ordinary cusp as its only singularity,*
(b) *C is a curve with a tacnode as its only singular point,*
(c) *C is a curve with a triple point as only singular point.*

Then, by applying the semistable reduction theorem, we can replace the fiber C over $t = 0$ with the stable curve of the following, respectively.

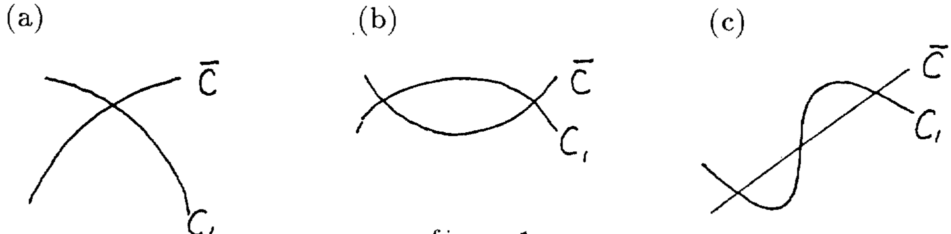

figure 1

Here $\bar{C}$ is the normalization of C and C_1 is an elliptic curve. In (a) *and* (c), *the j -invariant of C_1 is always zero.*

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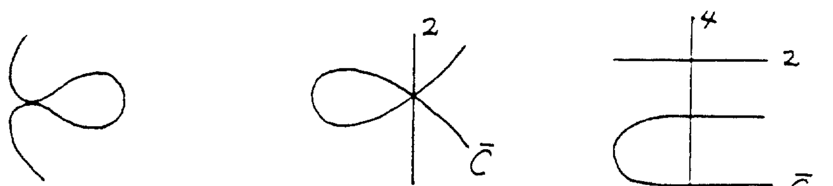
Proof. The above facts are not new. But the process of semistable reduction is very essential to guess the results of the next section, so we give the proofs for (b) and (c) here. For (a), the reader can see example 1.6 in [B].

(b) Since X is nonsingular we first resolve the singularity of C. Taking the blow-ups of X at the singularity of C and its infinitely near point, we get, over $t = 0$, the following figures (Figure 2). Each vertical line is the exceptional divisor of each step and the number next to each component the multiplicity, and the curve $\bar{C}$ is the (partial) normalization of C throughout this paper.



original center fiber first blowup second blowup

figure 2

Next process is to get rid of the multiple components of the fiber by base changes. The first change of order 2 gives us three singular points of a total surface of types $x^4y^2 = t^2$, $x^4y = t^2$ and $x^4y = t^2$, respectively. By Lemma 2 the normalization looks like Figure 3, where the vertical component is two to one cover over $\mathbb{P}^1$ branched at two points; Riemann-Hurwitz implies that its genus is zero. One more base change of order 2 is needed; again by Lemma 2, we have Figure 4. The vertical line this time is two to one cover branched at four points, therefore the genus is one by Riemann-Hurwitz. Two upper horizontal components in Figure 4 are rational and it is easy to check that the self-intersection number of each of them is -1. Contracting them, we get (b).

We give the proof of (c) rather quickly. To resolve a triple point of C, we blow up X at a triple point and get, over $t = 0$, Figure 5. Taking the base change of order three and normalizing at singular points, we get Figure 6. Then the vertical component is three to one cover over $\mathbb{P}^1$ totally branched at three points; therefore the genus is one and its j-invariant is zero [Ha].

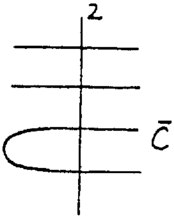


figure 3

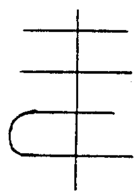


figure 4

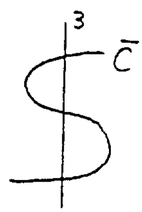


figure 5

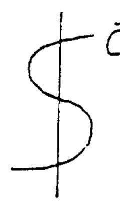


figure 6

3. Stable reduction of singular plane quartics

In this section we work on the following setting. Let $\mathbb{P}^{14}$ be the projective space parameterizing all plane quartics, C a singular quartic and E an equisingular stratum containing C. Let Δ be an open unit disk of $\mathbb{C}$. We embed Δ locally in $\mathbb{P}^{14}$ in such a way that $\Delta - 0$ is contained in the locus of smooth curves and 0 maps to C. Pulling back the universal family over $\mathbb{P}^{14}$ to a family over Δ we get a family $p : X \longrightarrow \Delta$ of smooth plane quartics degenerating to C. We call P the singular point of C we examine. Note that the total surface X is either nonsingular or singular at P according how we embed Δ in $\mathbb{P}^{14}$. We also take a family whose generic fibers are not projectively equivalent to each other. For a chosen family as above we make X nonsingular and do semistable (or stable) reduction as in Proposition 3 to get a stable curve. We call it *a stable model* of C. From it we can determine the map Δ into $\overline{\mathcal{M}_3}$, the Deligne-Mumford compactification of $\mathcal{M}_3$, i.e., the moduli space of all genus three stable curves. We note that the stable reductions of all plane quartics have been done for generic X. But it is very much out of hands to describe the rational map from $\mathbb{P}^{14}$ to $\overline{\mathcal{M}_3}$ because a stable model can be anything for a choice of a family $p : X \longrightarrow \Delta$

We now find singular plane quartics which have smooth stable models. As one can see from the proof of Proposition 3, the stable model of C contains, in the case that X is nonsingular, as its components the (partial) normalization of all components of C if it is not contracted and components produced by each singular point other than nodes of C. From proposition 3 one can expect that the latter have genus bigger than zero since a cusp, a tacnode, or a triple point is the simplest singular point of the similar type. We also note that the intersection number of two components is determined by the type of the tangent line of C at P. One may consider that therefore, for irreducible quartic to have a smooth stable model, it should be rational with only one singular point with unitangent line. From Proposition 1, there remain only two candidates among irreducible ones: *a plane quartic with a triple cusp*, and *one with an ordinary cusp of multiplicity three.*

THEOREM 1. *The above two candidates have smooth stable models.*

Proof. It is enough to give a family $p : X \longrightarrow \Delta_t$ of smooth quartics degenerating to each candidate, the stable reduction of which replaces C with a smooth curve of genus three. For a plane quartic with a triple cusp we take C as $y^2z^2 + 2x^2yz + x^4 + xy^3 = 0$. It has a triple cusp at $(0 : 0 : 1)$ with the tangent line $y = 0$. Let us take X as $y^2z^2 + 2x^2yz + x^4 + xy^3 + tz^4 = 0$ or $y^2z^2 + 2x^2yz + x^4 + xy^3 + t(z^4 + xz^3) = 0$ in $\mathbb{P}^2 \times \Delta_t$. Both of X are nonsingular. The first choice is natural for nonsingular X when X_t is a nonsingular quartic for small $t \neq 0$. The reason that the author put the second choice is that it is one way to make X_t projectively independent for nonzero t from the first choice if the generic fibers of the first choice are not projectively independent. For possible singular X and the corresponding stable reduction, see [K]. Since X is smooth, the semistable reduction process is same as that in Proposition 3. Just chasing the central fibers, we have the following figures.

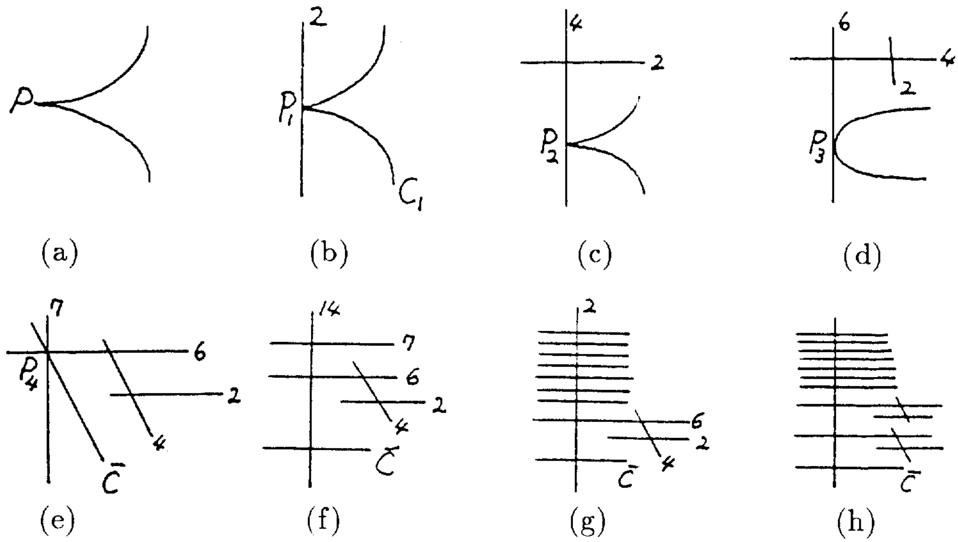


figure 7

The Figure (a) is the original central fiber with P a triple cusp: (b) is the blow-up of X at P with P_1 a double cusp: (c) is the blowup of the total surface of (b) at P_1 with P_2 an ordinary cusp of C_2: (d) is the blowup of the total surface of (c) at P_2: (e) is the blowup of the total surface of (d) at P_3: (f) is the blowup of the total surface of (e) at P_4: (g) is the base change of order seven of (f) with the vertical component rational: (h) is the base change of order two of (g). The vertical component is a smooth curve of genus three since it is the two to one cover over $\mathbb{P}^1$ totally branched at eight points. Contracting the first seven horizontal components and others after some base changes we get a smooth curve of genus three. At each step the base change is always followed by the normalization in the proof of Theorems 1 and 2.

For a plane quartic C with an ordinary cusp of multiplicity three, there are two up to projective equivalence [N] : $y^3z+x^4 = 0$ and $y^3z+y^4+x^4 = 0$. Here we take C as $y^3z + x^4 = 0$ and X as $y^3z + x^4 + t(z^4 + xz^3) = 0$. For other possible C and X see [K]. Doing exactly the same process as before, we get a smooth curve of genus three as a stable model. In this case it is trigonal.

REMARK. The above theorem does not claim that all possible stable models of two curves in theorem are smooth. In fact, the above cases are rather rare. If we take X as $y^2 + 2x^2y + x^4 + xy^3 + ty = 0$ when C is a quartic with a triple cusp, we get, as a stable model, a reducible curve consisting of a genus two curve and an elliptic curve meeting at one point. For a quartic with an ordinary cusp of multiplicity three, the similar example $y^3 + x^4 + ty = 0$ gives a smooth one too. But if we take X as $y^3 + x^4 + t^2x + tx^2 = 0$, we get a reducible curve consisting of a genus two curve and an elliptic curve meeting at one point.

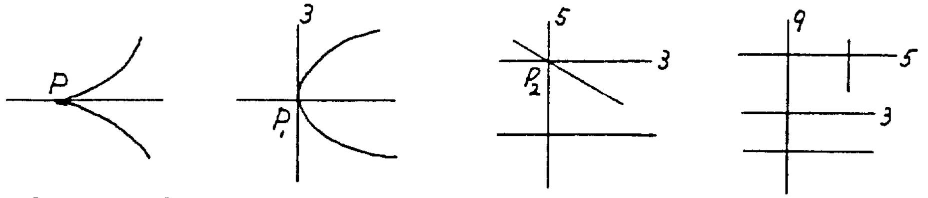
We now seek all reducible (including reduced ones) plane quartics C which can occur as fibers of the complete one dimensional family of plane quartics mentioned in the introduction. As mentioned before all components of the candidates are rational with only one singular point P other than nodes to produce a genus three smooth component. We also know that each single component of C through P should have unitangent line, otherwise it would produce a node to a stable model. Such candidates are

(i) a cuspidal cubic and a tangent line at the cusp
(ii) two conics which meet only one point
(iii) double conic
(iv) a conic and a double tangent line
(v) four concurrent lines
(vi) three concurrent lines one of which is a double line
(vii) two double lines
(viii) a triple line and a line
(ix) a quadruple line.

THEOREM 2. *Among the above* 9 *candidates, each curve of type* (i), (ii), (iii), (v), (viii) *and* (ix) *does admit a smooth stable model in* $\mathcal{M}_3$.

Proof. As in Theorem 1, it is enough to give a right example $X = \{X_t\}_{t\in\Delta}$ which satisfies that X_t is a nonsingular quartic and projectively independent for nonzero t for each curve C. Note that there is only one up to projective equivalence in each family.

When C is a cuspidal cubic and a tangent line, take $y(y^2z - x^3) = 0$ and $y(y^2z - x^3) + t(z^4 + xz^3) = 0$ as equations of C and X respectively. Since X is smooth surface for $t \in \Delta$, the semistable reduction process is same as that in Proposition 3. The following are the changes of the fibers over $t = 0$.



the curve C blowup of X at P blowup at P_1 blowup at P_2

figure 8

Then take order 9 base change and get a genus three curve at the vertical component. Blowing down four horizontal components and taking order 5 base change to remove remaining components, we are done. The result is a genus three smooth curve (which is trigonal).

If C is two conics meeting only one point, we take C and X as $(yz + x^2)(yz + x^2 + y^2) = 0$ and $(yz + x^2)(yz + x^2 + y^2) - tz^4 = 0$ respectively. The total surface X is smooth, So we can do it as before.

We here note some remarks about the choice of the equation X. The reason that we take X in (i) as $y(y^2z - x^3) + t(z^4 + xz^3) = 0$ instead of $y(y^2z - x^3) + t(z^4) = 0$ is that the latter choice is projectively equivalent for two different nonzero t and t'. If the chosen X in (ii) were projectively equivalent for two different nonzero t and t', one can add terms as in (i) to make general fibers projectively inequivalent.

For double conics, see [H]. Or, as we did, take C and X as $(x^2+yz)^2 = 0$ and $(x^2 + yz)^2 + t(y^4 + z^4) = 0$. This time X has eight singular points on C away P of type $\tilde{x}^2 = \tilde{t}\tilde{y}$ for an adequate coordinate $\tilde{x}$, $\tilde{y}$ and $\tilde{t}$. Desingularize X and get, over $t = 0$, a double conic with 8 exceptional lines (Figure 9). By taking the base change of order 2, we have a genus three curve with eight (-1) rational components. This time the stable model is a hyperelliptic genus three smooth curve.

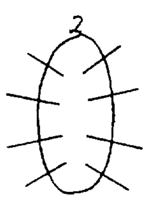


figure 9

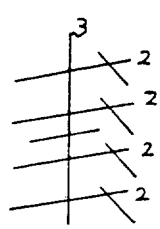


figure 10

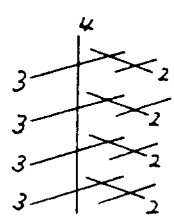


figure 11

Take C as $x^3y - xy^3 = 0$ and X as $x^3y - xy^3 + t(z^4 + xz^3) = 0$ for a curve in (v). Since X is smooth, taking the blowup of X at P and the base changes of order two twice (or order four once), we get a smooth curve of genus three after blowing down four (-1) rational components.

For a triple line and a line we take C as $x^3y = 0$ and X as $x^3y + \rho t(x^4 + y^4 + z^4) = 0, \rho = 1/2$. Here ρ is taken to make X_t nonsingular for $t \in \Delta$. The total surface X has four singular points ($x = 0$ and $y^4 + z^4 = 0$) over $t = 0$ away from P of type $\tilde{x}^3 + \tilde{t}\tilde{y} = 0$ for adequate coordinates $\tilde{x}$, $\tilde{y}$, and $\tilde{t}$. Smoothing X at the above four singular points, we replace the central fiber by Figure 10. The base change of order three and then order two give us a smooth components of genus three with nine (-1) rational components. In particular it is trigonal.

For a quadruple line $x^4 = 0$, we take X as $x^4 + t(x^4 + y^4 + z^4) = 0$. There are four singular points of X of type $\tilde{x}^4 + \tilde{t}\tilde{y} = 0$ when $t = 0$, $x = 0$ and $y^4 + z^4 = 0$. The normalization of X at these four singular point replaces C with Figure 11. Two base changes of order two and blowing down (-1) rational components, we get a smooth curve of genus three.

REMARK. We suspect that the remaining three components (iv), (vi) and (vii) never admit a smooth stable model.

References

[B] F. Bardelli, *Lectures on stable curves*, Lectures on Riemann Surface, World Scientific, 1989.

[D] S. Diaz, *A bound on the dimension of complete subvarieties of $\mathcal{M}_g$*, Duke Math. J. **51** (1984), 405–408.

[H] J. Harris, *Curves and their moduli*, Proc. Sympos. Pure Math. **46** (1987), 99–143.

[Ha] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in math., vol.52, Springer-Verlag, Heidelberg, 1977.

[GH] G. Diez and W. Harvey, *On complete curves in moduli space 1, 2*, Math. Proc. Cambridge Philos. Soc. **110** (1991), 461–466, 467–472.

[K] P-L. Kang, *Stable reduction of some families of plane quartics in order to get a complete curve of $\mathcal{M}_3$*, Proceedings of the second GARC Symposium on pure and applied mathematics, Part 3, 103–108.

[M] D. Mumford, *Semi-stable reduction*, Lecture Notes in Math., vol.339, Springer-Verlag, 53–108.

[N] M. Namba, *Geometry of projective algebraic curves*, Marcel Dekker, 1989.

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