Comm. Korean Math. Soc. 9 (1994), No. 4, pp. 891–896

SCALING FUNCTIONS SUPPORTED IN INTERVALS OF LENGTH $\leq$ 3

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Daubechies [1] discovered compactly supported scaling functions and corresponding wavelets with high regularities. It seems that there are no known compactly supported scaling functions other than Daubechies'. In this article, we will construct new scaling functions supported in intervals of length ≤ 3 without using deep analysis. While one of them is Daubechies' scaling function, others are less regular than Daubechies'. Also, we will show that Daubechies' scaling function is the unique one with highest regularity.

Suppose that a function ϕ in $L^2(\mathbf{R})$ is a scaling function for a multiresolution analysis. Then it follows from the definition (see [2]) that it satisfies the following conditions.

(a) There is a sequence $\{c_k\}$ such that

$$\phi(x) = \sum_{k\in\mathbf{Z}} c_k \phi(2x-k)$$

for every x, and

(b) $\{\phi(x-k)\}_{k\in\mathbf{Z}}$ is an orthonormal set.

The equation in the condition (a) is called the dilation equation and $\{c_k\}$ are called the dilation coefficients. We impose on the scaling function the normalization constraint.

(c) $\int_{\mathbf{R}} \phi(x)dx = 1$.

From now on we will assume that the dilation coefficients are real. It easily follows from (a), (b) and (c) that if $\{c_k\}$ are dilation coefficients for a scaling function then the followings hold.

$$\sum_{k\in\mathbf{Z}} c_k = 2, \tag{1}$$

Received July 19, 1994. Revised August 17, 1994.

This paper was supported by Nondirected Research Fund, Korea Research Foundation, 1993

(2) $$\sum_{k\in\mathbf{Z}} {c_k}^2 = 2,$$

and

(3) $$\sum_{k\in\mathbf{Z}} c_k c_{k+l} = 0$$

for every nonzero even integer l.

First, we will show that there is a unique scaling function supported in intervals of length not greater than 1 – the box function.

THEOREM 1. *The dilation coefficients for a scaling function with support in the interval* $[0,1]$ *are*

$$c_0 = c_1 = 1.$$

Proof. Since the scaling function ϕ has the support in $[0,1]$, the dilation equation looks like

$$\phi(x) = c_0\phi(2x) + c_1\phi(2x-1).$$

The sums in the conditions (1) and (2) become finite ones, and that in (3) is an empty one.

$$c_0 + c_1 = 2,$$
$$c_0^2 + c_1^2 = 2.$$

Solving these, we get $c_0 = c_1 = 1$.

It is well known that the characteristic function on the unit interval $[0,1]$ is really a scaling function and its dilation coefficients are as in the theorem. Hence this is the unique scaling function supported in $[0,1]$. A similar simple argument yields the following result.

THEOREM 2. *The dilation coefficients for a scaling function with support in the interval* $[0,2]$ *are either*

(4) $$c_0 = c_1 = 1,$$

or

(5) $$c_1 = c_2 = 1.$$

We note that the characteristic functions on the intervals $[0,1]$ and $[1,2]$ are scaling functions with dilation coefficients in (4) and (5) respectively. If we make the support length longer, some interesting scaling functions turn up.

THEOREM 3. *The dilation coefficients for a scaling function with support in the interval* $[0,3]$ *are either*

(6)
$$c_0 = p,\ c_1 = \frac{1+\sqrt{1+4p-4p^2}}{2},\ c_2 = 1-p,\ c_3 = \frac{1-\sqrt{1+4p-4p^2}}{2}$$

or

(7)
$$c_0 = p,\ c_1 = \frac{1-\sqrt{1+4p-4p^2}}{2},\ c_2 = 1-p,\ c_3 = \frac{1+\sqrt{1+4p-4p^2}}{2}$$

where p *is any real number in* $[(1-\sqrt{2})/2, (1+\sqrt{2})/2]$.

Proof. Let ϕ be a scaling function with support in $[0,3]$. Then the dilation equation is written as

$$\phi(x) = c_0\phi(2x) + c_1\phi(2x-1) + c_2\phi(2x-2) + c_3\phi(2x-3) \tag{8}$$

and the conditions (1), (2) and (3) yield

$$c_0 + c_1 + c_2 + c_3 = 2, \tag{9}$$

$$c_0^2 + c_1^2 + c_2^2 + c_3^2 = 2, \tag{10}$$

and

$$c_0c_2 + c_1c_3 = 0. \tag{11}$$

Successively substituting $x = 0, 1, 2,$ and 3 into (8), we obtain

$$\begin{aligned}
\phi(0) &= c_0\phi(0),\\
\phi(1) &= c_2\phi(0) + c_1\phi(1) + c_0\phi(2),\\
\phi(2) &= c_3\phi(1) + c_2\phi(2) + c_1\phi(3),\\
\phi(3) &= c_3\phi(3).
\end{aligned}$$

Suppose that $\phi(0) \neq 0$. Then $c_0 = 0$, and so the scaling function is as described in Theorem 2. Hence we may assume that $\phi(0) = \phi(3) = 0$. Also we assume that $\phi(1)\phi(2) \neq 0$ since otherwise the scaling function vanishes at every dyadic points. Then from the second and third equations in the above we have

$$c_0 c_3 - c_1 c_2 + c_1 + c_2 = 1. \tag{12}$$

Solving (9), (10), (11) and (12) simultaneously, we complete the proof.

For each set $\{c_0, c_1, c_2, c_3\}$ of coefficients given by (6) with parameter $1/2 \leq p \leq (1+\sqrt{2})/2$, define a new set $\{c_0', c_1', c_2', c_3'\}$ of coefficients by replacing the parameter in (7) by $p' = (1 - \sqrt{1+4p-4p^2})/2$. Then $(1-\sqrt{2})/2 \leq p' \leq 1/2$. We see that

$$c_0 = c_3', \quad c_1 = c_2', \quad c_2 = c_1', \quad c_3 = c_0',$$

and that the corresponding scaling functions, if they exist, are symmetric to each other with respect to the vertical line $x = 3/2$.

On substituting $p = (1+\sqrt{3})/4$ in (6), we obtain the coefficients

$$c_0 = \frac{1+\sqrt{3}}{4}, \; c_1 = \frac{3+\sqrt{3}}{4}, \; c_2 = \frac{3-\sqrt{3}}{4}, \; c_3 = \frac{1-\sqrt{3}}{4}. \tag{13}$$

To these coefficients there corresponds a celebrated scaling function due to Daubechies. Putting $p = (1-\sqrt{3})/4$ in (7), we obtain the coefficients

$$c_0 = \frac{1-\sqrt{3}}{4}, \; c_1 = \frac{3-\sqrt{3}}{4}, \; c_2 = \frac{3+\sqrt{3}}{4}, \; c_3 = \frac{1+\sqrt{3}}{4}. \tag{14}$$

The corresponding scaling function is symmetric to Daubechies'. Pollen [3] derived various properties, including the regularity, of Daubechies' scaling function from the dilation equation with coefficient in (13). In the following we prove that these two scaling functions are the unique ones with high regularity.

THEOREM 4. *Let ϕ be a scaling function with support in the interval $[0,3]$. Suppose that ϕ is left or right differentiable. Then the dilation coefficients are either as in* (13) *or* (14).

Proof. We proceed as in the proof of Theorem 3. The dilation coefficients should satisfy (9), (10), (11) and (12). Assume that ϕ is left differentiable and denote its left derivative by ϕ'. From (8), we have

$$\phi'(x) = 2c_0\phi'(2x) + 2c_1\phi'(2x-1) + 2c_2\phi'(2x-2) + 2c_3\phi'(2x-3).$$

By successively substituting $x = 0, 1, 2$ and 3, we obtain

$$\begin{aligned}
\phi'(0) &= 2c_0\phi'(0), \\
\phi'(1) &= 2c_2\phi'(0) + 2c_1\phi'(1) + 2c_0\phi'(2), \\
\phi'(2) &= 2c_3\phi'(1) + 2c_2\phi'(2) + 2c_1\phi'(3), \\
\phi'(3) &= 2c_3\phi'(3).
\end{aligned}$$

Again, we may assume that $\phi'(0) = \phi'(3) = 0$. From the second and third equations it easily follows that

$$2c_0c_3 - 2c_1c_2 + c_1 + c_2 = \frac{1}{2}. \tag{15}$$

Simultaneously solving (9), (10), (11), (12) and (15), we get the desired result.

Finally, we show that each set of coefficients given by Theorem 3 except for a single one really produces a scaling function for a multiresolution analysis.

THEOREM 5. *To each set of coefficients given by* (6) *or* (7) *there corresponds a scaling function for a multiresolution analysis, where* $p \neq 1$ *in* (7).

Proof. We will consider only the case when $\{c_k\}$ are given by (7) since the other case can be treated similarly. Consider the trigonometric polynomial

$$P(\xi) = c_0 + c_1e^{i\xi} + c_2e^{2i\xi} + c_3e^{3i\xi}.$$

By a theorem due to Mallat (see Theorem 2 in [2]), it is enough to show that this trigonometric polynomial is nonzero for $|\xi| \leq \pi/2$. Straightforward calculations using (10) and (11) lead to

$$\begin{aligned}
|P(\xi)|^2 &= c_0c_3(e^{3i\xi} + e^{-3i\xi}) + (c_0c_1 + c_1c_2 + c_2c_3)(e^{i\xi} + e^{-i\xi}) + 2 \\
&= (4p + 4p\sqrt{1 + 4p - 4p^2})\cos^3\xi \\
&\quad + (2 - 4p - 4p\sqrt{1 + 4p - 4p^2})\cos\xi + 2.
\end{aligned}$$

We see that $0 \leq \cos\xi \leq 1$ for $|\xi| \leq \pi/2$. Consider the cubic polynomial

$$f(t) = (4p + 4p\sqrt{1+4p-4p^2})t^3 + (2 - 4p - 4p\sqrt{1+4p-4p^2})t + 2$$

for $0 \leq t \leq 1$. If $0 \leq p \leq (1 - \sqrt{2}\sqrt[4]{3} + \sqrt{3})/4$ then the minimum of f is 2, and otherwise it is

$$\frac{2}{3}\sqrt{\frac{4p + 4p\sqrt{1+4p-4p^2} - 2}{3(4p + 4p\sqrt{1+4p-4p^2})}}(2 - 4p - 4p\sqrt{1+4p-4p^2}) + 2,$$

which is positive unless $p = 1$.

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