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ON THE BIRTH OF PB-CHAINS FOR GENERAL AREA-PRESERVING MAPS

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1. Introduction

A PB-chain(Poincaré-Birkhoff chain) is by definition a pair of elliptic and hyperbolic n-periodic orbits for a mapping and its existence has been well established numerically or analytically in many particular occasions such as in standard maps or twist maps [1, 8, 9] or Henon maps [1, 2, 12]. This paper gives focus on the investigation of the appearance of such a PB-chain in a one-parameter family of *general* area-preserving maps and is in fact a generalization of the results given in [12] for a one-parameter family of specific area-preserving maps,so called Henon maps.

We start studying the occurrence of a PB-chain in this general case by imposing the assumptions such as (2.2) and (2.3) in Section 2 below which imply that the complex conjugate eigenvalues of the linear part of a map depending on a parameter lie on the unit circle in the complex plane and move along the unit circle as the parameter varies through zero. Those conditions come naturally from the area-preserving property of the family of maps and are in contrast to the Hopf conditions in which the eigenvalues of a family of maps cross the unit circle transversally.

To investigate the occurrence of PB-chains from the origin for a one-parameter family of general area-preserving maps, we first give the normal forms, in Lemma 1, of the family of maps together with some constraint conditions due to the area-preserving property and briefly mention the method of Liapunov-Schmidt reduction in Lemma 2.

Even though the main idea of analysis is similar to the one given in [12], the actual analysis and calculation done in this general case are more involved and yield many notable results, which, of course, should cover the previous results [12] obtained for a typical Henon map.

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2. Preliminaries

Consider a one-parameter family of general area-preserving maps on $\mathbf{R}^2$,

$$F_\mu : \mathbf{R}^2 \longrightarrow \mathbf{R}^2, \tag{2.1}$$

where $F \in \mathcal{C}^\infty$ in μ and x, μ is a real parameter and by the definition of area-preservingness, $det D_x F_\mu(x) \equiv 1$ for any $x \in \mathbf{R}^2$ and $\mu \in \mathbf{R}$. We may assume that $F_\mu(0) = 0$ for any $\mu \in \mathbf{R}$.

Let $D_x F_\mu(0) = A_\mu \in \mathbf{R}^{2\times 2}$ and $\lambda(\mu)$, $\bar{\lambda}(\mu)$ be eigenvalues of A_μ for $|\mu|$ sufficiently small and let $\lambda_0 = \lambda(0)$, $\bar{\lambda}_0 = \overline{\lambda(0)}$ be eigenvalues of A_0. We assume that

$$|\lambda_0| = 1, \lambda_0 \neq \pm 1, \tag{2.2}$$

$$\frac{d}{d\mu} \arg \lambda(\mu)\bigg|_{\mu=0} > 0. \tag{2.3}$$

Notice that the condition (2.3) implies that the eigenvalues of the linear part of F_μ move along the unit circle as μ varies through 0 .

Since $F \in \mathcal{C}^\infty$, we can write

$$\lambda(\mu) = \lambda_0(1 + \lambda_1 \mu + \mathcal{O}(|\mu|^2)). \tag{2.4}$$

From (2.2), (2.3) and area-preserving property of F_μ, we can write

$$\begin{aligned} \lambda_1 &= 2\pi i a \quad (a > 0), \\ \lambda_0 &= e^{2\pi i \theta_0} \quad (\theta_0 \neq 0, 1/2 \ (\mathrm{mod}\ 1)), \end{aligned} \tag{2.5}$$

and

$$\lambda(\mu) = \lambda_0 e^{2\pi i a \mu + \mathcal{O}(|\mu|^2)}. \tag{2.6}$$

Putting A_μ in the real Jordan form by choosing a suitable basis and using complex conjugate coordinates $z = x_1 + ix_2$ and $\bar{z} = x_1 - ix_2$, we can rewrite the given real map (2.1) in the following complex form, again denoted by F_μ,

$$z' = F_\mu(z) = \lambda(\mu)z + \sum_{l\geq 2} R_l(\mu, z, \bar{z}), \tag{2.7}$$

where

$$R_l(\mu, z, \bar{z}) = \sum_{p+q=l} c_{pq}(\mu) z^p \bar{z}^q, \quad l \geq 2.$$

Note that here the area-preserving property requires that $\det \frac{\partial(z', \bar{z}')}{\partial(z, \bar{z})} \equiv 1$ for all $\mu \in \mathbf{R}$ and $z \in \mathbf{C}$.

Now, once again we put (2.7) in a normal form by successive applications of μ-dependent *area-preserving* transformations of the following form

$$z = w + \psi_l(\mu, w, \bar{w}) \equiv T_l(\mu, w), \quad l \geq 2, \tag{2.8}$$

where

$$\psi_l(\mu, w, \bar{w}) = \sum_{p+q=l} \gamma_{pq} w^p \bar{w}^q, \quad l \geq 2,$$

with a suitable choice of the coefficients $\gamma_{pq}(\mu)$. The resulting normal forms, again denoted by F_μ, of the map (2.7) must be also area-preserving and hence the coefficients in the normal forms must meet some constraint conditions, which will be given in each case of λ_0 in the following Lemma.

LEMMA 1. *Let $F_\mu(z)$ be the map given in (2.7) where $\theta_0 = \frac{1}{n}(n \geq 3)$. Then it has the following normal forms:*

(i) *when $n = 3$ and $c_{02}(0) \neq 0$,*

$$F_\mu(z) = \lambda(\mu) z + c_{02}(\mu) \bar{z}^2 + \mathcal{O}(|z|^3).$$

(ii) *when $n = 3$ and $c_{02}(0) = 0$,*

$$F_\mu(z) = \lambda(\mu) z + \alpha(\mu) z^2 \bar{z} + \beta(\mu) z^4 + \gamma(\mu) z \bar{z}^3 + \mathcal{O}(|z|^5),$$

where

$$\arg \alpha(0) = \pi/6 \pmod{\pi}. \tag{2.9}$$

(iii) *when $n = 4$,*

$$F_\mu(z) = \lambda(\mu) z + \alpha(\mu) z^2 \bar{z} + \beta(\mu) \bar{z}^3 + \mathcal{O}(|z|^5),$$

where

$$\arg\alpha(0) = \pi \pmod{\pi}. \tag{2.10}$$

(iv) *when* $n \geq 5$,

$$F_\mu(z) = \lambda(\mu)z + \alpha(\mu)z^2\bar{z} + \beta(\mu)\bar{z}^{n-1} + \gamma(\mu)z^3\bar{z}^2 + \mathcal{O}(|z|^7 + |z|^n),$$

where

$$\arg\alpha(0) = \frac{4-n}{2n}\pi \pmod{\pi} \quad (n \geq 5). \tag{2.11}$$

Proof. The normal forms can be found in many literatures, e.g., [3, 4, 6, 7, 8]. We need only to prove the constraint conditions (2.9), (2.10) and (2.11). Here we will only give the proof of (2.9) because other conditions can be proved in exactly the same way. By the definition of area-preservingness of the map $z' = F_\mu(z)$, we must have

$$\det \frac{\partial(z', \bar{z'})}{\partial(z, \bar{z})} \equiv 1 \quad \forall \mu \in \mathbf{R}, z \in \mathbf{C}.$$

This implies that

$$\begin{aligned} |\lambda(\mu) + 2\alpha(\mu)|z|^2 + 4\beta(\mu)\bar{z}^3 + \gamma(\mu)\bar{z}^3 \\ + \mathcal{O}(|z|^4)|^2 - |\alpha(\mu)z^2 + 3\gamma(\mu)z\bar{z}^2 + \mathcal{O}(|z|^4)|^2 \equiv 1 \end{aligned} \tag{2.12}$$

for all$\mu \in \mathbf{R}$ and $z \in \mathbf{C}$. Let $z = re^{2\pi i\phi}$ and write

$$\begin{aligned} \lambda(\mu) &= \lambda_0(1 + \lambda_1\mu + \mathcal{O}(\mu^2)) \\ \alpha(\mu) &= \alpha_0 + \alpha_1\mu + \mathcal{O}(\mu^2) \\ \beta(\mu) &= \beta_0 + \beta_1\mu + \mathcal{O}(\mu^2) \\ \gamma(\mu) &= \gamma_0 + \gamma_1\mu + \mathcal{O}(\mu^2) \end{aligned} \tag{2.13}$$

and

$$\mu = \mu_0 r^2 + \mathcal{O}(r^3), \quad \phi = \phi_0 + \mathcal{O}(r).$$

Substituting (2.13) into (2.12) and arranging in the power series of r, we have

$$1 + 2Re(\lambda_1\mu_0 + 2\bar{\lambda}_0\alpha_0)r^2 + 2Re\bar{\lambda}_0(4\beta_0 e^{6\pi i\phi_0} + \gamma_0 e^{-6\pi i\phi_0})r^3 + \mathcal{O}(r^4)$$
$$= 1 + \mathcal{O}(r^4).$$

Comparing the coefficients of equal powers of r on both sides, we have

$$Re(\lambda_1\mu_0 + 2\bar{\lambda}_0\alpha_0) = 0.$$

Since $\lambda_1 = 2\pi ai$ and $\lambda_0 = e^{2\pi i/3}$, this implies that $Re(e^{-2\pi i/3}\alpha_0) = 0$ and so we must have

$$\arg\alpha_0 = \frac{\pi}{6} + n\pi \quad (n \in \mathbf{Z}).$$

Now, following the Liapunov-Schmidt method used in [12], we can reduce the study of the occurrence of n-cycles for the map given in normal form into that of finding zeros of an algebraic function (so called bifurcation function) as stated in the following Lemma (For the proof, refer to the Lemma 2 in [12]).

LEMMA 2. *Assume that* $\lambda_0^n = 1$ $(n \geq 3)$ *and let* $x = (x_1, \ldots, x_n) \in \mathbf{C}^n$ *be a n-cycle of the map* F_μ *given in normal form. Let* S *be a right-shift operator* $(x_1, \ldots, x_{n-1}, x_n) \rightarrow (x_2, \ldots, x_n, x_1)$ *and* $\mathcal{F}_\mu(x) = (F_\mu(x_1), \ldots, F_\mu(x_n))$. *Let* $y = Px$, $y = (y_1, \ldots, y_n) \in \mathbf{C}^n$, *where each column of* P *consists of eigenvectors of* S. *Define a map* Φ: $\mathbf{C}^n \times \mathbf{R} \longrightarrow \mathbf{C}^n$ *by* $\Phi(y, \mu) = P\mathcal{F}_\mu(P^{-1}y) - \Lambda y$, *where* $\Lambda = diag(1, \bar{\lambda}_0, \ldots, \bar{\lambda}_0^{n-1})$. *Let* $L = D_y\Phi(0, 0)$ *and write* $y = y_n v_n + w$, *where* $v_n = (0, \ldots, 0, 1) \in Ker\ L$ *and* $w \in Im\ L$. *Let* $E : \mathbf{C}^n \rightarrow Im\ L$ *be a projection. Let* $z = \frac{1}{n}y_n$.

Then finding the n-cycle $(x_1, \ldots, x_n)$ *of* F_μ *is equivalent to solving the following equation in* $\mathbf{C}$:

$$\lambda_0 z = F_\mu(z) = \lambda(\mu)z + R(\mu, z, \bar{z}). \tag{2.14}$$

Moreover, if we write

$$x_1 \equiv \phi_\mu(z) \equiv z + \frac{1}{n}\sum_{j=1}^{n-1} w_j^*(nz, \mu), \tag{2.15}$$

where $w^* = (w_1^*, \ldots, w_{n-1}^*)$ *satisfies the equation*

$$E\Phi(y_n v_n + w^*(y_n, \mu)) \equiv 0$$

then the other n-periodic points $x_2, \ldots, x_n$ *are given by*

$$x_j = \phi_\mu(\lambda_0^{j-1} z) \qquad (j = 2, \ldots, n). \tag{2.16}$$

3. PB-chains at strong resonances

(i) The case $n = 3$ and $c_{02}(0) \neq 0$

In this case, $\lambda_0 = e^{2\pi i/3}$ and $F_\mu(z)$ has the normal form

$$F_\mu(z) = \lambda(\mu) z + c_{02}(\mu)\bar{z}^2 + \mathcal{O}(|z|^3) \tag{3.1}$$

where $\lambda(\mu) = \lambda_0(1 + \mu\lambda_1 + \mathcal{O}(|\mu|^2))$ with $\lambda_1 = 2\pi i a$ $(a > 0)$.
Then (2.14) becomes

$$\mu\lambda_1 z + \bar{\lambda}_0 c_{02}(0)\bar{z}^2 + \mathcal{O}(|\mu|^2|z| + |\mu||z|^2 + |z|^3) = 0. \tag{3.2}$$

Letting $z = re^{2\pi i\phi}$ and separating the trivial solution $r = 0$, we have

$$2\pi i a\mu + \bar{\lambda}_0 c_{02}(0) r e^{-6\pi i\phi} + g(\mu, r, \phi) = 0, \tag{3.3}$$

where $g(\mu, r, \phi) = \mathcal{O}(|\mu|^2 + |\mu| r + r^2)$ and $g(\mu, r, \phi + 1/3) = g(\mu, r, \phi)$.
Set

$$\begin{aligned} r &= 2\pi a \cdot \left|\frac{\mu}{c_{02}(0)}\right| \cdot (1 + r_1), \\ \phi &= \phi_0 + \phi_1, \\ \phi_0 &= -\frac{1}{36} - \frac{1}{6\pi}\arg\mu + \frac{1}{6\pi}\arg c_{02}(0) \quad (\mathrm{mod}\ 1/3). \end{aligned} \tag{3.4}$$

Substituting (3.4) in (3.3) and simplifying, we have

$$1 - e^{-6\pi i\phi_1}(1 + r_1) + g_2(\mu, r_1, \phi_1) = 0,$$

where

$$g_2(\mu, r_1, \phi_1) = (2\pi i a\mu)^{-1} g(\mu, 2\pi a|\frac{\mu}{c_{02}(0)}|(1+r_1), \phi_0 + \phi_1) = \mathcal{O}(|\mu|).$$

Let

$$h(\mu, r_1, \phi_1) = 1 - e^{-6\pi i\phi_1}(1+r_1) + g_2(\mu, r_1, \phi_1).$$

By the implicit function theorem, we have

$$r_1 = r_1(\mu), r_1(0) = 0, \phi_1 = \phi_1(\mu), \phi_1(0) = 0.$$

Consequently, we have from (3.4),

$$\begin{aligned} r &= 2\pi a \cdot \left|\frac{\mu}{c_{02}(0)}\right| \cdot (1 + \mathcal{O}(|\mu|)) = 2\pi a \cdot \left|\frac{\mu}{c_{02}(0)}\right| + \mathcal{O}(|\mu|^2), \\ \phi &= -\frac{1}{36} - \frac{1}{6\pi}\arg\mu + \frac{1}{6\pi}\arg c_{02}(0) + \mathcal{O}(|\mu|) \pmod{1/3}, \end{aligned} \tag{3.5}$$

and the coordinates of the 3-periodic points for the area-preserving map $F_\mu(z)$ in normal form are given, from (2.15), (2.16) and (3.5), by

$$\begin{aligned} x_1 &= \phi_\mu(z) \equiv z(\mu) + \mathcal{O}(|\mu||z| + |z|^2) \\ &= r(\mu)e^{2\pi i\phi(\mu)} + \mathcal{O}(|\mu|^2) = 2\pi a \cdot \left|\frac{\mu}{c_{02}(0)}\right| \cdot e^{2\pi i\phi_0} + \mathcal{O}(|\mu|^2), \\ x_2 &= \phi_\mu(\lambda_0 z), \\ x_3 &= \phi_\mu(\lambda_0^2 z). \end{aligned} \tag{3.6}$$

Notice that as μ varies from $\mu < 0$ to $\mu > 0$, $\arg(\mu)$ changes by π, and hence the orientation of the 3-cycle is reversed as μ crosses 0.

To examine the stability of the 3-cycle for the map

$$F_\mu(z) = \lambda(\mu)z + c_{02}(\mu)\bar{z}^2 + \mathcal{O}(|z|^3)$$

We consider the map

$$F_\mu^3(z) = (1 + 3\mu\lambda_1 + \mathcal{O}(|\mu|^2))z + 3\bar{\lambda}_0 c_{02}(0)\bar{z}^2 + \mathcal{O}(|\mu||z|^2 + |z|^3).$$

Then, we can easily see that the eigenvalues of the Jacobian $\partial(F_\mu^3(z), \bar{F}_\mu^3(z))/\partial(z, \bar{z})$ are real and reciprocal and hence the 3-cycle is hyperbolic (saddle) on both sides of $\mu = 0$. Thus, we have the following conclusion:

THEOREM 1. *Let $F_\mu : \mathbf{C} \longrightarrow \mathbf{C}$ be the general area-preserving map given in normal form (3.1) and assume that the conditions (2.2) and (2.3) hold and that $\lambda_0^3 = 1$ and $c_{02}(0) \neq 0$.*

Then a one-parameter family of 3-cycles $\{(x_1(\mu), x_2(\mu), x_3(\mu)) | \mu \in \mathbf{R}\}$ undergoes transcritical bifurcation from the origin in the fashion given by (3.5) and (3.6) and they are hyperbolic (saddle) on both sides of $\mu = 0$ and reverses the orientation as μ crosses 0. Hence in this case we have no PB-chains bifurcating from the origin.

(ii) The case $n = 3$ and $c_{02}(0) = 0$.

In this case, we can remove the second order term because the coefficient $\gamma_{02}(\mu)$ of the transformation

$$z = w + \Psi(\mu, w, \bar{w}), \Psi(\mu, w, \bar{w}) = \sum_{p+q \geq 2} \gamma_{pq}(\mu) w^p \bar{w}^q$$

becomes regular for μ near 0. Hence, after the change of variables, F_μ takes the form:

$$F_\mu(z) = \lambda(\mu)z + \alpha(\mu)z^2\bar{z} + \beta(\mu)z^4 + \gamma(\mu)z\bar{z}^3 + \mathcal{O}(|z|^5), \tag{3.7}$$

where the coefficient $\alpha_0 \equiv \alpha(0)$ must satisfy the condition (2.9), i.e.,

$$\arg \alpha_0 = \pi/6 \qquad (\text{mod } \pi)$$

and we assume that $\beta_0 \equiv \beta(0) \neq 0$ and $\gamma_0 \equiv \gamma(0) \neq 0$. From (2.14), we have

(3.8)

$$\mu\lambda_1 z + \bar{\lambda}_0\alpha_0 z^2\bar{z} + \bar{\lambda}_0\beta_0 z^4 + \bar{\lambda}_0\gamma_0 z\bar{z}^3 + \mathcal{O}(|\mu|^2|z| + |\mu||z|^3 + |\mu||z|^4 + |z|^5) = 0.$$

Setting $z = re^{2\pi i\phi}$ and separating the trivial solution $r = 0$, we have

$$\begin{aligned} 2\pi i a\mu + \bar{\lambda}_0\alpha_0 r^2 + \bar{\lambda}_0\beta_0 r^3 e^{6\pi i\phi} + \bar{\lambda}_0\gamma_0 r^3 e^{-6\pi i\phi} \\ + \mathcal{O}(|\mu|^2 + |\mu|r^2 + |\mu|r^3 + r^4) \\ = 0. \end{aligned} \tag{3.9}$$

Set

$$\mu = \mu_0 r^2 + \mu_1 r^3, \quad \phi = \phi_0 + \phi_1, \tag{3.10}$$

where μ_0, μ_1, ϕ_0 and ϕ_1 are to be determined. Substituting (3.10) into (3.9), we have
(3.11)

$$(2\pi i a\mu_0 + \bar{\lambda}_0\alpha_0)r^2 + (2\pi i a\mu_1 + \bar{\lambda}_0(\beta_0 e^{6\pi i\phi} + \gamma_0 e^{-6\pi i\phi}))r^3 + \mathcal{O}(r^4) = 0.$$

First, in order to choose μ_0 so that $2\pi i a\mu_0 + \bar{\lambda}_0\alpha_0 = 0$, we must have $\arg\alpha_0 = \pi/6 \pmod{\pi}$ and notice that this condition is automatically satisfied by (2.9) in Lemma 1. Hence we choose

$$\mu_0 = \begin{cases} \frac{|\alpha_0|}{2\pi a}, & \text{if } \arg\alpha_0 = \frac{\pi}{6} \pmod{2\pi} \\ \frac{-|\alpha_0|}{2\pi a}, & \text{if } \arg\alpha_0 = \frac{7\pi}{6} \pmod{2\pi}. \end{cases} \tag{3.12}$$

With this choice of μ_0, (3.11) becomes

$$\bar{\lambda}_0(\beta_0 e^{6\pi i\phi} + \gamma_0 e^{-6\pi i\phi}) + 2\pi i a\mu_1 + \mathcal{O}(r) = 0. \tag{3.13}$$

In order to choose ϕ_0 so that $\beta_0 e^{6\pi i\phi_0} + \gamma_0 e^{-6\pi i\phi_0} = 0$, we must have $|\beta_0| = |\gamma_0|$ and then

$$\phi_0 = \frac{1}{12\pi}\arg(-\frac{\gamma_0}{\beta_0}) \pmod{1/6}. \tag{3.14}$$

Hence if $|\beta_0| \neq |\gamma_0|$, there is no 3-cycle bifurcating from the origin. So, here, we assume that

$$|\beta_0| = |\gamma_0| \neq 0. \tag{3.15}$$

Note that ϕ_0 in (3.15) has two values

$$\begin{aligned} \phi_0^{(1)} &= \frac{1}{12\pi}\arg(-\frac{\gamma_0}{\beta_0}) \pmod{1/3}, \\ \phi_0^{(2)} &= \frac{1}{12\pi}\arg(-\frac{\gamma_0}{\beta_0}) + 1/6 \pmod{1/3}. \end{aligned} \tag{3.16}$$

Now from (3.13), we let

$$h(\mu_1, r, \phi) = \bar{\lambda}_0(\beta_0 e^{6\pi i\phi} + \gamma_0 e^{-6\pi i\phi}) + 2\pi i a\mu_1 + \mathcal{O}(r).$$

Then by the implicit function theorem, we know that

$$\mu_1 = \mu_1^{(j)}(r) = \mathcal{O}(r), \phi = \phi^{(j)}(r) = \phi_0^{(j)} + \mathcal{O}(r) \qquad (j = 1, 2).$$

Thus, we have a pair of 3-cycles $z = re^{2\pi i\phi^{(j)}(r)}$ $(j = 1, 2)$ on one side of $\mu = 0$, where r is regarded as a parameter which is related to μ as

$$(3.17) \qquad \begin{aligned} \mu^{(1)} &= \mu_0 r^2 + \mathcal{O}(r^4), \\ \phi^{(1)} &= \frac{1}{12\pi}\arg(-\frac{\gamma_0}{\beta_0}) + \mathcal{O}(r) \pmod{1/3}, \\ \mu^{(2)} &= \mu_0 r^2 - \mathcal{O}(r^4), \\ \phi^{(2)} &= \frac{1}{12\pi}\arg(-\frac{\gamma_0}{\beta_0}) + 1/6 + \mathcal{O}(r) \pmod{1/3}. \end{aligned}$$

Note that if $\arg\alpha_0 = \pi/6 \pmod{2\pi}$, we have a supercritical bifurcation and if $\arg\alpha_0 = 7\pi/6 \pmod{2\pi}$, subcritical bifurcation.

To study the stability of the pair of 3-cycles for the map

$$F_\mu(z) = \lambda(\mu)z + \alpha(\mu)z^2\bar{z} + \beta(\mu)z^4 + \gamma(\mu)z\bar{z}^3 + \mathcal{O}(|z|^5),$$

we consider the map

(3.18)
$$\begin{aligned} F_\mu^3(z) = (1 + 6\pi i a\mu)z &+ 3\bar{\lambda}_0\alpha_0 z^2\bar{z} + 3\bar{\lambda}_0\beta_0 z^4 \\ &+ 3\bar{l}d_0\gamma_0 z\bar{z}^3 + \mathcal{O}(|\mu|^2|z| + |\mu||z|^3 + |\mu||z|^4 + |z|^5). \end{aligned}$$

Then we can easily check that one of the two 3-cycles on one side must be hyperbolic(saddle) and the other must be elliptic. Therefore, we can state the following theorem.

THEOREM 2. *Let $F_\mu : \mathbf{C} \longrightarrow \mathbf{C}$ be the general area-preserving map given in (2.7) and assume that $\lambda_0^3 = 1 (\lambda_0 \neq \pm 1), c_{02}(0) = 0$ and $F_\mu(z)$ is put into a normal form (3.7).*

If $|\beta_0| = |\gamma_0|$ $(\neq 0)$, then a one-parameter family of PB-chains of 3-cycles $\{(x_1^{(j)}, x_2^{(j)}, x_3^{(j)}) \mid r \in \mathbf{R}^+,\ j = 1, 2\}$ undergoes a supercritical (if $\arg\alpha_0 = \frac{\pi}{6} \pmod{2\pi}$) or subcritical (if $\arg\alpha_0 = \frac{7\pi}{6} \pmod{2\pi}$) bifurcation from the origin and the parameter r is related to μ as in (3.17).

(iii) The case $n = 4$.

Let $\lambda_0 = e^{2\pi i/4} = i$. The normal form of $F_\mu(z)$ is

$$F_\mu(z) = \lambda(\mu)z + \alpha(\mu)z^2\bar{z} + \beta(\mu)\bar{z}^3 + \mathcal{O}(|z|^5). \tag{3.19}$$

where $\alpha(0) \equiv \alpha_0$ must satisfy the condition (2.10), i.e.,

$$\arg \alpha_0 = \pi \quad (\text{mod } \pi)$$

and we assume that $\beta(0) \equiv \beta_0 \neq 0$.

From (2.14), we have

$$\mu\lambda_1 z + \bar{\lambda}_0\alpha_0 z^2\bar{z} + \bar{\lambda}_0\beta_0\bar{z}^3 + g_1(\mu, z, \bar{z}) = 0, \tag{3.20}$$

where $g_1(\mu, z, \bar{z}) = \mathcal{O}(|\mu|^2|z| + |\mu||z|^3 + |z|^5)$. Setting $z = re^{2\pi i\phi}$ and separating the trivial solution $r = 0$, we have

$$2\pi a\mu - \alpha_0 r^2 - \beta_0 r^2 e^{-8\pi i\phi} + g(\mu, r, \phi) = 0, \tag{3.21}$$

where $g(\mu, r, \phi) = \mathcal{O}(|\mu|^2 + |\mu|r^2 + r^4)$.

To look for the principal part, put

$$\begin{aligned} \mu &= \mu_0 r^2 + \mu_1 r^2, \\ \phi &= \phi_0 + \phi_1, \end{aligned} \tag{3.22}$$

where μ_0, μ_1, ϕ_0 and ϕ_1 are to be determined. Substituting (3.22) in (3.21) and dividing by r^2. We have

$$(2\pi a\mu_0 - \alpha_0 - \beta_0 e^{-8\pi i\phi_0}) + 2\pi a\mu_1 + f_1(\mu, r, \phi) = 0, \tag{3.23}$$

where $f_1(\mu, r, \phi) = \mathcal{O}(r^2)$, $f_1(\mu, r, \phi + 1/4) = f_1(\mu, r, \phi)$.

Choose μ_0, ϕ_0 so that

$$2\pi a\mu_0 - \alpha_0 - \beta_0 e^{-8\pi i\phi_0} = 0. \tag{3.24}$$

Then we must have

$$\begin{aligned} |2\pi a\mu_0 - \alpha_0| &= |\beta_0|, \\ \phi_0 &= -\frac{1}{8\pi}\arg\left(\frac{2\pi a\mu_0 - \alpha_0}{\beta_0}\right) \quad (\text{mod } 1/4). \end{aligned} \tag{3.25}$$

Let $\mu_0^{(1)}, \mu_0^{(2)}$ be two solutions of $|2\pi a\mu_0 - \alpha_0| = |\beta_0|$. Notice that the condition (2.9) in Lemma 1 implies that α_0 is real and so we have

$$\mu_0^{(1),(2)} = \frac{1}{2\pi a}(\alpha_0 \pm |\beta_0|). \tag{3.26}$$

Once μ_0 is determined from (3.26), then we know from (3.25) that ϕ_0 has also two values

$$\phi_0^{(j)} = -\frac{1}{8\pi}\arg\left(\frac{2\pi a\mu_0^{(j)} - \alpha_0}{\beta_0}\right) \quad (\text{mod } 1/4) \qquad (j = 1, 2). \tag{3.27}$$

From (3.23), let

$$h(\mu_1, r, \phi) = (2\pi a\mu_0 - \alpha_0 - \beta_0 e^{-8\pi i\phi}) + 2\pi a\mu_1 + f_1(\mu, r, \phi).$$

Then by the implicit function theorem, we know that

$$\mu_1 = \mu_1(r), \phi = \phi(r), \mu_1(0) = 0, \phi(0) = \phi_0,$$

since $f_1(\mu, r, \phi)$ is an even function of r from the property in (3.23). We also know that $\mu(r)$, $\phi(r)$ are even functions of r. Thus, we have

$$\begin{aligned} \mu^{(j)}(r) &= \mu_0^{(j)} r^2 + \mathcal{O}(r^4) \qquad (j = 1, 2) \\ \phi^{(j)}(r) &= \phi_0^{(j)} + \mathcal{O}(r^2) \end{aligned} \tag{3.28}$$

where $\mu_0^{(1),(2)}$, $\phi_0^{(1),(2)}$ are given in (3.26) and (3.27). From (3.26), we know that

$$\mu_0^{(1)} \cdot \mu_0^{(2)} = \frac{|\alpha_0|^2 - |\beta_0|^2}{4\pi^2 a^2}.$$

Hence we know that if $|\alpha_0| > |\beta_0|$, then $\mu_0^{(1)} \cdot \mu_0^{(2)} > 0$ and so two families of 4-cycles bifurcate on the same side of $\mu = 0$. And if $|\alpha_0| < |\beta_0|$, then $\mu_0^{(1)} \cdot \mu_0^{(2)} < 0$ and so they bifurcate on the opposite sides of $\mu = 0$ (i.e., transcritical bifurcation). If $|\alpha_0| = |\beta_0|$, then $\mu_0^{(1)} = 0$, $\mu_0^{(2)} = \frac{1}{\pi a}\alpha_0 \neq 0$ and hence we know that

$$\begin{aligned} \mu^{(1)}(r) &= \mathcal{O}(r^4), \\ \mu^{(2)}(r) &= \mu_0^{(2)} r^2 + \mathcal{O}(r^4). \end{aligned} \tag{3.29}$$

To study the stability of the 4-cycles for the map

$$F_\mu(z) = \lambda(\mu)z + \alpha(\mu)z^2\bar{z} + \beta(\mu)\bar{z}^3 + \mathcal{O}(|z|^5),$$

we consider the map

$$F_\mu^4(z) = [1 + 8\pi i a\mu + \mathcal{O}(|\mu|^2)]z - 4i\alpha_0 z^2\bar{z} - 4i\beta_0\bar{z}^3 + \mathcal{O}(|\mu||z|^3 + |z|^5).$$

If σ_1, σ_2 are the eigenvalues of the linear part of $F_\mu^4(z)$ at one of the 4 fixed points of one family, then we can easily see that if $|\alpha_0| < |\beta_0|$, then σ_1,σ_2 are real reciprocal and if $|\alpha_0| > |\beta_0|$, then σ_1,σ_2 are complex conjugate on the unit circle. Hence we have the following conclusion :

THEOREM 3. *Let $F_\mu : \mathbf{C} \longrightarrow \mathbf{C}$ be the general area-preserving map given in (2.7) and assume that $\lambda_0^4 = 1\,(\lambda_0 \neq \pm 1)$ and $F_\mu(z)$ is put into the normal form (3.19), where α_0 satisfies the condition (2.9) and $\beta_0 \neq 0$.*

Then we have two one-parameter families of 4-cycles $\{(x_1^{(j)}(r), x_2^{(j)}(r), x_3^{(j)}(r), x_4^{(j)}(r)) \mid r \in \mathbf{R}^+,\ j = 1,2\}$ bifurcating from the origin and those 4-cycles are given by $x_k^{(j)} = x_k^{(j)}(r) = re^{2\pi i(\phi_0^{(j)} + \frac{k-1}{4})} + \mathcal{O}(r^3)$ $(j = 1,2,\ k = 1,2,3,4)$, where the parameter r is related to μ as in (3.22), (3.26) and (3.27).

Moreover, if $|\alpha_0| > |\beta_0|$, then a one-parameter family of PB-chains of 4-cycles bifurcates on the same sides of $\mu = 0$ and if $|\alpha_0| < |\beta_0|$, the two families bifurcate on the opposite side of $\mu = 0$ and both are hyperbolic.

4. Bifurcation Analysis of n-cycles $(n \geq 5)$

When $\lambda_0 = e^{2\pi i/n} (n \geq 5)$, the normal form of $F_\mu(z)$ is given as

(4.1) $$F_\mu(z) = \lambda(\mu)z + \alpha(\mu)z^2\bar{z} + \beta(\mu)\bar{z}^{n-1} + \gamma(\mu)z^3\bar{z}^2 + \mathcal{O}(|z|^7 + |z|^n),$$

where $\alpha(0) \equiv \alpha_0$ must satisfies the condition (2.11), i.e,

$$\arg\alpha_0 = \frac{4-n}{2n}\pi \pmod{\pi} \quad (n \geq 5),$$

and we assume that $\beta_0 \neq 0$. From (2.14), we have

(4.2)
$$\mu\lambda_1 z + \bar{\lambda}_0\alpha_0 z^2\bar{z} + \bar{\lambda}_0\beta_0\bar{z}^{n-1} + \mathcal{O}(|z|^5 + |\mu|^2|z| + |\mu||z|^3 + |\mu||z|^{n-1}) = 0.$$

Letting $z = re^{2\pi i\phi}$ and separating the trivial solution $r = 0$, we have

(4.3)
$$2\pi i a\mu + \bar{\lambda}_0\alpha_0 r^2 + \bar{\lambda}_0\beta_0 r^{n-2}e^{-2n\pi i\phi} + \mathcal{O}(|\mu|^2 + |\mu|r^2 + |\mu|r^{(n-2)} + r^4) = 0.$$

Set

$$\mu = \mu_0 r^2 + \mu_1 r^3 + \mu_2 r^3, \quad \phi = \phi_0 + \phi_1, \tag{4.4}$$

where $\mu_0, \phi_0, \mu_1, \mu_2$ and ϕ_1 are to be determined. Putting (4.4) into (4.3), we have

(4.5)
$$(\mu_0 r^2 + \mu_1 r^3 + \mu_2 r^3)2\pi i a + \bar{\lambda}_0\alpha_0 r^2 + \bar{\lambda}_0\beta_0 r^{n-2}e^{-2n\pi i(\phi_0+\phi_1)} + \mathcal{O}(r^4) = 0.$$

For $n = 5$, (4.5) becomes

$$(2\pi i a\mu_0 + \bar{\lambda}_0\alpha_0)r^2 + (2\pi i a\mu_1 + \bar{\lambda}_0\beta_0 e^{-10\pi i\phi})r^3 + \mu_2 r^3 + \mathcal{O}(r^4) = 0. \tag{4.6}$$

Notice that since α_0 satisfies the condition (2.11), we can always determine μ_0 in each case of $n \geq 5$ so that $2\pi i a\mu_0 + \bar{\lambda}_0\alpha_0 = 0$, i.e.,

$$\mu_0 = \begin{cases} \frac{|\alpha_0|}{2\pi a} & \text{if } \arg_0 = (4-n)\pi/2n \pmod{2\pi} \\ -\frac{|\alpha_0|}{2\pi a} & \text{if } \arg_0 = (4+n)\pi/2n \pmod{2\pi}. \end{cases} \tag{4.7}$$

From (4.6), we have

$$2\pi i a\mu_1 + \bar{\lambda}_0\beta_0 e^{-10\pi i\phi_0} + 2\pi i a\mu_0 + \mathcal{O}(r) = 0. \tag{4.8}$$

Choose again μ_1, ϕ_0 so that

$$2\pi i a\mu_1 + \bar{\lambda}_0\beta_0 e^{-10\pi i\phi_0} = 0. \tag{4.9}$$

Then we have

$$\mu_1^{(1),(2)} = \pm\frac{|\beta_0|}{2\pi a}, \quad \phi_0^{(1)} = \frac{1}{100} + \frac{1}{10\pi}\arg\beta_0 \pmod{1/5}, \quad \phi_0^{(2)} = -\frac{9}{100} + \frac{1}{10\pi}\arg\beta_0 \pmod{1/5}. \tag{4.10}$$

If $\arg(\alpha_0) = -\pi/10 \pmod{2\pi}$, then

$$(4.11)\qquad \begin{aligned} \mu &= \frac{|\alpha_0|}{2\pi a} r^2 \pm \frac{|\beta_0|}{2\pi a} r^3 + \mu_2 r^3, \\ \phi &= \phi_0 + \phi_1, \\ \phi_0^{(1),(2)} &= \frac{1}{100} - \frac{1}{10\pi} \arg \mu_1^{(1),(2)} + \frac{1}{10\pi} \arg \beta_0 \pmod{1/5}. \end{aligned}$$

If $\arg(\alpha_0) = -\frac{9\pi}{10} \pmod{2\pi}$, then

$$(4.12)\qquad \begin{aligned} \mu &= -\frac{|\alpha_0|}{2\pi a} r^2 \pm \frac{|\beta_0|}{2\pi a} r^3 + \mu_2 r^3, \\ \phi &= \phi_0 + \phi_1, \\ \phi_0 &= \phi_0^{(1)} \text{or} \phi_0^{(2)}. \end{aligned}$$

From (4.6), we define

$$h(\mu_2, r, \phi) = 2\pi i a \mu_1 + \bar{\lambda}_0 \beta_0 e^{-10\pi i \phi} + 2\pi i a \mu_2 + \mathcal{O}(r).$$

By the implicit function theorem, we have

$$\mu_2 = \mu_2(r) = \mathcal{O}(r), \phi_1 = \phi_1(r) = \mathcal{O}(r).$$

Therefore we have two one-parameter families of 5-cycles bifurcating from the origin, given by

$$(4.13)\qquad \begin{aligned} \mu^{(j)} &= \mu_0 r^2 \pm \mu_1^{(j)} r^3 + \mathcal{O}(r^4) \qquad (j = 1, 2), \\ \phi^{(j)} &= \phi_0^{(j)} + \mathcal{O}(r), \end{aligned}$$

where $\mu_0, \mu_1^{(j)}, \phi_0^{(j)}$ $(j = 1, 2)$ are given by (4.7) and (4.10) respectively.

For $n \geq 6$, we can proceed similarly as before and expect to have two one-parameter families of n-cycles whose stabilities depend on the coefficients in the normal forms. However, as long as we deal with the general normal form (4.1), we can only say the following conclusion.

THEOREM 4. *Let $F_\mu : \mathbf{C} \longrightarrow \mathbf{C}$ be the general area-preserving map given in (2.7) and assume that $\lambda_0^n = 1 (\lambda_0 \neq \pm 1)(n \geq 5)$ and $F_\mu(z)$ is put into the normal form (4.1), where α_0 satisfies the condition (2.11).*

Then we have at least one-parameter family of n-cycles bifurcating from the origin.

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