Comm. Korean Math. Soc. 9 (1994), No. 4, pp. 853–855

A PROOF OF STIRLING'S FORMULA

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The object of present note is to give a very short proof of Stirling's formula which uses only a formula for the generalized zeta function. There are several proofs for this formula. For example, Dr. E. J. Routh gave an elementary proof using Wallis' theorem in lectures at Cambridge ([5, pp.66-68]). We can find another proof which used the Maclaurin summation formula ([5, pp.116-120]). In [1], they used the Central Limit Theorem or the inversion theorem for characteristic functions. In [2], P. Diaconis and D. Freeman provided another proof similarly as in [1]. J. M. Patin [7] used the Lebesgue dominated convergence theorem.

For our purpose we introduce a known formula ([6, pp.22-25]):

$$\zeta'(0,a) = \log\Gamma(a) - \frac{1}{2}\log(2\pi), \tag{1}$$

where $\zeta(s,a)$ is the generalized (or Hurwitz) zeta function and $\zeta'(s,a) = \frac{\partial}{\partial s}\zeta(s,a)$.

We may use the Bohr-Mollerup theorem ([3], p.179; [4], p.868) to obtain the formula (1). The theorem is as follows:

The Euler gamma function $\Gamma(a)$ is the only function defined for $a > 0$ which is positive, is 1 at $a = 1$, satisfies the functional equation $a\Gamma(a) = \Gamma(a+1)$, and is logarithmically convex.

THEOREM. *Let $\zeta(s,a) = \sum_{k=0}^{\infty}(k+a)^{-s}$ be the Hurwitz ζ-function, where $a > 0$ and $\mathrm{Re}(s) > 1$, then we have*

$$\Gamma(a) = \frac{e^{\zeta'(0,a)}}{R},$$

where R is a constant.

Received April 25, 1994. Revised July 21, 1994.

Proof. It is well known ([6], p.22) that $\zeta(s,a)$ is analytically continued for all $s \neq 1, (a > 0)$. We observe that

$$\begin{aligned}\zeta(s,a+1) &= \zeta(s,a) - a^{-s},\\ \zeta'(s,a+1) &= \zeta'(s,a) + a^{-s}\log a,\\ \zeta'(0,a+1) &= \zeta'(0,a) + \log a.\end{aligned}$$

Letting $H(a) = e^{\zeta'(0,a)}$, we have $H(a+1) = aH(a),\ a > 0$, and

$$(2) \qquad \frac{d^2}{da^2}\log H(a) = \frac{d^2}{da^2}\frac{d}{ds}\zeta(s,a)|_{s=0} = \sum_{k=0}^{\infty}\frac{1}{(k+a)^2} > 0,\ \ a > 0,$$

which implies that $H(a)$ is logarithmically convex.

And by the analytic continuation of $\zeta(s,a)$ one sees that $H(a)$ is in C^∞ on R^+. So by the Bohr-Mollerup theorem we obtain

$$H(a) = \Gamma(a)R,$$

where R is a constant which is equal to $e^{\zeta'(0)} = (2\pi)^{-1/2}$ from (1) since $\zeta(s,1) = \zeta(s)$, where $\zeta(s)$ is the Riemann zeta function.

It should be **noted** in passing that the equation (2) holds for $\mathrm{Re}(s) > 1,\ a > 0$ and so formally we cannot substitute $s = 0$ for the second equality of (2), but since the right-hand side series converges absolutely for $\mathrm{Re}(s) > -1$ after differentiating, we can obtain the last equality of the equation (2) by analytic continuation.

Recall the Hermite's formula for $\zeta(s,a)$ ([6], p.23):

(3)

$$\zeta(s,a) = \frac{1}{2}a^{-s} + \frac{a^{1-s}}{s-1} + 2\int_0^\infty (a^2+t^2)^{-\frac{1}{2}s}\left\{\sin\left(s\arctan\frac{t}{a}\right)\right\}\frac{dt}{e^{2\pi t}-1},$$

where $s \neq 1,\ \mathrm{Re}(a) > 0$. Differentiating both sides of the equation (3) with respect to s, and letting $s = 0$ in the resulting equation, we obtain

$$(4) \qquad \zeta'(0,a) + a + \frac{1}{2}\log a - a\log a = 2\int_0^\infty \arctan\frac{t}{a}\frac{dt}{e^{2\pi t}-1}.$$

Taking the limit as $a \to \infty$ in (4), we have

$$\lim_{a\to\infty}\left[\zeta'(0,a)+a+\frac{1}{2}\log a - a\log a\right]=0. \tag{5}$$

Taking the exponential on both sides of (5) and considering the equation (1), we can finally obtain the Stirling's formula:

$$\lim_{a\to\infty}\frac{\Gamma(a)e^a\sqrt{a}}{a^a}=\sqrt{2\pi}. \tag{6}$$

Replacing a by a positive integer n in (6) with $\Gamma(n)=(n-1)!$, we have the familiar Stirling's formula:

$$\lim_{n\to\infty}\frac{e^n n!}{n^{n+\frac{1}{2}}}=\sqrt{2\pi}.$$

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