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ON A DECOMPOSITION OF MINIMAL COISOMETRIC EXTENSIONS

KUN WOOK CHOI

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. A *dual algebra* is a subalgebra of $\mathcal{L}(\mathcal{H})$ that contains the identity operator $I_{\mathcal{H}}$ and is closed in the ultraweak operator topology on $\mathcal{L}(\mathcal{H})$.

The study of the property $(\mathbb{A}_{m,n})$ which will be defined below appearing frequently in the theory of dual algebras has been applied to the topics of invariant subspace, dilation theory and reflexivity. In particular, Chevreau-Exner-Pearcy obtained some interesting results concerning the class $\mathbb{A}_{1,\aleph_0}$ and minimal coisometric extensions. In this paper, we study a decomposition theorem concerning a minimal coisometric extension of an absolutely continuous contraction.

We shall denote by $\mathbb{D}$ the open unit disc in the complex plane $\mathbb{C}$ and we write $\mathbb{T}$ for the boundary of $\mathbb{D}$. For $1 \le p < \infty$, we denote by $L^p = L^p(\mathbb{T})$ the Banach space of complex valued, Lebesgue measurable functions f on $\mathbb{T}$ for which $|f|^p$ is Lebesgue integrable, and by $L^\infty = L^\infty(\mathbb{T})$ the Banach algebra of all complex valued Lebesgue measurable, essentially bounded functions on $\mathbb{T}$. For $1 \le p \le \infty$ we denote by $H^p = H^p(\mathbb{T})$ the subspace of L^p consisting of those functions whose negative Fourier coefficients vanish. One knows that the preannihilator ${}^\perp(H^\infty)$ of H^∞ in L^1 is the subspace H_0^1 consisting of those functions g in H^1 whose analytic extension $\widetilde{g}$ to $\mathbb{D}$ satisfies $\widetilde{g}(0) = 0$. It is well known that H^∞ is the dual space of L^1/H_0^1, where the duality is given by the pairing

$$(1) \qquad < f, [g] >= \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) g(e^{it})\, dt, \quad f \in H^\infty,\ [g] \in L^1/H_0^1.$$

Supposes $\mathcal{A}$ is a dual algebra in $\mathcal{L}(\mathcal{H})$. Let $\mathcal{C}_1 = \mathcal{C}_1(\mathcal{H})$ is the von Neumann-Schatten ideal of trace class operators in $\mathcal{L}(\mathcal{H})$ under the trace

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norm and let ${}^{\perp}\mathcal{A}$ denote the preannihilator of $\mathcal{A}$ in $\mathcal{C}_1$. Let $\mathcal{Q}_{\mathcal{A}}$ denote the quotient space $\mathcal{C}_1/{}^{\perp}\mathcal{A}$.

The following provides a good relationship between the function space H^∞ and the dual algebra $\mathcal{A}_T$ generated by a single absolutely continuous contraction.

SZ.-NAGY-FOIAS FUNCTIONAL CALCULUS [1, Theorem 4.1]. *Let T be an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$. Then there is an algebra homomorphism*

$$\Phi_T : H^\infty \longrightarrow \mathcal{A}_T \tag{2}$$

defined by $\Phi_T(f) = f(T)$ such that

(a) $\Phi_T(1) = 1_{\mathcal{H}}$, $\Phi_T(\xi) = T$, (b) $\|\Phi_T(f)\| \leq \|f\|_\infty, f \in H^\infty$, (c) Φ_T *is continuous if both H^∞ and $\mathcal{A}_T$ are given weak*-topologies,* (d) *the range of Φ_T is weak*-dense in $\mathcal{A}_T$,* (e) *there exists a bounded, linear, one-to-one map*

$$\phi_T : \mathcal{Q}_T \longrightarrow L^1/H_0^1 \tag{3}$$

such that $\Phi_T = \phi_T^*$, and (f) *if Φ_T is an isometry, then Φ_T is a weak* homeomorphism of H^∞ onto $\mathcal{A}_T$ and ϕ_T is an isometry of $\mathcal{Q}_T$ onto L^1/H_0^1.*

For x and y in $\mathcal{H}$, we denote a rank one operator $(x \otimes y)(u) = (u, y)x$ for all u in $\mathcal{H}$.

Suppose m and n are any cardinal numbers such that $1 \leq m, n \leq \aleph_0$. A dual algebra $\mathcal{A}$ will be said to have property $(\mathbb{A}_{m,n})$ if every $m \times n$ system of simultaneous equations of the form

$$[x_i \otimes y_j] = [L_{ij}],\ 0 \leq i < m,\ 0 \leq j < n, \tag{4}$$

where $\{[L_{ij}]\}_{\substack{0\leq i<m\\ 0\leq j<n}}$ is an arbitrary $m \times n$ array from $\mathcal{Q}_{\mathcal{A}}$, has a solution $\{x_i\}_{0\leq i<m}$, $\{y_j\}_{0\leq j<n}$ consisting of a pair of sequences of vectors from $\mathcal{H}$. Furthermore, if $m,\ n \in \mathbb{N}$ and r is a fixed real number satisfying $r \geq 1$, then a dual algebra $\mathcal{A}$ has property $(\mathbb{A}_{m,n})(r)$ if for every $s > r$ and every $m \times n$ array from $\mathcal{Q}_{\mathcal{A}}$, there exist sequences $\{x_i\}_{0 \le i < m}$, $\{y_j\}_{0 \le j < n}$ that satisfy (4) and also satisfy the following conditions:

$$\|x_i\| \le \left(s \sum_{0 \le j < n} \|[L_{ij}]\| \right)^{1/2}, \quad 0 \le i < m \tag{5a}$$

and

$$\|y_j\| \le \left(s \sum_{0 \le i < m} \|[L_{ij}]\| \right)^{1/2}, \quad 0 \le j < n. \tag{5b}$$

Finally, a dual algebra $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ has property $(\mathbb{A}_{m,\aleph_0}(r))$ for some real number $r \ge 1$, if for every $s > r$ and every array $\{[L_{ij}]\}_{\substack{0 \le i < m \\ 0 \le j < \infty}}$ from $\mathcal{Q}_{\mathcal{A}}$ with summable rows, there exist sequences $\{x_i\}_{0 \le i < m}$ and $\{y_j\}_{0 \le j < \infty}$ of vectors from $\mathcal{H}$ that satisfy (4), (5a) and (5b) with the replacement of n by $\aleph_0$. Properties $(\mathbb{A}_{\aleph_0,n}(r))$ and $(\mathbb{A}_{\aleph_0,\aleph_0}(r))$ are defined similarly. For brevity, we shall denote $(\mathbb{A}_{n,n})$ by $(\mathbb{A}_n)$.

We denote by $\mathcal{Q}_T$ the predual space $\mathcal{Q}_{\mathcal{A}_T}$ of $\mathcal{A}_T$.

For an arbitrary Borel subset Σ of $\mathbb{T}$, we define the subspace $L^p(\Sigma)$ of $L^p(\mathbb{T})$, $1 \le p \le \infty$, as the set of all functions f in $L^p(\mathbb{T})$ such that $f = 0$ almost everywhere on $\mathbb{T} \setminus \Sigma$. The space $H^2(\Sigma)$ is the closure in $L^2(\Sigma)$ of the linear manifold consisting of those functions that agree with some polynomials on Σ (cf. [2]). Let T be an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$. Let

$$B = U^* \oplus R \tag{6}$$

denote a minimal coisometric extension of T, where U is an unilateral shift operator of some multiplicity in $\mathcal{L}(\mathcal{U})$ and R is an unitary operator in $\mathcal{L}(\mathcal{R})$. Then it follows from [3, Lemma 3.2] that R is absolutely continuous. Hence there exists a Borel subsets Σ of $\mathbb{T}$ such that the measure $m|\Sigma$ defined on a Borel subset Δ of $\mathbb{T}$ by

$$(m|\Sigma)(\Delta) = m(\Sigma \cap \Delta) \tag{7}$$

is a scalar spectral measure for R. For any vectors x and y in $\mathcal{R}$, if we define a complex Borel measure $\mu_{x,y}$ on $\mathbb{T}$ defined by

$$\mu_{x,y}(\Delta) = (E(\Delta)x, y), \tag{8}$$

where E is the spectral measure corresponding to the unitary operator$R$, then it is absolutely continuous with respect to $m|\Sigma$. Hence there is the Radon-Nikodym derivative $x \cdot y \in L^1(\Sigma)$ of $\mu_{x,y}$ with respect to $m|\Sigma$ (cf. [2, p 32]). We write $[x \cdot y]$ for the equivalence class of $x \cdot y$ in the quotient space $L^1(\mathbb{T})/H_0^1(\mathbb{T})$.

LEMMA 1. *Suppose that T is an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$ and has coisometric extension*

$$B = U^* \oplus R \tag{9}$$

in $\mathcal{L}(\mathcal{U} \oplus \mathcal{R})$ with $\mathcal{R} \neq (0)$. Assume that $\mathcal{R}_1$ and $\mathcal{R}_2$ are reducing subspaces for R with $\mathcal{R}_1 \subset \mathcal{R}_2^{\perp}$. If $w \in \mathcal{R}_1$ and $z \in \mathcal{R}_2$, then $[w \cdot z] = 0$.

Proof. It follows from [3, Lemma 3.9] that

$$[w \cdot z] = \phi_B([w \otimes z]). \tag{10}$$

For any $h \in H^\infty$, we have

$$\begin{aligned} < h, [w \cdot z] > &= < h, \phi_B([w \otimes z]_B) > \\ &= < \Phi_B(h), [w \otimes z]_B > \\ &= (h(B)w, z) \\ &= (h(R)w, z) = 0. \end{aligned} \tag{11}$$

By hypothesis, we have the lemma.

The following lemma comes from [5, Theorem 1].

LEMMA 2. *Let $\mathcal{A}$ be a von Neumann algebra. Then the following are equivalent:*

(a) *$\mathcal{A}$ has a separating vector;*
(b) *$\mathcal{A}$ has property $(\mathbb{A}_1)$,and;*
(c) *$\mathcal{A}$ has property $(\mathbb{A}_{1,\aleph_0})(1)$.*

We write $\mathcal{W}^*(T)$ for the von Neumann algebra generated by a contraction operator T. The following decomposition theorem is an improvement of [3, Proposition 3.10].

THEOREM 3. *Suppose that T is an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$ and has minimal coisometric extension*

$$B = U^* \oplus R \tag{12}$$

in $\mathcal{L}(\mathcal{U} \oplus \mathcal{R})$ with $\mathcal{R} \neq (0)$. Then there exist some cardinal number m with $1 \leq m \leq \aleph_0$, a decreasing sequence $\{\Sigma_i\}_{0 \leq i < m}$ of Borel sets in $\mathbb{T}$, and a decomposition

$$R = \underbrace{R_0 \oplus R_1 \oplus R_2 \oplus \cdots}_{(m)} \tag{13}$$

of unitary operators $R_i \in \mathcal{L}(\mathcal{R}_i), 0 \leq i < m$ such that
(a) *$m|\Sigma_i$ is a scalar spectral measure for R_i,*
(b) *$\mathcal{W}^*(R_i)$ has a property $(\mathbb{A}_{1,\aleph_0}(1))$,*
(c) *R_i is unitarily equivalent to the operator $M_{e^{it}}$ of multiplication by the position function on $L^2(\Sigma_i)$,*
(d) *if we denote by $\mathcal{R}_i^+$ the subspace of $\mathcal{R}_i$, corresponding to $H^2(\Sigma)$ under the unitary equivalence in (c), then*

$$\bigcup_{o \leq i < m} \mathcal{R}_i^+ \subset \overline{(A\mathcal{H})}, \tag{14}$$

where A is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{R}$,
(e) *for any $r = \underbrace{r_0 \oplus r_1 \oplus \cdots}_{(m)} \in \sum \oplus_{0 \leq i < m} R_i$, we have $[r_i \cdot r_j] = 0$ if $i \neq j$.*

Proof. Since $\mathcal{R} \neq (0)$, it is obvious that $\overline{A\mathcal{H}} \neq (0)$.
Case 1. Assume that $\overline{A\mathcal{H}} = \mathcal{R}$. Let e be a nonzero element of $\mathcal{R}$. By [4, Proposition 10.4], there is a separating vector e_0 for $\mathcal{W}^*(R)$ such that

$$e \in \overline{\mathcal{W}^*(R)e_0}. \tag{15}$$

Set $\mathcal{R}_0 = \overline{\mathcal{W}^*(R)e_0}$.Then $\mathcal{R}_0$ is a reducing subspace for R. By the multiplicity theory of normal operators (cf. [4, p.299-p308]), if we denote $R_0 = R|\mathcal{R}_0$, then R_0 is unitarily equivalent to the operator $M_{e^{it}}$ of multiplication by the position function on $L^2(\Sigma_0)$, where $\Sigma_0 = \sigma(R)$.

Moreover, it follows easily that R_0 satisfies $(a),(b)$ and $\mathcal{R}_0^+ \subset \overline{A\mathcal{H}}$. If $\mathcal{R}_0 = \mathcal{R}$, we stop here and take $m = 1$. Otherwise, since $R|\mathcal{R}_0^\perp$ is an unitary operator, we can repeat this process. Then by the mathematical induction, we obtain a decomposition

$$R = \underbrace{R_0 \oplus R_1 \oplus \cdots}_{(m)} \tag{16}$$

which satisfies (a), (b), (c), and (d).

Case 2. Assume that $\overline{A\mathcal{H}} \neq \mathcal{R}$. Let us follow the method of [3, Proposition 3.10]. Then we can obtain a reducing subspace $\mathcal{R}_0 \supset \mathcal{R}_0^+$ for R such that $\sigma(R) = \mathbb{T}$ and $R_0 = R|\mathcal{R}_0$ is a bilateral shift operator multiplicity one. Moreover, if we denote $R_0 = R|\mathcal{R}_0$ by the proof of [3, Proposition 3.10] R_0 satisfies (a) and $\mathcal{R}_0^+ \subset \overline{A\mathcal{H}}$. But it is obvious that the von Neumann algebra generated by a bilateral shift operator has a separating vector. If $\mathcal{R}_0 = \mathcal{R}$, we stop here as above. Otherwise, since $R|\mathcal{R}_0^\perp$ is an unitary operator,we can repeat this process and we obtain a required decomposition. Furthermore, Lemma 1 shows (e). Finally, we shall show (b). Since $\mathcal{W}^*(R_i), 0 \leq i < m$, has a separating vector, by the Lemma 2, $\mathcal{W}^*(R_i)$ has property $(\mathbb{A}_{1,\aleph_0}(1))$. Hence the proof is complete.

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Department of Mathematics
College of Natural Sciences
Kyungpook National University
Taegu 702-701, Korea