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JOINT NUMERICAL RANGES IN NON UNITAL NORMED ALGEBRAS

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1. Introduction

Let A denote a unital normed algebra over a field $K = \mathbb{R}$ or $\mathbb{C}$ and let e be the identity of A. Given $a \in A$ and $x \in A$ with $\|x\| = 1$, let

$$V(A, a, x) = \{f(ax) : f \in A', f(x) = 1 = \|f\|\}.$$

Then the (Bonsall and Duncan) numerical range of an element $a \in A$ is defined by

$$V(a) = \cup\{V(A, a, x) : x \in A, \|x\| = 1\},$$

where A' denotes the dual of A. In [2], $V(a) = \{f(a) : f \in A', f(e) = 1 = \|f\|\}$. (see [1, 2] for details.)

We have two limitations in this numerical range as well as joint numerical range: First this definition of $V(A, a)$ is dependent on the identity. There are many normed algebras which do not possess an identity. Therefore it is of some interest to make the notion of relative numerical range identity-free.

The second limitation is in the definition itself. For $a \in A$, a normed algebra, the scalars comprising the numerical range of a are of the form $f(ax)$ where $x \in A, f \in A'$, and $1 = \|x\| = \|f\| = f(x)$. No consideration is given to scalars of the form $f(xa)$, and as will be seen, these are significant if progress is to be made.

In this paper we introduce the notion of right(left) relative (joint) numerical range $V_x^R(A, \mathbf{a})(V_x^L(A, \mathbf{a}))$ of an n-tuple $\mathbf{a} = (a_1, \ldots, a_n)$ of elements in a non unital normed algebra A relative to $x \in A$. (See Definition 2.1) If $x = e$, the identity of A and $\|e\| = 1$, then $V_x^R(A, \mathbf{a})$

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coincides with $V(\mathbf{a})$. Thus this concept extends the (Bonsall and Duncan) joint numerical range. Among the results, it is shown that our numerical ranges $V_x^L(A,\mathbf{a})$ and $V_x^R(A,\mathbf{a})$ are compact convex subsets of K^n. Also we give a sufficient condition for our numerical range to be a singleton set.

Further, we show that the relative (joint) numerical range of an n-tuple of elements in a normed algebra is invariant under certain algebra homomorphism. An example is given to show that the invariance of the relative numerical range under homomorphism ϕ does not imply that ϕ is an isometry. Also we introduce the concept of regular norm on a normed algebra and study the invariance of the relative numerical range under a homomorphism in terms of this concept.

Throughout this paper let A be a non unital normed algebra over a field K ($\mathbb{R}$ or $\mathbb{C}$).

2. Relative numerical ranges of elements

DEFINITION 2.1. Let A be a normed algebra over the field $K=\mathbb{R}$ or $\mathbb{C}$, and A' its dual. For $x \in A$, we write

$$D(A,x)=\{f\in A' : \|f\|=1, f(x)=\|x\|\}.$$

Let $\mathbf{a}=(a_1,\dots,a_n)\in A^n$ denote an n-tuple of elements in A. The right relative numerical range of $\mathbf{a}=(a_1,\dots,a_n)\in A^n$ relative to x is defined to be $V_x^R(A,\mathbf{a})=\{(f(a_1x),\dots,f(a_nx)) : f\in D(A,x)\}$. The left relative numerical range of $\mathbf{a}=(a_1,\dots,a_n)\in A^n$ relative to x is defined to be $V_x^L(A,\mathbf{a})=\{(f(xa_1),\dots,f(xa_n)) : f\in D(A,x)\}$. The relative numerical range of $\mathbf{a}=(a_1,\dots,a_n)\in A^n$ relative to x is defined to be $V_x(A,\mathbf{a})=V_x^R(A,\mathbf{a})\cup V_x^L(A,\mathbf{a})$. The right relative numerical radius of $\mathbf{a}$ relative to x is defined by $v_x^R(\mathbf{a})=\sup\{|\lambda| : \lambda=(\lambda_1,\dots,\lambda_n)\in V_x^R(A,\mathbf{a})\}$. The left relative numerical radius of $\mathbf{a}$ relative to x is defined by $v_x^L(\mathbf{a})=\sup\{|\lambda| : \lambda=(\lambda_1,\dots,\lambda_n)\in V_x^L(A,\mathbf{a})\}$. The relative numerical radius of $\mathbf{a}$ relative to x is defined by $v_x(\mathbf{a})=\max\{v_x^R(\mathbf{a}),v_x^L(\mathbf{a})\}$.

Note that the set $D(A,x)$ is nonempty by the Hahn-Banach Theorem, and so $V_x^R(A,\mathbf{a})$ and $V_x^L(A,\mathbf{a})$ are nonempty. If A is commutative, then $V_x^R(A,\mathbf{a})=V_x^L(A,\mathbf{a})=V_x(A,\mathbf{a})$ as $f(a_ix)=f(xa_i)$, $(i=1,2,\dots,n)$. If $x=e$(identity of A) with $\|e\|=1$, then $V_e(A,\mathbf{a})=V(\mathbf{a})$, where $V(\mathbf{a})$

denotes the (Bonsall and Duncan) joint numerical range of $\mathbf{a}$ [1]. Thus the concept of joint numerical range is a special case of that of relative numerical range.

LEMMA 2.2. *Let* $\mathbf{a} = (a_1, \ldots, a_n) \in A^n$ *and* $\mathbf{b} = (b_1, \ldots, b_n) \in A^n$ *be n-tuples of elements in A, and let* $x \in A$ *and* $\alpha, \beta \in K$. *Then*

(1) $V_x(A, \alpha\mathbf{a} + \beta\mathbf{b}) \subseteq \alpha V_x(A, \mathbf{a}) + \beta V_x(A, \mathbf{b})$, *and* $V_x(A, \alpha\mathbf{a}) = \alpha V_x(A, \mathbf{a})$,

(2) $v_x(\mathbf{a} + \mathbf{b}) \leq v_x(\mathbf{a}) + v_x(\mathbf{b})$ *and* $v_x(\alpha\mathbf{a}) = |\alpha| v_x(\mathbf{a})$,

(3) $v_x(\mathbf{a}) \leq \max\{\|\mathbf{a}x\|, \|x\mathbf{a}\|\}$, *where* $\mathbf{a}x$ *denotes* $(a_1x, \ldots, a_nx)$.

Proof. (1) Let $f \in D(A, x)$. Since $f((\alpha a_i + \beta b_i)x) = \alpha f(a_ix) + \beta f(b_ix)$ and $f((\alpha a_i)x) = f(\alpha(a_ix)) = \alpha f(a_ix)$ for $i = 1, 2, \ldots, n$, $V_x^R(A, \alpha\mathbf{a} + \beta\mathbf{b}) \subseteq \alpha V_x^R(A, \mathbf{a}) + \beta_x^R(A, \mathbf{a})$ and $V_x^R(A, \alpha\mathbf{a}) = \alpha V_x^R(A, \mathbf{a})$.

Similar statements hold in terms of V^L, hence taking unions

$$V_x(A, \alpha\mathbf{a} + \beta\mathbf{b}) \subseteq \alpha V_x(A, \mathbf{a}) + \beta V_x(A, \mathbf{b}) \text{ and } V_x(A, \alpha\mathbf{a}) = \alpha V_x(A, \mathbf{a}).$$

(2) These follow from (1).

(3) $\lambda = (\lambda_1, \ldots, \lambda_n) \in V_x(A, \mathbf{a})$ implies $\lambda = (f(a_1x), \ldots, f(a_nx))$ or $(f(xa_1), \ldots, f(xa_n))$ for some $f \in D(A, x)$. Hence

$$\begin{aligned} |\lambda| = |(f(a_1x), \ldots, f(a_nx))| &= \left(\sum_{n=1}^{n} |f(a_ix)|^2\right)^{1/2} \\ &\leq \left(\sum_{i=1}^{n} \|f\|^2 \|a_ix\|^2\right)^{1/2} \\ &= \|(a_1x, \ldots, a_nx)\| = \|\mathbf{a}x\| \end{aligned}$$

or

$$\begin{aligned} |\lambda| = |(f(xa_1), \ldots, f(xa_n))| &= \left(\sum_{n=1}^{n} |f(xa_i)|^2\right)^{1/2} \\ &\leq \left(\sum_{i=1}^{n} \|f\|^2 \|xa_i\|^2\right)^{1/2} \\ &= \|(xa_1, \ldots, xa_n)\| = \|x\mathbf{a}\|. \end{aligned}$$

We note that the inclusion relation in (1) cannot be replaced by the equality in general e.g. take $\mathbf{a} = -\mathbf{b}$.

LEMMA 2.3. *Let* $\mathbf{a} = (a_1, \ldots, a_n) \in A^n$ *and* $x \in A$. *Then*

(1) $D(A,x)$ *is a weak* compact convex subset of* A'.
(2) $V_x^R(A,\mathbf{a})$ *and* $V_x^L(A,\mathbf{a})$ *are compact convex subsets of* K, *hence* $V_x(A,\mathbf{a})$ *is a compact subset of* K.

Proof. (1) Let $f, g \in D(A,x)$ and let λ be any number in [0,1]. Then $\|\lambda f + (1-\lambda)g\| \le \lambda\|f\| + (1-\lambda)\|g\| = 1$ and $(\lambda f + (1-\lambda)g)(x) = \|x\|$. So $\|\lambda f + (1-\lambda)g\| = 1$ and $\lambda f + (1-\lambda)g \in D(A,x)$. Therefore $D(A,x)$ is convex.

Define $e_x(f) = f(x)$. Then e_x is *weak** continuous, i.e., continuous in the pointwise convergence topology on A'. By [3], $D \equiv \{f \in A' : \|f\| \le 1\}$ is *weak** compact. Hence

$$D(A,x) = D \cap e_x^{-1}(\{\|x\|\})$$

is a *weak** closed subset of D and so is *weak** compact.

(2) Define $e_{a_i x}(f) = f(a_i x)\,(i = 1, \ldots, n)$. Then $e_{a_i x}$ is weak* continuous for each $i = 1, \ldots, n$ and so the function $F : D(A,x) \longrightarrow K^n$ defined by $F(f) = (e_{a_1 x}(f), \ldots, e_{a_n x}(f)) = (f(a_1 x), \ldots, f(a_n x))$ is weak* continuous. Hence $V_x^R(A,\mathbf{a}) = F(D(A,x))$ is a compact subset of K^n. As F is linear and $D(A,x)$ is convex, $V_x^R(A,\mathbf{a})$ is convex. Similarly $V_x^L(A,\mathbf{a})$ is a compact convex subset of K^n. Hence $V_x(A,a) = V_x^R(A,\mathbf{a}) \cup V_x^L(A,\mathbf{a})$ is a compact subset of K^n.

THEOREM 2.4. *Let* $\mathbf{a} = (a_1, \ldots, a_n) \in A^n$ *and let* x *be any nonzero element of* A. *Then*

(1) *If* $a_i x = x$ *for all* $i = 1, \ldots, n$, *then* $V_x^R(A,\mathbf{a}) = \{\|x\|(1, \ldots, 1)\}$.
(2) *If* $V_x^R(A,\mathbf{a}) = \{\|x\|(1, \ldots, 1)\}$, *then either* $a_i x = x$ *or* $0 < \operatorname{dist}(x, Ka_i x) < \|x\|$ *for all* i.

Proof. (1) This follows from the definition.

(2) Suppose that $V_x^R(A,\mathbf{a}) = \{\|x\|(1, \ldots, 1)\}$. First we note that $\operatorname{dist}(x, Ka_i x) = \inf_\lambda \|x - \lambda a_i x\| \le \|x\|$ for $i = 1, \ldots, n$.

If $\operatorname{dist}(x, Ka_i x) = \|x\|$ for some $i (1 \le i \le n)$, then since $\|x\| \ne 0$, $x \notin Ka_i x$, and so by ([3, p. 82] or [4, p.64]) there exists $f \in A'$ such that $\|f\| = 1, f(x) = \|x\|$ and $f(a_i x) = 0$. Thus $(f(a_1 x), \ldots, f(a_i x), \ldots, f(a_n x)) = (f(a_1 x), \ldots, 0, \ldots, f(a_n x)) \ne \|x\|(1, \ldots, 1, \ldots, 1)$ This is a contradiction to our hypothesis. Hence $\operatorname{dist}(x, Ka_i x) < \|x\|$ for all i.

If $0 = \operatorname{dist}(x, Ka_i x)$ for some $i (1 \leq i \leq n)$, then by our hypothesis $x = a_i x$.

We have the similar statement for the left relative numerical range:

THEOREM 2.5. *Let* $\mathbf{a} = (a_1, \ldots, a_n) \in A$ *and let* x *be any nonzero element of* A. *Then*

(1) *If* $xa_i = x$ *for each* $i = 1, \ldots, n$, *then* $V_x^L(A, \mathbf{a}) = \{\|x\|(1, \ldots, 1)\}$.
(2) *If* $V_x^L(A, \mathbf{a}) = \{\|x\|(1, \ldots, 1)\}$, *then either* $xa_i = x$ *or* $0 < \operatorname{dist}(x, Kxa_i) < \|x\|$ *for all* i.

COROLLARY 2.6. *If* $a \in A$, *and* $a^2 = a$, *then*

$$V_a(A, (a, \ldots a)) = \{\|a\|(1, \ldots, 1)\}.$$

LEMMA 2.7. *Let* $\mathbf{a} = (a_1, \ldots, a_n), \mathbf{b} = (b_1, \ldots, b_n)$ *be n-tuples of elements in* A, *and let* $x \in A$. *Let* $N_\epsilon = N(0, \epsilon)$ *denote the open ball at 0 of radius* ϵ *in* K^n. *If* $\|\mathbf{a} - \mathbf{b}\| = (\sum_{i=1}^n \|a_i - b_i\|^2)^{1/2} < \epsilon$, *then* $V_x^R(A, \mathbf{b}) \subseteq V_x^R(A, \mathbf{a}) + \|x\| N_\epsilon$ *and* $V_x^R(A, \mathbf{a}) \subseteq V_x^R(A, \mathbf{b}) + \|x\| N_\epsilon$.

Proof. Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in V_x^R(A, \mathbf{b})$. There exists $f \in D(A, x)$ such that $\lambda = (f(b_1 x), \ldots, f(b_n x))$. Thus

$$\begin{aligned}
|\lambda - (f(a_1 x), \ldots, f(a_n x))| &= |(f(b_1 x), \ldots, f(b_n x)) - (f(a_1 x), \ldots, f(a_n x))|. \\
&= \left(\sum_{i=1}^n |f((b_i - a_i)x)|^2 \right)^{1/2} \\
&\leq \left(\sum_{i=1}^n \|b_i - a_i\|^2 \|x\|^2 \right)^{1/2} \\
&= \|\mathbf{b} - \mathbf{a}\| \|x\| < \|x\| \epsilon.
\end{aligned}$$

So $\lambda \in V_x^R(A, \mathbf{a}) + \|x\| N_\epsilon$. Similarly $V_x^R(A, \mathbf{a}) \subseteq V_x^R(A, \mathbf{b}) + \|x\| N_\epsilon$.

REMARK 2.8. The previous lemma is true for the left relative numerical range by a similar proof.

THEOREM 2.9. *Let* $\mathbf{a} = (a_1, \ldots, a_n), \mathbf{b} = (b_1, \ldots, b_n)$ *be n-tuples of elements in* A, *and let* $x \in A$ *and* $N_\epsilon = N(0, \epsilon)$. *If* $\|\mathbf{a} - \mathbf{b}\| < \epsilon$, *then* $V_x(A, \mathbf{b}) \subseteq V_x(A, \mathbf{a}) + \|x\| N_\epsilon$ *and* $V_x(A, \mathbf{a}) \subseteq V_x(A, \mathbf{b}) + \|x\| N_\epsilon$.

Proof. By the Lemma 2.7 and Remark 2.8, $V_x^R(A, \mathbf{a}) \subseteq V_x^R(A, \mathbf{b}) + \|x\| N_\epsilon$ and $V_x^L(A, \mathbf{a}) \subseteq V_x^L(A, \mathbf{b}) + \|x\| N_\epsilon$. Hence

$$\begin{aligned} V_x(A, \mathbf{a}) &= V_x^R(A, \mathbf{a}) \cup V_x^L(A, \mathbf{a}) \\ &\subseteq (V_x^R(A, \mathbf{b}) + \|x\| N_\epsilon) \cup (V_x^L(A, \mathbf{b}) + \|x\| N_\epsilon) \\ &= \{V_x^L(A, \mathbf{b}) \cup V_x^R(A, \mathbf{b})\} + \|x\| N_\epsilon \\ &= V_x(A, \mathbf{b}) + \|x\| N_\epsilon. \end{aligned}$$

Therefore

$$V_x(A, \mathbf{a}) \subseteq V_x(A, \mathbf{b}) + \|x\| N_\epsilon.$$

Exchanging $\mathbf{a}$ and $\mathbf{b}$,

$$V_x(A, \mathbf{b}) \subseteq V_x(A, \mathbf{a}) + \|x\| N_\epsilon.$$

Consider a pair of compact subsets of the n-dimensional Euclidean space, M and N and define $d(M, N) = \inf\{\epsilon : M \subseteq N + N_\epsilon, N \subseteq M + N_\epsilon\}$. Then for $\mathbf{a} = (a_1, \ldots, a_n), \mathbf{b} = (b_1, \ldots, b_n) \in A^n$ and $x \in A$ we can consider $d(V_x^R(A, \mathbf{a}), V_x^R(A, \mathbf{b}))$ as a metric, the "Hausdorff metric" on sets associated with $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$.

THEOREM 2.10. *For each* $x \in A$, $V_x^R(\cdot)$ *is a continuous function from* A^n *endowed with the norm topology to the family of compact subsets of* K^n, *endowed with the Hausdorff metric topology. Also* $v_x^R(\cdot)$ *is a continuous real-valued function on* A^n.

Proof. Let $\mathbf{a} = (a_1, \ldots, a_n), \mathbf{b} = (b_1, \ldots, b_n) \in A^n$ with $\|\mathbf{a} - \mathbf{b}\| < \epsilon$. Then by Lemma 2.7,

$$d(V_x^R(A, \mathbf{a}), V_x^R(A, \mathbf{b})) \le \epsilon \|x\|,$$

and so $V_x^R(\cdot)$ is continuous.

Also $v_x^R(\mathbf{a}) \le v_x^R(\mathbf{b}) + \epsilon\|x\|$ and $v_x^R(\mathbf{b}) \le v_x^R(\mathbf{a}) + \epsilon\|x\|$ imply $|v_x^R(\mathbf{a}) - v_x^R(\mathbf{b})| \le \epsilon\|x\|$. So $v_x^R(\cdot)$ is a continuous function.

REMARK 2.11.

(1) The previous theorem is true for the left relative numerical range V_x^L and numerical radius v_x^L.
(2) The previous theorem is true for the relative numerical range V_x and numerical radius v_x.

The following theorem gives the invariance of relative numerical ranges under isometric algebraic homomorphism.

THEOREM 2.12. *Let ϕ be an isometric algebraic homomorphism of a normed algebra A into a normed algebra B and let $x \in A$. Then*

$$V_{\phi(x)}^R(B, (\phi(a_1), \dots, \phi(a_n))) = V_x^R(A, \mathbf{a})$$

for all $\mathbf{a} = (a_1, \dots, a_n) \in A^n$.

Proof. Let $\lambda = (\lambda_1, \dots, \lambda_n) \in V_{\phi(x)}^R(B, (\phi(a_1), \dots, \phi(a_n)))$. Then there exists $g \in D(B, \phi(x))$ such that $\lambda = (g(\phi(a_1 x)), \dots, g(\phi(a_n x)))$. Define f on A by $f(z) = g(\phi(z)), z \in A$. Clearly, f is linear and $\|f\| \leq 1$. Since ϕ is an isometry, $\|\phi(z)\| = \|z\|$ implies $\|f\| = 1$ and so $\lambda = (f(a_1 x), \dots, f(a_n x)) \in V_x^R(A, \mathbf{a})$.

Conversely if $\mu = (\mu_1, \dots, \mu_n) \in V_x^R(A, \mathbf{a})$, then there exists $f \in D(A, x)$ such that $\mu = (f(a_1 x), \dots, f(a_n x))$. Define g on $\phi(A) = \{\phi(z) : z \in A\}$ by $g(\phi(z)) = f(z), z \in A$. Then again we see that g is a bounded linear functional on $\phi(A)$ with $\|g\| = 1$ because ϕ is an isometry. By ([3, p.81] or [4, p. 63]) g can be extended to a bounded linear functional h on B with $\|h\| = \|g\| = 1$ and $h(\phi(z)) = f(z)$ for any $z \in A$. Hence

$$\begin{aligned}
\mu &= (f(a_1 x), \dots, f(a_n x)) \\
&= (g(\phi(a_1 x)), \dots, g(\phi(a_n x))) \\
&= (h(\phi(a_1 x)), \dots, h(\phi(a_n x))) \in V_{\phi(x)}^R(B, (\phi(a_1), \dots, \phi(a_n))).
\end{aligned}$$

REMARK 2.13. The previous theorem is true for the left relative numerical range.

COROLLARY 2.14. *Let ϕ be an isometric algebraic homomorphism of a normed algebra A into a normed algebra B, and let $x \in A$. Then*

$$V_{\phi(x)}(B, (\phi(a_1), \dots, \phi(a_n))) = V_x(A, \mathbf{a})$$

for all $\mathbf{a} = (a_1, \dots, a_n) \in A^n$.

We note that the invariance of relative numerical ranges under an algebraic homomorphism in Theorem 2.12 does not imply isometry. For we consider an algebra A having divisors of zero, $a \neq 0, b \neq 0, ab = 0$ and a zero homomorphism ϕ of A into an arbitrary algebra B. Then $V_b^R(A, a) = \{0\} = V_{\phi(b)}^R(B, \phi(a))$ but ϕ is not an isometry. Another example is

EXAMPLE 2.15. Let X be a normed space. Consider

$$Y = \{ \begin{bmatrix} x \\ \lambda \end{bmatrix} : x \in X, \lambda \in \mathbb{C} \}.$$

Let $B(Y)$ denote the algebra of all bounded linear operators on Y. If

$$B_1 = \{ \begin{bmatrix} 0 & \omega \\ 0 & 0 \end{bmatrix} : \omega \in X \},$$

then B_1 is a subalgebra of $B(Y)$. Let $\mathbf{a} = (a_1, \dots, a_n) \in B_1^n$ and $x \in B_1$. Then by Theorem 2.12 we have

$$V_x^R(B_1, \mathbf{a}) = V_{T_x}^R(B(Y), (T_{a_1}, \dots, T_{a_n})) = \{(0, \dots, 0)\}$$

and

$$V_x^L(B_1, \mathbf{a}) = V_{T_x}^L(B(Y), (T_{a_1}, \dots, T_{a_n})) = \{(0, \dots, 0)\}$$

where $\phi : B(Y) \longrightarrow B(Y), \phi(z) = T_z$ is an algebraic homomorphism. (T_z is an operator given by $T_z y = zy$ for $y \in Y$.) Hence

$$V_x(B_1, \mathbf{a}) = V_{T_x}(B(Y), (T_{a_1}, \dots, T_{a_n})) = \{(0, \dots, 0)\}.$$

But the mapping $\phi|_{B_1} : B_1 \longrightarrow B(Y)$ is an algebra homomorphism, though not an isometry because for $u, v \in X$,

$$\begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The following is a good generalization of the Bonsall and Duncan's result which is a particular case of Corollary 2.14 (put $\phi = i : A \longrightarrow B$ identity map).

COROLLARY 2.16. *If B is a subalgebra of a normed algebra A, $x \in B$, and $\mathbf{b} = (b_1, \ldots, b_n) \in B^n$, then $V_x(B, \mathbf{b}) = V_x(A, \mathbf{b})$.*

DEFINITION 2.17. Let A be a normed algebra over K. An element $a \in A$ is said to have right(left) regular norm if

$$\|a\| = \sup_{\|x\| \leq 1} \|ax\| \quad (\|a\| = \sup_{\|x\| \leq 1} \|xa\|).$$

If each $a \in A$ has right(left) regular norm, then A is said to have right(left) regular norm.

The following theorem is an application of Theorem 2.12.

THEOREM 2.18. *Let A be a normed algebra with right regular norm. Suppose $B(A)$ is the algebra of all bounded linear operators on A. Then for $\mathbf{x} = (x_1, \ldots, x_n) \in A^n$ and $y \in A$,*

$$V_y^R(A, \mathbf{x}) = V_{T_y}^R(B(A), (T_{x_1}, \ldots, T_{x_n})),$$

where T_z is the left regular representation on A.

Proof. The function $\phi : A \longrightarrow B(A)$ defined by $\phi(a) = T_a$ is an algebraic homomorphism. Since A has right regular norm, $\|T_a\| = \sup_{\|x\| \leq 1} \|ax\| = \|a\|$, and so ϕ is an isometric. Hence by Theorem 2.12, the theorem follows.

REMARK 2.19. The previous theorem is true for the left relative numerical range.

COROLLARY 2.20. *Let A be a normed algebra with right regular norm. Suppose $B(A)$ is the algebra of all bounded linear operators on A. Then for $\mathbf{x} = (x_1, \ldots, x_n) \in A^n$ and $y \in A$,*

$$V_y(A, \mathbf{x}) = V_{T_y}(B(A), (T_{x_1}, \ldots, T_{x_n})),$$

where T_z is the left regular representation or on A.

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