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ON A GENERALIZATION OF THE PÓLYA-WIMAN CONJECTURE

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1. Introduction

This paper is concerned with the zeros of successive derivatives of real entire functions. In order to state our results, we introduce the following notations: An entire function which assumes only real values on the real axis is said to be a *real entire function.* Thus, if a complex number is a zero of a real entire function, then its conjugate is also a zero of the same function. An entire function $g(z)$ is said to be of genus 0, if it can be expressed in the form

$$g(z) = cz^n \prod_j \left(1 - \frac{z}{a_j}\right),$$

where c is a constant, n is a nonnegative integer, and $a_1, a_2, \ldots$ are nonzero complex numbers with $\sum |a_j|^{-1} < \infty$. On the other hand, an entire function $g(z)$ is said to be of genus 1, if it is not of genus 0 and if it can be expressed in the form

$$g(z) = cz^n e^{\gamma z} \prod_j \left(1 - \frac{z}{a_j}\right) e^{\frac{z}{a_j}},$$

where c, γ are constants, n is a nonnegative integer, and $a_1, a_2, \ldots$ are nonzero complex numbers with $\sum |a_j|^{-2} < \infty$. A real entire function $f(z)$ is said to be of *genus* 1^*, if it can be represented in the form $f(z) = e^{-\alpha z^2} g(z)$ where $\alpha \geq 0$ and $g(z)$ is a real polynomial or a real entire function of genus 0 or a real entire function of genus 1.

In 1930, Pólya and Wiman conjectured that *if a real entire function $f(z)$ of genus 1^* has only a finite number of nonreal zeros, then there*

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is a positive integer m_0 *such that if* $m \geq m_0$, *then* $f^{(m)}(z)$ *has only real zeros* [P1, P2, P3, W1, W2]. This conjecture has been completely proved by T. Craven, G. Csordas, W. Smith, and the author [CCS1, CCS2, K1, K2]. In this paper, we will prove the following generalization of the Pólya–Wiman conjecture.

THEOREM. *Let* C *be an arbitrary nonnegative real number. If a real entire function* $f(z)$ *of genus* 1^* *has only a finite number of zeros outside the infinite strip* $|\operatorname{Im} z| \leq C$, *then there is a positive integer* m_0 *such that if* $m \geq m_0$, *then all the zeros of* $f^{(m)}(z)$ *are distributed in the infinite strip* $|\operatorname{Im} z| \leq C$.

2. Proof of the theorem

Before proving our theorem, let us introduce some terminologies. The *order* ρ of an entire function $f(z)$ is defined by

$$\rho = \overline{\lim_{r\to\infty}} \frac{\log\log M(r;f)}{\log r},$$

where $M(r;f)$ is the maximum modulus of $f(z)$ on the circle $|z| = r$, that is

$$M(r;f) = \max_{|z|=r} |f(z)|.$$

If an entire function $f(z)$ is of order ρ and if $0 < \rho < \infty$, then the *type* τ of $f(z)$ is defined by

$$\tau = \overline{\lim_{r\to\infty}} \frac{\log M(r;f)}{r^\rho}.$$

If $\tau = 0$, the function $f(z)$ is said to be of *minimal type*, if $0 < \tau < \infty$ of *mean type*, and if $\tau = \infty$ of *maximal type*. If the entire function $f(z)$ is represented by $\sum a_n z^n$, then its order ρ and type τ satisfy the following equations [L, p. 4, Theorem 2].

$$\rho = \overline{\lim_{n\to\infty}} \frac{n\log n}{-\log|a_n|}, \quad (e\tau\rho)^{\frac{1}{\rho}} = \overline{\lim_{n\to\infty}} n^{\frac{1}{\rho}} |a_n|^{\frac{1}{n}}.$$

In particular, order and type are unchanged by differentiation.

Now we can prove our theorem. Let $f(z)$ be a nonconstant real entire function of genus 1^*. Then $f(z)$ can be expressed in the form

$$(1) \qquad \begin{aligned} f(z) = cz^n e^{-\alpha z^2+\beta z} \prod_k \left(1 - \frac{z}{a_k}\right) e^{\frac{z}{a_k}} \\ \times \prod_j \left(1-\frac{z}{c_j}\right)\left(1 - \frac{z}{\bar{c}_j}\right) e^{\left(\frac{1}{c_j}+\frac{1}{\bar{c}_j}\right)z}, \end{aligned}$$

where n is a nonnegative integer, $\alpha \geq 0$, c and β are real constants, a_k are the real zeros of $f(z)$ which are different from 0, and c_j, $\bar{c}_j$ are the nonreal zeros of $f(z)$. Of course we have $\sum |a_k|^{-2} < \infty$ and $\sum |c_j|^{-2} < \infty$. For each pair $(c_j, \bar{c}_j)$ of nonreal zeros of $f(z)$ the closed disk

$$(x - \operatorname{Re} c_j)^2 + y^2 \leq (\operatorname{Im} c_j)^2$$

is called the *Jensen disk* of $f(z)$ associated with the pair $(c_j, \bar{c}_j)$ of nonreal zeros of $f(z)$, and the union of all the Jensen disks of $f(z)$ will be denoted by $\mathcal{J}(f)$.

From (1), the logarithmic derivative of $f(z)$ is given by

$$\begin{aligned} \frac{f'(z)}{f(z)} = \frac{n}{z} - 2\alpha z + \beta + \sum_k \left(\frac{1}{z-a_k} + \frac{1}{a_k}\right) \\ + \sum_j \left(\frac{1}{z-c_j} + \frac{1}{z-\bar{c}_j} + \frac{2\operatorname{Re} c_j}{|c_j|^2}\right), \end{aligned}$$

and hence we have the following:

$$z \notin \mathbb{R} \cup \mathcal{J}(f) \Rightarrow (\operatorname{Im} z)\left(\operatorname{Im} \frac{f'(z)}{f(z)}\right) < 0.$$

In particular, all the nonreal zeros of $f'(z)$ are in the set $\mathcal{J}(f)$. This fact was first announced by Jensen, and later proved by Nagy and Walsh.

JENSEN'S THEOREM. *Let $f(z)$ be a nonconstant real entire function of genus 1^* and let z_1 be a nonreal zero of $f'(z)$. Then there is a nonreal zero z_0 of $f(z)$ such that*

$$|z_1 - \operatorname{Re} z_0| \leq \operatorname{Im} z_0.$$

For each nonnegative real number C let $\mathcal{LP}^C$ be the class of all real entire functions of genus 1^* whose zeros are distributed in the infinite strip $|\operatorname{Im} z| \leq C$. As an immediate consequence of [L, p. 331, Theorem 3], we have the following: *a real entire function $f(z)$ is in the class $\mathcal{LP}^C$, if and only if there is a sequence $\{P_n(z)\}$ of real polynomials such that* (a) *for all n the zeros of $P_n(z)$ are distributed in the infinite strip $|\operatorname{Im} z| \leq C$, and* (b) *$\{P_n(z)\}$ converges to $f(z)$ uniformly on compact sets in the complex plane.* In particular, the Gauss–Lucas theorem implies that the class $\mathcal{LP}^C$ is closed under differentiation for each nonnegative real number C.

Now assume that C is a nonnegative real number, and that $f(z)$ is a nonconstant real entire function of genus 1^* which has only a finite number of zeros outside the infinite strip $|\operatorname{Im} z| \leq C$. We can find a positive real number C' such that $f \in \mathcal{LP}^{C'}$. Since the class $\mathcal{LP}^{C'}$ is closed under differentiation, $f^{(n)} \in \mathcal{LP}^{C'}$ for all $n = 1, 2, \ldots$, and we wish to show that there is a positive integer m_0 such that $f^{(m_0)} \in \mathcal{LP}^C$. To obtain a contradiction, suppose that $f^{(n)} \notin \mathcal{LP}^C$ for all $n = 1, 2, \ldots$.

For $n = 0, 1, 2, \ldots$, let $X_n = \{z : \operatorname{Im} z > C, f^{(n)}(z) = 0\}$. From the assumption that X_0 is a finite set, the set $J(f) \setminus \{z : |\operatorname{Im} z| \leq C\}$ is bounded. On the other hand, Jensen's theorem implies that $X_1 \subset J(f) \setminus \{z : |\operatorname{Im} z| \leq C\}$. In particular, the set X_1 is bounded. Since X_1 is discrete, X_1 is a finite set. Using the same argument, we can show that X_2 is finite, and then X_3 is finite, and so on.

Since each X_n is a nonempty finite set, $X = \prod_{n=0}^{\infty} X_n$ is a nonempty compact space with respect to the product topology. Define E_n, $n = 1, 2, \ldots$, as follows.

$$E_n = \{(\zeta_0, \zeta_1, \zeta_2, \ldots) \in X : |\zeta_{j+1} - \operatorname{Re} \zeta_j| \leq \operatorname{Im} \zeta_j\,, \quad j = 0, 1, \ldots, n\}.$$

Then each E_n is a closed subset of the compact space X and $E_1 \supset E_2 \supset \cdots$. Moreover Jensen's theorem implies that $E_n \neq \emptyset$, $n = 1, 2, \ldots$. Therefore $\bigcap_{n=1}^{\infty} E_n \neq \emptyset$. This means that there is an infinite sequence $z_0,\ z_1,\ z_2, \ldots,$ of complex numbers such that for all $n = 0, 1, 2, \ldots,$ $\operatorname{Im} z_n > C$, $f^{(n)}(z_n) = 0$ and

$$|z_{n+1} - \operatorname{Re} z_n| \leq \operatorname{Im} z_n. \tag{2}$$

Let $z_n = \alpha_n + i\beta_n$, $n = 0, 1, 2, \dots$. Then (2) implies that $\{\beta_n\}$ is a nonincreasing sequence of positive real numbers. Moreover, by an induction, we have

$$\tag{3} \begin{aligned} &|z_m - z_{m+1}| + |z_{m+1} - z_{m+2}| + \cdots + |z_{m+n-1} - z_{m+n}| \\ &\le \beta_m - \beta_{m+n} + \sqrt{n({\beta_m}^2 - {\beta_{m+n}}^2)}, \end{aligned}$$

for $m = 0, 1, 2, \dots$, and for $n = 1, 2, \dots$.

Let $\beta = \lim_{n\to\infty} \beta_n$. From (3), we have

$$\tag{4} \begin{aligned} &\overline{\lim_{n\to\infty}} \frac{|z_m - z_{m+1}| + |z_{m+1} - z_{m+2}| + \cdots + |z_{m+n-1} - z_{m+n}|}{n^{\frac{1}{2}}} \\ &\le \sqrt{{\beta_m}^2 - \beta^2} \longrightarrow 0, \qquad \text{as} \qquad m \to \infty. \end{aligned}$$

To complete the proof of our theorem, we need the following theorem of Gontcharoff.

GONTCHAROFF'S THEOREM. *Let $f(z)$ be an entire function and assume the following:*

(a) $M(r; f) = O(\exp(A + \varepsilon)r^\rho)$ *for all* $\varepsilon > 0$,

(b) $f^{(n)}(z_n) = 0, \quad n = 0, 1, 2, \dots,$

(c) $\displaystyle \overline{\lim_{n\to\infty}} \frac{|z_0 - z_1| + |z_1 - z_2| + \cdots + |z_{n-1} - z_n|}{n^{\frac{1}{\rho}}} = \tau,$

(d) $\rho A \tau^\rho < \omega^\rho (1 + \omega)^{1-\rho}$, *where ω is the positive root of the equation* $\omega^\rho e^{\omega+1} = 1$.

Then $f(z) \equiv 0$.

Proof. See [G, pp. 29–31].

From (1), we see that $f(z)$ is at most of order 2 and mean type. Hence there is a positive real number A such that $M(r; f) = O(\exp(Ar^2))$ as $r \to \infty$. Choose a positive real number τ so that $2A\tau^2 < \omega^2(1 + \omega)^{-1}$, where ω is the positive root of the equation $\omega^2 e^{\omega+1} = 1$. From (4), there is a positive integer m such that

$$\overline{\lim_{n\to\infty}} \frac{|z_m - z_{m+1}| + |z_{m+1} - z_{m+2}| + \cdots + |z_{m+n-1} - z_{m+n}|}{n^{\frac{1}{2}}} < \tau.$$

Since order and type are unchanged by differentiation, $M(r; f^{(m)}) = O(\exp(Ar^2))$ as $r \to \infty$. Therefore, Gontcharoff's theorem implies that $f^{(m)}(z) \equiv 0$. In particular, $f(z)$ is a real polynomial. But if $f(z)$ is a real polynomial, then there must be a positive integer n such that $f^{(n)} \in \mathcal{LP}^C$.

From this contradiction, we see that our theorem is true.

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