Comm. Korean Math. Soc. 9 (1994), No. 4, pp. 809–817

# ON MINIMAL STABLE RATIONAL INTERPOLANTS

JEONGOOK KIM

## 1. Introduction

Let $z_1, \dots, z_n$ be an n-tuple of distinct points in the open unit disk $\mathcal{D}$ in the complex plane. The classical Nevannlina-Pick interpolation problem is to find a function $f(z)$ satisfying given interpolation conditions

$$f(z_i) = \omega_i, \quad i = 1, \dots, n \tag{1.1}$$

and the stability constraint

$$|f(z)| \leq 1, \qquad \text{for } z \in \mathcal{D}. \tag{1.2}$$

An interpolation problem considered in [AA2] and [B] is a problem of finding a rational function $f(z)$ which satisfies (1.1) and (1.2) together with interpolation conditions at so called *mirror image array* of points, i.e.

$$f(\bar{z}_i^{-1}) = \bar{\omega}_i^{-1}, \quad i = 1, \dots, n-1 \tag{1.3}$$

(we assume $z_i \neq 0$, $\omega_i \neq 0$ for all $1 \leq i \leq n$). On the other hand papers such as [AA1], [ABKW], [K] study rational interpolants (without considering stability condition (1.2)) which has the minimal possible *complexity*. Here, the complexity of a rational function is measured by *McMillan degree*. The McMillan degree of a rational function $f(z)$ represented as

$$f(z) = \frac{n(z)}{d(z)}, \quad n(z), d(z) \text{ coprime}$$

Received March 4, 1994.

The author was partially supported by KOSEF(project number:931-0100-019-1) and BSRI-94-1426, Ministry of Education, Korea

is defined by

$$\delta(f) := \max\{\deg n(z),\ \deg d(z)\}. \tag{1.4}$$

By a *stable minimal solution*, let us mean a rational solution satisfying (1.1)-(1.3) and has the minimal possible McMillan degree.

In this paper the problem of finding a stable minimal solution is studied and a relationship between a minimal solution of (1.1), (1.3) and a stable minimal solution is found. The central result is the following. If a minimal solution $f_{\min}$ of (1.1), (1.3) is unique, then it is automatically stable (i.e., (1.2) is fulfilled) and $\delta(f_{\min}) = n-1$. Otherwise, a minimal solution of (1.1) and (1.3) is not necessarily stable and $\delta(f_{\min}) = n$. But, one can find a minimal stable solution of McMillan degree $n$ in the latter case. These results are followed by applying [ABKW] to the parameterization of all Nevannlina-Pick interpolants given by [N] and corrects [AA2].

## 2. The recursive algorithm of Nevannlina

The contribution of Pick is that solutions of (1.1), (1.2) exist if and only if the so called Pick matrix

$$\Lambda = \left[\frac{1-\omega_i\bar{\omega}_j}{1-z_i\bar{z}_j}\right]_{1\le i,j\le n} \tag{2.1}$$

is nonnegative definite (see [P](1916)). In the rest of this paper we assume that the Pick matrix $\Lambda$ is positive definite and $\beta$ denotes the set of analytic functions bounded by 1 on $\mathcal{D}$. Nevannlina (see [N](1919)), on the other hand, derived a recursive algorithm which leads to a linear fractional parameterization for the set of all solutions (when there exist more than one solution). The recursive algorithm for finding all solutions given by Nevannlina based on an inductive method can be formulated as follows (see [N] for the details). Let

$$\Theta(z) = \Theta_1(z)\Theta_2(z)\cdots\Theta_n(z) \tag{2.2}$$

where for $1 \le i \le n$,

$$\Theta_i(z) = \begin{bmatrix} \dfrac{z-z_i}{\overline{\omega_i^{i-1}}\,(z-z_i)} & \dfrac{\omega_i^{i-1}(1-z\bar{z}_i)}{1-z\bar{z}_i} \end{bmatrix} \tag{2.3}$$

with $\omega_j^0 = \omega_j$ and with

$$\omega_j^i = \frac{\omega_j^{i-1} - \omega_i^{i-1}}{1 - \omega_j^{i-1}\overline{\omega_i^{i-1}}} \Big/ \frac{z_j - z_i}{1 - z_j \bar{z}_i} \tag{2.4}$$

for $1 \le i \le n$ and $(i+1) \le j \le n$. Then $f$ is a stable rational solution if and only if there exists a rational function $g \in \beta$ so that

$$f(z) = \frac{n_g(z)}{d_g(z)}, \tag{2.5}$$

where

$$\begin{bmatrix} n_g(z) \\ d_g(z) \end{bmatrix} = \Theta(z) \begin{bmatrix} g(z) \\ 1 \end{bmatrix}.$$

REMARK. Nevannlina's solution algorithm given in [N] is for the analytic functions in $\beta$ rather than for rational functions in $\beta$. But the difference between the algorithms is only the choice of $f_n$ at the end of $n$ inductive steps. If we choose $f_n$ as an analytic function in $\beta$, then we get a function in $\beta$ and interpolating all the n-points, but the function is not necessarily a rational function.

## 3. Minimal stable interpolation

First, some terminologies are introduced. By an $m \times m$ *regular matrix polynomial*, we mean an $m \times m$ matrix whose each entry is a polynomial and $\det P(z)$ is not identically zero. For a pair of *controllable* matrices $(A, B)$ ((see [Ka] for the definition of controllability) of sizes $n \times n$, $n \times m$ given by

$$A = \begin{bmatrix} z_1 & & 0 \\ & \ddots & \\ 0 & & z_n \end{bmatrix}, \quad B = \begin{bmatrix} x_1 & -y_1 \\ \vdots & \vdots \\ x_n & -y_n \end{bmatrix}$$

$(z_i \neq z_j$ if $i \neq j)$, it is said that $(A, B)$ is a *left null pair* of $P(z)$ if

$$[x_i \ -y_i]P(z_i) = 0, \text{ for } i = 1, \dots, n$$

and $\det P(z) \neq 0$ for all $z \notin \{z_1, \dots, z_n\}$. ( For a definition of a left null pair for a matrix polynomial for more general case, the cases where if $z_i = z_j$ or det $P(z)$ has a zero of high multiplicities, see [BGR].) By the degree of a $n \times 1$ polynomial vector, we mean the highest degree of its components. Let $k_i$ be the degree of the $i$-th column of $P(z)$. Then, $P(z)$ is said to be *column reduced* if

$$\sum k_i = \text{ deg det } P(z).$$

Now we suppose any of $z_i$, $\omega_j$ in (1.1) is not zero. Let

$$A_\zeta = \begin{bmatrix} A'_\zeta & 0 \\ 0 & \bar{z}_n^{-1} \end{bmatrix}, \quad B_\zeta = \begin{bmatrix} B_\zeta^{+\prime} & B_\zeta^{-\prime} \\ \bar{\omega}_n & -1 \end{bmatrix} \tag{3.1}$$

where

$$A'_\zeta = \begin{bmatrix} z_1 & & & & & 0 \\ & \ddots & & & & \\ & & z_n & & & \\ & & & \bar{z}_1^{-1} & & \\ & & & & \ddots & \\ 0 & & & & & \bar{z}_{n-1}^{-1} \end{bmatrix},$$

$$[B_\zeta^{+\prime}\ B_\zeta^{-\prime}] = \begin{bmatrix} 1 & -\omega_1 \\ \vdots & \vdots \\ 1 & -\omega_n \\ \bar{\omega}_1 & -1 \\ \vdots & \vdots \\ \bar{\omega}_{n-1} & -1 \end{bmatrix}. \tag{3.2}$$

Now, we present the next theorem which describes the properties of $\Theta$.

THEOREM 3.1. *The matrix polynomial $\Theta$ in (2.5) has the following properties.*

(3.3) *$(A_\zeta, B_\zeta)$ is a left null pair for $\Theta$*

(3.4) *$\Theta$ is column reduced*

*Proof.* Upon substituting (2.4) for $\omega_j^i$, we get

$$[1\ -\omega_i^{k-1}]\Theta_k(z_i) = \alpha_{ik}[1\ -\omega_i^k] \tag{3.5}$$

and

$$[-\overline{\omega_i^{k-1}} \;\; 1]\Theta_k(\frac{1}{\bar{z}_i}) = \beta_{ik}[-\overline{\omega_i^k} \;\; 1] \tag{3.6}$$

for some complex numbers $\alpha_{ik}$, $\beta_{ik}$. By applying (3.5) and (3.6) repeatedly to
$[1 \; -\omega_i](\Theta_1\Theta_2\cdots\Theta_i)(z_i)$ and to $[-\overline{\omega}_i \; 1](\Theta_1\Theta_2\cdots\Theta_i)(\frac{1}{\bar{z}_i})$ we can obtain

$$[1 \; -\omega_i]\Theta(z_i) = [0\,0], \quad [-\bar{\omega}_i \; 1]\Theta(\frac{1}{\bar{z}_i}) = [0\,0]$$

for all $1 \leq i \leq n$. Since deg det $\Theta(z)$ is at most $2n$, $\Theta(z)$ has no more zeros. Thus $(A_\zeta, B_\zeta)$ is a left null pair of $\Theta(z)$ and deg det$\Theta(z) = 2n$.

Next we prove (3.4). We note that the column degree of $\theta^i$ is at most $n$ by the construction of $\Theta$ in (2.2), where $\theta^i$ is the $i^{th}$ column of $\Theta$. But the fact that

$$2n = \deg \det\Theta(z) \leq \text{ the sum of column degrees of } \Theta(z)$$

insures

$$deg\ \theta^1 = deg\ \theta^2 = n \tag{3.7}$$

that is, the sum of column degrees of $\Theta$ is the same as deg det$\Theta(z)$. Thus, $\Theta$ is column reduced by the definition.

Let us represent $\Theta$ by

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}. \tag{3.8}$$

Suppose

$$\sigma = \{z_i, \bar{z}_j^{-1} | 1 \leq i \leq n, \; 1 \leq j \leq n-1\}$$

and $f_{\min}$ denotes a minimal solution of (1.1), (1.3). The next theorem is derived from Theorem 3.5 of [ABKW] or [AA1].

THEOREM 3.2. *A rational function $f$ is a solution of (1.1), (1.3) if and only if there exist polynomials $p(z)$, $q(z)$ for which*

$$f = (\Theta_{11}p + \Theta_{12}q)(\Theta_{21}p + \Theta_{22}q)^{-1} \tag{3.9}$$

*where $\Theta_{21}p + \Theta_{22}q$ has no zeros in $\sigma$. Moreover if $\Theta_{22}$ has no zeros in $\sigma$, then*

$$f_{\min} = \Theta_{12}\Theta_{22}^{-1} \tag{3.10}$$

*is a unique minimal solution with $\delta(f_{\min}) = n-1$. Otherwise $\delta(f_{\min}) = n$ and for constants $p$, $q$ such that $(\Theta_{21}p + \Theta_{22}q)$ has no zeros in $\sigma$*

$$f_{\min} = (\Theta_{11}p + \Theta_{12}q)(\Theta_{21}p + \Theta_{22}q)^{-1}$$

*is a minimal solution.*

*Proof.* Since $\Theta(z)$ has $(A'_\zeta, B'_\zeta)$ given by (3.2) as its $\sigma$-left null pair, all solutions of (1.1), (1.3) can be parametrized as (3.9) by Theorem 16.4.1 of [BGR]. If we set

$$\hat{\Theta} = \Theta \begin{bmatrix} 1 & 0 \\ 0 & (1 - z\bar{z}_n)^{-1} \end{bmatrix} = \begin{bmatrix} \hat{\Theta}_{11} & \hat{\Theta}_{12} \\ \hat{\Theta}_{21} & \hat{\Theta}_{22} \end{bmatrix}$$

then $\hat{\Theta}$ is column reduced having $(A'_\zeta, B'_\zeta)$ as its left null pair because

$$\deg \det\hat{\Theta} = 2n - 1 = \deg \hat{\theta}^1 + \deg\hat{\theta}^2$$

where $\hat{\theta}^i$ is the $i^{th}$ column of $\hat{\Theta}(z)$. If $\Theta_{22}$ has no zeros in $\sigma$, then $\hat{\Theta}_{22}$, by its construction, has no zeros in $\sigma$ and $f_{\min}(z) = \hat{\Theta}_{12}\hat{\Theta}_{22}^{-1}$ is a unique minimal solution with $\delta(f_{\min}) = n - 1$ by Theorem 3.5 of [ABKW]. Noting that $\hat{\Theta}_{12}\hat{\Theta}_{22}^{-1} = \Theta_{12}\Theta_{22}^{-1}$, we conclude that $f_{\min} = \Theta_{12}\Theta_{22}^{-1}$ is a unique minimal solution with $\delta(f_{\min}) = n - 1$. If $\Theta_{22}$ has a zero in $\sigma$, then $\hat{\Theta}_{22}$ has a zero in $\sigma$. In this case, by Theorem 3.5 of [ABKW] a minimal solution of (1.1), (1.3) $f_{\min}$ is not unique with

$$\delta_{\min} = n, \tag{3.11}$$

where $\delta_{\min}$ represents the McMillan degree of a minimal solution. If

$$f(z) = (\Theta_{11}p + \Theta_{12}q)(\Theta_{21}p + \Theta_{22}q)^{-1}$$

for constants $p$, $q$ such that $(\Theta_{21}p+\Theta_{22}q)$ has no zeros in $\sigma$, by Theorem 16.4.1 of [BGR], $f$ is a solution of (1.1), (1.3). To find the McMillan degree of $f$ we note that $\delta(f)$ is at most $n$ because deg $\theta^1 =$ deg $\theta^2 = n$. But (3.11) says $\delta(f)$ is at least $n$. Thus $f = f_{\min}$ is a minimal solution with $\delta(f_{\min}) = n$.

The following theorem is our main result which gives the McMillan degree of a minimal stable solution.

THEOREM 3.3. *A rational function $f$ is a stable solution if and only if there exists $g \in \beta$ for which*

$$\Theta_{21}g + \Theta_{22} \tag{3.12}$$

*has no zeros in $\sigma$ and*

$$f(z) = (\Theta_{11}g + \Theta_{12})(\Theta_{21}g + \Theta_{22})^{-1}. \tag{3.13}$$

*If $\Theta_{22}$ has no zeros in $\sigma$ then $f_{\min}$ given by (3.10) is automatically stable. Otherwise, $f_{\min}$ is not necessarily stable but there exists $\alpha \in C$ such that $|\alpha| < 1$ and*

$$\hat{f}_{\min} = (\Theta_{11}\alpha + \Theta_{12})(\Theta_{21}\alpha + \Theta_{22})^{-1}$$

*is a minimal stable solution with $\delta(\hat{f}_{\min}) = n$.*

*Proof.* Recall that by (2.5) all solutions satisfying (1.1), (1.2) are parametrized as (3.13) (without the extra constraint (3.12)). But for $f$ in form of (3.13) to be a solution of (1.1) and (1.3), the condition (3.12) should be fulfilled by Theorem 3.2. Thus all solutions of (1.1)-(1.3) can be parametrized as (3.13) together with the constraint (3.12). Let $\hat{f}_{\min}$ denote a stable solution of (1.1), (1.3). Suppose $\Theta_{22}$ has no zeros in $\sigma$. Upon considering

$$\begin{bmatrix} \Theta_{12} \\ \Theta_{22} \end{bmatrix} = \Theta \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$f_{\min} = \Theta_{12}\Theta_{22}^{-1}$ also satisfies (1.2) by (2.7), that is, $f_{\min}$ is automatically stable.

Now we consider the case where $\Theta_{22}$ has a zero in $\sigma$. Let

$$\begin{aligned} A =& \{(a,b) \in C^2 \,|(\Theta_{21}a + \Theta_{22}b)(\mu) = 0 \text{ for some } \mu \in \sigma\} \\ B =& \{(a,b) \in C^2| \ |a| < |b|\}. \end{aligned}$$

Since $A^C$ is dense in $C^2$ and $B$ is an open set, there exists $(p,q) \in A^C \cap B$. If we set $\alpha = p/q$, then $|\alpha| < 1$ and hence $\hat{f}(z) = (\Theta_{11}\alpha + \Theta_{12})(\Theta_{21}\alpha + \Theta_{22})^{-1}$ is a solution of (1.1), (1.2) by (2.7). But the fact that $(p,q) \in A^C$ and Theorem 3.2 insure $\hat{f}(z)$ is a solution of (1.3). Now we compute $\delta(\hat{f})$. We easily see that $\delta(\hat{f}) \leq n$. But Theorem 3.2 says that $n$ is the minimal possible McMillan degree for the stable solutions. Hence $\delta(\hat{f}) = n$. This completes the proof.

COROLLARY 3.4. *If $\Theta_{22}$ has no zeros in $\sigma$, then a stable minimal solution $\hat{f}_{\min}$ is unique with $\delta(\hat{f}_{\min}) = n - 1$. Otherwise, $\hat{f}_{\min}$ is not unique and $\delta(\hat{f}_{\min}) = n$.*

## References

[AA1] A. C. Antoulas and B. D. O. Anderson, *On the scalar rational interpolation problem*, IMA J. Math. Control Inform. **3** (1986), 61–88.

[AA2] ———, *On the problem of stable rational interpolation*, Linear Algebra Appl. **122/123**, (1989), 301–328.

[ABKW] A. C. Antouals, J. A. Ball, J. Kang (Kim), and J. C. Willems, *On the solution of minimal rational interpolation problem*, Lin. Algebra Appl. **137/138** (1990), 511–573.

[B] V. Belevitch, *Interpolation matrices*, Philips Research Reports **25** (1970), 337–369.

[BGR] J. A. Ball, I. Gohberg and L. Rodman, *Interpolation of Rational Matrix Functions*, monograph, Birkhäuser, OT 45, Basel, 1990.

[K] J. Kang(Kim), *Interpolation by rational matrix functions with minimal McMillan dgree*, Ph. D Thesis, Virginia Tech, 1990.

[Ka] T. Kailath, *Linear Systems*, Prentice-Hall, 1980.

[N] R. Nevannlina, *Über beschränkte analytische funktionen*, Ann. Acad. Sci. Fenn. **32** (7), (1929).

[P] G. Pick, *Über die Beschrankungen analytischer Funktionne welche durch vorgegebene Funktionswerte bewirkt wird*, Math. Ann. **77**(1916), 7–23.

Department of Mathematics
Chonnam National University
Kwangju 500-757, Korea