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MODULES OF QUOTIENTS OVER COMMUTATIVE RINGS*

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In [3] Goldman introduced the notion of modules of quotients of a ring with respect to an idempotent kernel functor, which is a generalization of the locallization of a module with respect to a multiplicative subset of a commutative ring. For an idempotent kernel functor σ on the category of R-modules and for an R-module M, let $Q_\sigma(M)$ denote the module of quotients with respect to σ.
In this note we compute $Q_\sigma(M)$ in terms of usual localization if σ is determined by a set of prime ideals. We first recall the following definitions and basic properties. For more detailed discussion we refer [2, 3, 4]. We assume all rings in this paper are commutative with identity.

DEFINITION 1. A functor σ on R-mod, the category of R-modules, is called an idempotent kernel functor if the following properties hold:

(1) For every R-module M, $\sigma(M)$ is a submodule of M.
(2) If $f\colon M' \to M$ is a homomorphism then $f(\sigma(M')) \subset \sigma(M)$ and $\sigma(f)$ is a restriction of f to $\sigma(M')$.
(3) If M' is a submodule of M then $\sigma(M') = \sigma(M) \cap M'$.
(4) $\sigma(M/\sigma(M)) = 0$.

We say M is a σ-torsion (resp. σ-torsion free) R-module if $\sigma(M) = M$ (resp. $\sigma(M) = 0$). We denote by $\mathcal{L}(\sigma)$ the set of ideals I of R with the property R/I is σ-torsion. The set $\mathcal{L}(\sigma)$ is called the Gabriel topology associated to σ. A non-empty set $\mathcal{L}$ of ideals of R is a Gabriel topology if it satisfies the following two conditions [2]:

(1) (C1) If J is in $\mathcal{L}$ and I is an ideal of R such that $(I : b) \in \mathcal{L}$ for all $b \in J$ then I is in $\mathcal{L}$.
(2) (C2) If I is in $\mathcal{L}$ and $I \subset J$ then $I \in \mathcal{L}$.

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Given Gabriel topology $\mathcal{L}$, one can define an idempotent kernel functor σ by

$$\sigma(M) = \{m \in M \mid Im = 0 \text{ for some I in } \mathcal{L}\}$$

for each R-module M. We write $C(\sigma)$ for the set of prime ideals of R not contained in $\mathcal{L}$. Then

$$\{I \lhd R \mid I_p = R_p \text{ for all } p \in C(\sigma)\} = \mathcal{L}.$$

Conversely, for each set $\mathcal{P}$ of prime ideals, the set

$$\mathcal{L}(\mathcal{P}) = \{I \lhd R \mid I_p = R_p \text{ for all } p \in \mathcal{P}\}$$

defines a Gabriel topology, and hence an idempotent kernel functor. A Gabriel topology defined by one prime ideal is called a principal topology. The intersection of principal topologies is again a Gabriel topology and called a primal topology [2].
In [3] definition of Gabriel topology for any ring is given. By this definition, if I and J are in Gabriel topology $\mathcal{L}$, $I \cap J$ is in $\mathcal{L}$.
For any R-module M and a prime ideal p of R, let

$$M_{(p)} = \{\frac{m}{s} | s \text{ is a regular element not in } p \text{ and } m \in M\}.$$

Then $R_{(p)}$ is a localization of R and $M_{(p)}$ is an $R_{(p)}$-module [1].

LEMMA 2. *Let $\mathcal{P}$ be a set of prime ideals of R and let*

$$\mathcal{L}((\mathcal{P})) = \{I \lhd R \mid I_{(p)} = R_{(p)} \textit{ for all } p \in \mathcal{P}\}.$$

Then $\mathcal{L}((\mathcal{P}))$ is a Gabriel topology.

Proof. For each prime p in $\mathcal{P}$, $I_{(p)} = R_{(p)}$ if and only if there exists a regular element in I not in p. To show $\mathcal{L}((\mathcal{P}))$ is a Gabriel topology, it suffice to show $\mathcal{L}((p))$ is a Gabriel topology for each p in $\mathcal{P}$. We note that $I_{(p)} = R_{(p)}$ if and only if there exists a regular element s in I which is not in p. To show $\mathcal{L}((p))$ satisfies C1, let J be in $\mathcal{L}((p))$. By definition, there exists a regular element s in J not in p. If I is an ideal of R such that $(I : b) \in \mathcal{L}((p))$ for all b in J, then there exists a regular element t in $(I : b)$ not in p. Since ts is a regular element in I not in p, I is in $\mathcal{L}((p))$. Clearly, $\mathcal{L}((p))$ satisfies condition C2.

REMARKS. (1) If R is an integral domain, then $\mathcal{L}(\mathcal{P}) = \mathcal{L}((\mathcal{P}))$.

(2) It is not clear whether the Gabriel topology $\mathcal{L}((p))$ is a principal topology, but $\mathcal{L}((p))$ and $\mathcal{L}(\mathcal{P}))$ are primal topologies.

For M an arbitrary R-module, the module of quotients of M with respect to an idempotent kernel functor σ, denoted by $Q_\sigma(M)$ is a faithfully σ-injective module containing $M/\sigma(M)$ as a submodule unique up to isomorphism.

We note that if σ is defined by a multiplicative set S of R i.e., $\mathcal{L}(\sigma) = \{I \mid I \cap S \neq \emptyset\}$ then $Q_\sigma(M) = S^{-1}M$. We recall the construction of $Q_\sigma(M)$ from [3]. Since $Q_\sigma(M) = Q_\sigma(M/\sigma(M))$, for σ-torsion free R-module M, let

$$\Omega = \{(I, f) \mid I \in \mathcal{L}(\sigma), f\colon I \to M \text{ is an } R\text{-homomorphism}\}.$$

We define (I, f) and (I', f') are equivalent and denote by $(I, f) \sim (I', f')$ if there exists J in $\mathcal{L}(\sigma)$ with $J \subset I \cap I'$ such that $f|_J = f'|_J$. Then the set $Q_\sigma(M)$ of equivalence classes $[(I, f)]$ is an abelian group under the operation

$$[(I, f)] + [(J, g)] = [(I \cap J, f + g)].$$

In general $Q_\sigma(M)$ is not easy to compute and $Q_\sigma(\ )$ is not right exact functor in general [3, 4].

THEOREM. *Let $\mathcal{P}$ be a set of prime ideals of R, $\mathcal{L}((\mathcal{P}))$ be the Gabriel topology as in the lemma and let σ be the idempotent kernel functor determined by $\mathcal{L}((\mathcal{P}))$. Then for any torsion free R-module M,*

$$Q_\sigma(M) \cong \bigcap_{p \in \mathcal{P}} M_{(p)},$$

$M_{(p)}$ is viewed as a subset of $T^{-1}M$, where T is the set of all regular elements.

Proof. For any $z \in \bigcap_{p \in \mathcal{P}} M_{(p)}$, $z = \frac{x(p)}{s(p)}$ for some regular element $s(p)$ of R, which is not in p and $x(p) \in M$. Since M is regular, $x(p)$ is defined by z and $s(p)$ uniquely. If both $\frac{x(p)}{s(p)}$ and $\frac{x(q)}{s(q)}$ become z in $T^{-1}M$, $x(p)s(q) = s(p)x(q)$. For each z in $\bigcap_{p \in \mathcal{P}} M_{(p)}$, we define

$$I(z) \text{ to be the ideal generated by the set } \{s(p) \mid z = \frac{x(p)}{s(p)} \in \bigcap_{p \in \mathcal{P}} M_{(p)}\ \}.$$

Then $I(z) \in \mathcal{L}((\mathcal{P}))$ by definition of $\mathcal{L}((\mathcal{P}))$. Define a map

$$\alpha\colon \bigcap_{p\in\mathcal{P}} M_{(p)} \to Q_\sigma(M)$$

by $\alpha(z) = [(I(z), f)]$, where $f(s(p)) = x(p)$. Then α is well defined. It can be checked easily that α is a homomorphism of R-modules. Conversely, to define a map from $Q_\sigma(M)$ to $\bigcap_{p\in\mathcal{P}} M_{(p)}$, let $[(I,f)]$ be in $Q_\sigma(M)$. Then for each p in $\mathcal{P}$, there exists a regular element $s(p)$ in I not in p. If $f(s(p)) = x(p)$ in M. We define a map

$$\beta\colon Q_\sigma(M) \to \bigcap_{p\in\mathcal{P}} M_{(p)}$$

by $\beta([(I,f)]) = \frac{x(p)}{s(p)}$, which is viewed as an element in $T^{-1}M$. Then $\frac{x(p)}{s(p)}$ is in $\bigcap_{p\in\mathcal{P}} M_{(p)}$. To prove this, let $s(p)$ and $s(q)$ be regular elements in I not in p, and let $f(s(p)) = x(p)$ and $f(s(q)) = x(q)$. Then $s(p)x(q) = s(p)f(s(q)) = s(q)f(s(p)) = s(q)x(p)$, since f is an R-homomorphism. It follows that $\frac{x(p)}{s(p)} = \frac{x(q)}{s(q)}$ for all p and q in $\mathcal{P}$ and hence $\frac{x(p)}{s(p)}$ is in $\bigcap_{p\in\mathcal{P}} M_{(p)}$.
It is clear from definition that β is well-defined. It can be seen easily that β is a R-homomorphism and $\alpha\beta =$ identity on $Q_\sigma(M)$ and $\beta\alpha =$ identity on $\bigcap_{p\in\mathcal{P}} M_{(p)}$. This proves the theorem.

NOTE. Let σ be as in the theorem. If M is a torsion free R-module then M is σ-torsion free. For if $Im = 0$ for some $I \in \mathcal{L}(\sigma)$, then $m = 0$, since I contains a regular element.

COROLLARY. *Let σ be as in the above theorem. Then Q_σ is an exact functor on the category of torsion free R-modules, i.e., if*

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence of torsion free R-modules then

$$0 \to Q_\sigma(M') \to Q_\sigma(M) \to Q_\sigma(M'') \to 0$$

is also an exact sequence of abelian groups.

Proof. The corollary follows easily from theorem, since

$$0 \to M'_{(p)} \to M_{(p)} \to M''_{(p)} \to 0$$

is an exact sequence if

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence.

References

1. D. D. Anderson, *Commutative Rings with Zero Divisors*, Marcel Dekker, New York, 1982.
2. M. Beattie and M. Orzech, *Prime ideals and Finiteness conditions for Gabriel Topologies over commutative Rings*, Rocky Mountain J. Math. **22** (1992), 423–440.
3. O. Goldman, *Rings and Modules of Quotients*, J. Algebra **13** (1969), 10–47.
4. J. Hahn and H. Lee, *Torsion Theory and Local Cohomology*, Comm. Korean Math. Soc. **4** (1989), 279–287.
5. F. Van Oystaeyan and A. Verschoren, *Relative Invariants of Rings*, Monographs in pure and applied Mathematics, vol.79 Marcel Dekker Inc., New York, 1984.

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