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REGULARIZED EISENSTEIN SERIES ON METAPLECTIC GROUPS

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1. Introduction

Let V be a vector space of dimension m over $\mathbb{Q}$, and let $(\ ,\ )$ be a non-degenerate bilinear form on V. Let r be the Witt index of V, and let $V = V' + V_0 + V''$ be the Witt decomposition, where V_0 is anisotropic and V', V'' are paired non-singularly. Let $H = O(m-r, r)$ be the isometry group of V, $(\ ,\ )$, viewed as an algebraic group over $\mathbb{Q}$. Let $G = Sp(n)$ be the symplectic group of rank n defined over $\mathbb{Q}$.

Assume that m is *even*. Then G and H form a dual reductive pair, and $G(\mathbb{A})$ and $H(\mathbb{A})$ act in the Schwartz space $\mathcal{S}(V(\mathbb{A})^n)$ by the oscillator representation ω for the fixed additive character ψ of $\mathbb{A}$. Here $\mathbb{A}$ denotes the ring of adeles over $\mathbb{Q}$ as usual. For $\varphi \in \mathcal{S}(V(\mathbb{A})^n)$ we have the theta kernel

$$\theta(g, h; \varphi) = \sum_{x \in V(\mathbb{Q})^n} \omega(g, h)\varphi(x) = \sum_{x \in V(\mathbb{Q})^n} \omega(g)\varphi(h^{-1}x).$$

The theta integral

$$I(g; \varphi) = \int_{H(\mathbb{Q})\backslash H(\mathbb{A})} \theta(g, h; \varphi)\, dh$$

is absolutely convergent if V is anisotropic or $m - r > n + 1$. It is shown in [5] that there exists a certain differential operator D in the center of the universal enveloping algebra of $H(\mathbb{R})$ such that for all $\varphi \in \mathcal{S}(V(\mathbb{A})^n)$, the function $h \mapsto \theta(g, h; \omega(D)\varphi)$ is rapidly decreasing and hence $I(g; \omega(D)\varphi)$ is well-defined even when V is isotropic. As a consequence, we may lift Eisenstein series from $H(\mathbb{A})$ to $G(\mathbb{A})$. In this paper we study the metaplectic case (m is odd), in particular, the case $m = 3$, $n = 2$ and obtain analogous results.

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2. Oscillator representation

Let $V = Sym(2)$ be the vector space of 2×2 symmetric matrices over $\mathbb{Q}$. Let $(\ ,\ )$ be the bilinear form on V given by $(T,T) = -\det(T)$ for $T \in V$. We fix a basis e', e_0, e'' for V with $(e',e') = (e'',e'') = 0$ and $(e',e'') = 1$, and view V as a 3 dimensional space of column vectors over $\mathbb{Q}$. We have a Witt decomposition

$$V = V' + V_0 + V'' \tag{2.1}$$

where $V' + V''$ is a hyperbolic plane of dimension 2 and V_0 is anisotropic of dimension 1. Let $H = O(V)$ be the isometry group of $V, (\ ,\ )$. Note that $H(\mathbb{R}) = O(2,1)$ and that there is an isomorphism

$$\begin{gathered} M_{3,2} \simeq V^2 \to V_0^2 \oplus W \\ \begin{pmatrix} x \\ x_0 \\ y \end{pmatrix} \mapsto (x_0, (y,x)) \end{gathered} \tag{2.2}$$

where $W = \mathbb{Q}^4$ (row vectors).

Let $G = Sp(2)$ be the symplectic group of rank 2 over $\mathbb{Q}$, and $\widetilde{G} = \widetilde{Sp}(2)$ be the 2-fold metaplectic covering group of G. For any subgroup G_1 of G, we let $\widetilde{G}_1$ be the full inverse image of the projection $\widetilde{G} \to G$. The groups $\widetilde{G}$ and H form a dual reductive pair in $\widetilde{Sp}(6)$. The oscillator representation ω may be realized in a standard Schrödinger model $(\omega, \mathcal{S}_{\mathbb{A}})$, $\mathcal{S}_{\mathbb{A}} = \mathcal{S}(V(\mathbb{A})^2)$, or in a mixed model $(\hat{\omega}, \hat{\mathcal{S}}_{\mathbb{A}})$, where

$$\hat{\mathcal{S}}_{\mathbb{A}} = \mathcal{S}(V_0(\mathbb{A})^2) \otimes \mathcal{S}(W(\mathbb{A}))$$

and W is considered as a symplectic space with right G action. Recall that $H(\mathbb{A})$ acts linearly and commutes with $\widetilde{G}(\mathbb{A})$ in the Schrödinger model. If $P = MN$ is the Siegel parabolic subgroup G, where

$$M = \left\{ m(a) = \begin{pmatrix} a & \\ & {}^t a^{-1} \end{pmatrix} \mid a \in GL(2) \right\}$$

and

$$N = \left\{ n(b) = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \mid b = {}^t b \in Sym(2) \right\},$$

then the actions of $\widetilde{P}(\mathbb{A})$ in the Schrödinger model are given as follows: For $[m(a),\epsilon] \in \widetilde{M}(\mathbb{A})$,

$$\omega([m(a),\epsilon])\varphi(x) = \chi([m(a),\epsilon)] \, |\det(a)|^{\frac{m}{2}} \varphi(xa),$$

and for $n(b) \in N(\mathbb{A}) \hookrightarrow \widetilde{G}(\mathbb{A})$,

$$\omega(n(b))\varphi(x) = \psi(\frac{1}{2}tr(b(x,x)))\varphi(x),$$

where $\chi = \chi_V$ be the character of $\widetilde{M}(\mathbb{A})/M(\mathbb{Q})$ given by

$$\chi([m(a),\epsilon]) = \frac{\epsilon(\det(a),(-1)^{\frac{m-1}{2}}\det(V))_{\mathbb{A}}}{\gamma(\det(a),\frac{1}{2}\psi)}.$$

and where $(\ ,\ )_{\mathbb{A}}$ is the global Hilbert symbol. We refer [7, 8] for more notations and details.

If we define the partial Fourier transform

$$\mathcal{F}\varphi(x_0,x,y) = \hat{\varphi}(x_0,x,y) = \int_{\mathbb{A}^2} \psi(y^t u)\varphi\begin{pmatrix} u \\ x_0 \\ x \end{pmatrix} du \tag{2.3}$$

for $x_0 \in V_0(\mathbb{A})^2$ and $x, y, u \in M_{1,2}(\mathbb{A})$, then two models are related by

$$\hat{\omega}(g)\hat{\varphi}(x_0,w) = \omega_0(g)\hat{\varphi}(x_0,wg), \tag{2.4}$$

where ω_0 is the oscillator representation for the dual reductive pair $(\widetilde{G}, O(V_0))$, so that $\widetilde{G}(\mathbb{A})$ acts linearly in the mixed model. By abuse of notation we will again write ω for $\hat{\omega}$ or for the derived representation of the Lie algebra $\mathfrak{sp}(6,\mathbb{R}))$ of $Sp(6,\mathbb{R})$ on $\hat{S}_{\mathbb{A}}$.

Let P_H be the maximal parabolic subgroup of H associated to the decomposition (2.1), i.e., the stabilizer of isotropic line V''. The Levi decomposition of P_H is given by $P_H = LU$ where

$$L = \left\{ m(a,h_0) = \begin{pmatrix} a & & \\ & h_0 & \\ & & a^{-1} \end{pmatrix} \mid a \in GL(1),\ h_0 \in O(V_0) \right\},$$

$$U = \left\{ u(b) = \begin{pmatrix} 1 & b & -\frac{1}{2}b^2 \\ & 1 & -b \\ & & 1 \end{pmatrix} \right\}.$$

The actions of P_H on $\hat{\varphi} \in \hat{S}$ are computed as follows:

$$\omega(m(a, h_0))\hat{\varphi}(x_0, x, y) = |\det(a)|\hat{\varphi}(h_0 x_0, ax, ay), \tag{2.5}$$

$$\omega(u(b))\hat{\varphi}(x_0, x, y) = \psi(y^t(bx_0 + \tfrac{1}{2}b^2 x)\hat{\varphi}(x_0 + bx, x, y). \tag{2.6}$$

3. A central differential operator

$SL(2,\mathbb{R})$ acts on $V(\mathbb{R}) = Sym(2,\mathbb{R})$ by $h \cdot T = hT^t h$. This action preserves the bilinear form (,) and we have an exact sequence

$$1 \to \{\pm 1\} \to SL(2,\mathbb{R}) \to H(\mathbb{R})^0 \to 1,$$

where $H(\mathbb{R})^0$ is the connected component of identity in $H(\mathbb{R})$. Hence $H(\mathbb{R})^0 \simeq PSL(2,\mathbb{R})$ and the Lie algebra of $H(\mathbb{R})$ is $\mathfrak{h} = \mathfrak{sl}(2,\mathbb{R})$. The derived representation of ω_∞ for $\mathfrak{h}$ is defined by

$$\omega_\infty(X)\varphi = \frac{d}{dt}\omega_\infty(exp(tX))\varphi|_{t=0}.$$

Let $H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ be the generators of $\mathfrak{sl}(2,\mathbb{R})$. It is then easy to check that

$$\omega_\infty(H)\varphi \begin{pmatrix} x \\ x_0 \\ y \end{pmatrix} = \sum_{i=1}^{2} \left(-2x_i \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial y_i} \right) \varphi \begin{pmatrix} x \\ x_0 \\ y \end{pmatrix} \tag{3.1}$$

$$\omega_\infty(X)\varphi \begin{pmatrix} x \\ x_0 \\ y \end{pmatrix} = \sum_{i=1}^{2} \left(-2x_{0i} \frac{\partial}{\partial x_i} - y_i \frac{\partial}{\partial x_{0i}} \right) \varphi \begin{pmatrix} x \\ x_0 \\ y \end{pmatrix} \tag{3.2}$$

$$\omega_\infty(Y)\varphi \begin{pmatrix} x \\ x_0 \\ y \end{pmatrix} = \sum_{i=1}^{2} \left(-x_i \frac{\partial}{\partial x_{0i}} - 2x_{0i} \frac{\partial}{\partial y_i} \right) \varphi \begin{pmatrix} x \\ x_0 \\ y \end{pmatrix}, \tag{3.3}$$

where $x = (x_1, x_2)$, etc. Let

$$C = \tfrac{1}{2}H^2 + H + 2YX \tag{3.4}$$

be the Casimir element.

LEMMA 3.1. *Let $D = C - 4$. Then*

(1) *$\omega_\infty(D)$ commutes with the actions of $\widetilde{G}(\mathbb{R})$ and $H(\mathbb{R})$*

(2) *For all $\hat{\varphi} \in \mathcal{S}(V_0(\mathbb{R})^2) \otimes \mathcal{S}(W(\mathbb{R})) = \hat{\mathcal{S}}(V(\mathbb{R})^2)$, we have*

$$\omega_\infty(D)\hat{\varphi}(x_0, 0) = 0$$

for all $x_0 \in V_0(\mathbb{R})^2$.

Proof. For any $\varphi \in \mathcal{S}(V(\mathbb{R})^2)$, we have

$$(\frac{\partial \varphi}{\partial y_i})\hat{}\,(x_0, x, y) = -a_i y_i \hat{\varphi}(x_0, x, y)$$

$$(y_i \frac{\partial \varphi}{\partial y_i})\hat{}\,(x_0, x, y) = -\hat{\varphi}(x_0, x, y) - y_i \frac{\partial \hat{\varphi}}{\partial y_i}(x_0, x, y),$$

where $a_i = \frac{\partial \psi}{\partial y_i}(0)$. Therefore, we get

$$\omega_\infty(H)\hat{\varphi}(x_0, 0, 0) = -4\hat{\varphi}(x_0, 0, 0), \quad \omega(Y)\hat{\varphi}(x_0, 0, 0) = 0.$$

and hence, by (3.4), we have $\omega(C)\hat{\varphi}(x_0, 0, 0) = (8 - 4)\hat{\varphi}(x_0, 0, 0)$. This shows (ii). Of course, $\omega_\infty(D)$ commutes with actions of $\widetilde{G}(\mathbb{R})$ and of $H(\mathbb{R})^0$. Note that $H(\mathbb{R}) = O(2, 1)$ has 4 connected components and is generated by $H(\mathbb{R})^0$, h_1, h_2, where $h_1 = \mathrm{diag}(-1, 1, -1)$ and $h_2 = \mathrm{diag}(1, -1, 1)$. Thus it is enough to show that $\omega_\infty(C)$ commutes with $\omega_\infty(h_1)$ and $\omega_\infty(h_2)$. But the conjugation by $\omega_\infty(h_i)$ preserves $\omega_\infty(H)$, while it replaces $\omega_\infty(X)$ and $\omega_\infty(Y)$ by $-\omega_\infty(X)$ and $-\omega_\infty(Y)$ respectively, and hence it preserves $\omega_\infty(C)$.

As a consequence we have [5]

PROPOSITION 3.2. *As a function on $H(\mathbb{Q})\backslash H(\mathbb{A})$, $\theta(g, h, \omega(D)\varphi)$ is rapidly decreasing.*

Let K_H be the compact subgroup of $H(\mathbb{A})$ so that we have a global Iwasawa decomposition

$$H(\mathbb{A}) = P_H(\mathbb{A}) \cdot K_H. \tag{3.5}$$

We write any element $h \in H(\mathbb{A})$, via (3.5), as $h = u(b)m(a, h_0)k$, and let $|a(h)| = |a| = |\det(a)|$. Define an Eisenstein series

$$E(h, s) = \sum_{\gamma \in P_H(\mathbb{Q})\backslash H(\mathbb{Q})} |a(\gamma h)|^{s+\frac{1}{2}}.$$

This series converges absolutely for $Re(s) > \frac{1}{2}$, has a meromorphic analytic continuation, and its only pole is $s_0 = \frac{1}{2}$, which is simple [2]. By Proposition 3.2 above, we may define the integral

$$(3.6) \qquad I(g, s; \omega(D)\varphi) = \int_{H(\mathbb{Q})\backslash H(\mathbb{A})} \theta(g, h, \omega(D)\varphi)E(h, s)\, dh.$$

for $g \in \widetilde{G}(\mathbb{A})$, $\varphi \in \mathcal{S}_{\mathbb{A}}$. This integral is absolutely convergent whenever the Eisenstein series is holomorphic and hence defines a meromorphic function of s. Our goal is to identify this function with an Eisenstein series on $\widetilde{G}(\mathbb{A})$, closely following [5] or [6].

4. Regularized theta lift of $E(h, s)$

Unfolding the integral (3.6), we have

$$I(g, s; \omega(D)\varphi) = \int_{P_H(\mathbb{Q})\backslash H(\mathbb{A})} \theta(g, h, \omega(D)\varphi)|a(h)|^{s+\frac{1}{2}}\, dh.$$

This is

$$\int_{K_H}\int_{L(\mathbb{Q})\backslash L(\mathbb{A})}\int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \theta(g, um(a, h_0)k, \omega(D)\varphi)\times, \quad |a|^{s+\frac{1}{2}}|a|^{-1}\, du\, dm\, dk$$

$$= \int_{L(\mathbb{Q})\backslash L(\mathbb{A})}\int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \theta(g, um(a, h_0), \omega(D)pr_{K_H}(\varphi))\times \quad |a|^{s-\frac{1}{2}}\, du\, dm$$

where $pr_{K_H}(\varphi) = \int_{K_H} \omega(k)\varphi\, dk$ is the projection of φ to the K_H invariants in $\mathcal{S}_{\mathbb{A}}$. As usual, let

$$\theta_U(g, h, \varphi) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \theta(g, uh, \varphi)\, du$$

be the U-constant term of $\theta(g, \cdot, \varphi)$.

PROPOSITION 4.1. *For* $\varphi \in \mathcal{S}_{\mathbb{A}}$,

$$\theta_U(g,h,\omega(D)\varphi) = \sum_{\substack{\gamma \in P_1(\mathbb{Q})\backslash G(\mathbb{Q}) \\ t \in \mathbb{Q}^\times \\ x_{02} \in V_0(\mathbb{Q})}} \omega(\gamma g)\omega(h)\omega(D)\hat{\varphi}(0, x_{02}, tw_0),$$

where $P_1 \subset G$ *is the maximal parabolic subgroup which stabilizes the line generated by* $w_0 = (0,0,1,0)$.

Proof. By Poisson summation formula and (2.4), and by Lemma 3.1,

$$\begin{aligned}\theta(g,h,\omega(D)\varphi) &= \sum_{v \in V(\mathbb{Q})^2} \omega(g,h)\omega(D)\hat{\varphi}(v) \\ &= \sum_{0 \neq w \in W(\mathbb{Q})} \sum_{x_0 \in V_0(\mathbb{Q})^2} \omega_0(g)\omega(h)\omega(D)\varphi(x_0, wg).\end{aligned}$$

Since $G(\mathbb{Q})$ acts transitively on $W(\mathbb{Q}) - \{0\}$, we have

$$\begin{aligned}&\theta(g,h,\omega(D)\varphi) \\ &= \sum_{\gamma \in P_1(\mathbb{Q})\backslash G(\mathbb{Q})} \sum_{\substack{t \in \mathbb{Q}^\times \\ x_0 \in V_0(\mathbb{Q})^2}} \omega_0(g)\omega(h)\omega(D)\hat{\varphi}(x_0, tw_0\gamma g) \\ &= \sum_{\gamma \in P_1(\mathbb{Q})\backslash G(\mathbb{Q})} \sum_{\substack{t \in \mathbb{Q}^\times \\ x_0 \in V_0(\mathbb{Q})^2}} \omega_0(\gamma g)\omega(h)\omega(D)\hat{\varphi}(x_0, tw_0\gamma g) \\ &= \sum_{\gamma \in P_1(\mathbb{Q})\backslash G(\mathbb{Q})} \sum_{\substack{t \in \mathbb{Q}^\times \\ x_0 \in V_0(\mathbb{Q})^2}} \omega(\gamma g, h)\omega(D)\hat{\varphi}(x_0, tw_0).\end{aligned}$$

Here we have used the invariance of the sum on x_0 under $G(\mathbb{Q})$. From (2.6), by letting $\hat{\varphi}' = \omega(\gamma g)\omega(h)\omega(D)\hat{\varphi}$, we have

$$\begin{aligned}\theta_U(g,h,\omega(D)\varphi) &= \int_{\mathbb{Q}\backslash\mathbb{A}} \theta(g, u(b)h, \omega(D)\varphi)\, db \\ &= \sum_{\gamma,t,x_0} \left(\int_{\mathbb{Q}\backslash\mathbb{A}} \psi(bx_{01}t)\, db \right) \hat{\varphi}'(x_0, tw_0) \\ &= \sum_{\gamma,t} \sum_{x_{02} \in V_0(\mathbb{Q})} \hat{\varphi}'(0, x_{02}, tw_0)\end{aligned}$$

By Proposition 4.1 and (2.5), with $\hat{\varphi}'' = \omega(\gamma g)\omega(D)pr_{K_H}(\hat{\varphi})$, we obtain

$$\begin{aligned} &I(g,s,\omega(D)\varphi) \\ &= \int_{\mathbb{Q}^\times\backslash\mathbb{A}^\times}\int_{H_0(\mathbb{Q})\backslash H_0(\mathbb{A})}\sum_{t\in\mathbb{Q}^\times}\sum_{\gamma,x_{02}} |a|^{s+\frac{1}{2}}\hat{\varphi}''(0,x_{02},atw_0)\,dh_0\,da \\ &= \sum_{\gamma}\int_{\mathbb{A}^\times}|a|^{s+\frac{1}{2}}\int_{H_0(\mathbb{Q})\backslash H_0(\mathbb{A})}\sum_{x_{02}\in V_0(\mathbb{Q})}\hat{\varphi}''(0,x_{02},aw_0)\,dh_0\,da. \end{aligned}$$

Define a map $T : \mathcal{S}(V(\mathbb{A})^2) \to \mathcal{S}(V_0(\mathbb{A}))$ by

$$T\varphi(x_{02}) = \int_{\mathbb{A}^\times}|a|^{s+\frac{1}{2}}\hat{\varphi}(0,x_{02},aw_0)\,da.$$

and for $\varphi_0 \in \mathcal{S}(V_0(\mathbb{A}))$, define a theta series

$$I_0(\varphi_0) = \int_{H_0(\mathbb{Q})\backslash H_0(\mathbb{A})}\sum_{x_{02}\in V_0(\mathbb{Q})}\omega(h_0)\varphi_0(x_{02})\,dh_0.$$

If we let

$$F(g,s,\varphi) = I_0(T\omega(g)\varphi) = \int_{\mathbb{A}^\times}|a|^{s+\frac{1}{2}}I_0(\omega(g)\hat{\varphi}(0,\cdot,aw_0))\,da^\times,$$

then we have

$$I(g,s,\omega(D)\varphi) = \sum_{\gamma\in P_1(\mathbb{Q})\backslash G(\mathbb{Q})}F(\gamma g,s,\omega(D)pr_{K_H}(\varphi)).$$

We have the Levi decomposition $P_1 = M_1N_1$, where

$$\begin{aligned} M_1 &= \left\{ m_1(t,g_0) = \begin{pmatrix} t & & & \\ & a & & b \\ & & t^{-1} & \\ & c & & d \end{pmatrix} \,\middle|\, t\in GL(1),\quad g_0 = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in Sp(1) \right\} \\ &\cong GL(1)\times Sp(1) \end{aligned}$$

is the Levi factor, and

$$N_1 = \left\{ n(x,y,z) = \begin{pmatrix} 1 & x & y & z \\ & 1 & z & \\ & & 1 & \\ & & -x & 1 \end{pmatrix} \right\}$$

is the unipotent radical of P_1. Since the covering $\widetilde{G} \to G$ splits over N_1, we have $\widetilde{P_1} = \widetilde{M_1} N_1$.

Recall that $P = MN$ is the Siegel parabolic subgroup of G and the multiplication in $\widetilde{M}$ is given by

$$[m(a_1), \epsilon_1][m(a_2), \epsilon_2] = [m(a_1 a_2), \epsilon_1 \epsilon_2 (\det a_1, \det a_2)_{\mathbb{A}}].$$

If we let $\widetilde{GL}(1) := GL(1) \times \{\pm 1\}$, where the multiplication is given by

$$[t, \epsilon][t', \epsilon'] = [tt', \epsilon\epsilon'(t, t')],$$

then the map $\widetilde{GL}(1) \to \widetilde{G}$, $[t, \epsilon] \mapsto [m_1(t, 1), \epsilon]$ is thus an injective homomorphism. Also the map $\widetilde{Sp}(1) \to \widetilde{G}$, $[g_0, \epsilon] \mapsto [m_1(1, g_0), \epsilon]$ gives an injective homomorphism [8]. Using these identifications, we may write $\widetilde{M_1} = \widetilde{GL}(1)\widetilde{Sp}(1)$, with $\widetilde{GL}(1) \cap \widetilde{Sp}(1) = \{[1, \pm 1]\}$. Note that $\widetilde{GL}(1)$ commutes with $\widetilde{Sp}(1)$.

A representation π of a subgroup of $\widetilde{G}$ is called *genuine* if $\pi([1, \epsilon]) = \epsilon$. Suppose $G_1, G_2 \subset \widetilde{G}$ such that they commute and $G_1 \cap G_2 = \{[1, \pm 1]\}$. If π_1, π_2 are genuine representations of G_1, G_2 on V^{π_1}, V^{π_2}, respectively, then we can define a new representation $\pi_1 \otimes \pi_2$ of $G_1 G_2$ on the space $V^{\pi_1} \otimes V^{\pi_2}$ by the formula $\pi_1 \otimes \pi_2(g_1 g_2) := \pi_1(g_1) \otimes \pi_2(g_2)$.

Returning to our business, we study more on the functions $F(\cdot, s, \varphi)$. We denote by θ_0 the automorphic representation of $\widetilde{Sp}(1, \mathbb{A})$ on the space generated by theta functions $g \mapsto I_0(\cdot, \varphi_0)$, for φ_0 running over $\mathcal{S}(V_0(\mathbb{A}))$. From the formulas given in §2 for the oscillator representations, it is not difficult to check

LEMMA 4.2. *Let $\delta_{\widetilde{P}_1}$ be the modular character of $\widetilde{P}_1$, and*

$$\chi_s([t, \epsilon]) = \chi([m_1(t, 1), \epsilon])|t|^s,$$

which is a genuine character of $GL(1,\mathbb{Q})\backslash\widetilde{GL}(1,\mathbb{A})$. *Then*

(1) $F(n_1 g, s, \varphi) = F(g, s, \varphi)$ *for* $n_1 \in N_1(\mathbb{A})$.

(2) $F([m_1(t,1),\epsilon]g, s, \varphi) = \chi_s([t,\epsilon])\delta_{\widetilde{P}_1}(m_1(t,1))^{\frac{1}{2}}F(g,s,\varphi)$ *for* $t \in GL(1,\mathbb{A})$.

Thus we can conclude that for all $\hat{\varphi} \in S_{\mathbb{A}}$,

$$F(\cdot, s, \varphi) \in I(s,\theta_0) := \mathrm{Ind}_{\widetilde{P}_1(\mathbb{A})}^{\widetilde{G}(\mathbb{A})}(\chi_s \otimes \theta_0).$$

Here $I(s,\theta_0)$ is realized on the space of smooth and right $\widetilde{K}$-finite functions

$$\Psi : N_1(\mathbb{A})M_1(\mathbb{Q})\backslash\widetilde{G}(\mathbb{A}) \to \mathbb{C}$$

such that

(i) for all $g \in \widetilde{G}(\mathbb{A})$, the function $g_0 \mapsto \Psi(g_0 g, s)$ ($g_0 \in \widetilde{Sp}(1,\mathbb{A})$) lies in θ_0.

(ii) for $\tilde{t} \in \widetilde{GL}(1,\mathbb{A}) \hookrightarrow \widetilde{P}_1(\mathbb{A})$,

$$\Psi(\tilde{t}g, s) = \chi_s(\tilde{t})\delta_{\widetilde{P}_1}(\tilde{t})^{\frac{1}{2}}\Psi(g,s),$$

where K is a fixed maximal compact subgroup of $G(\mathbb{A})$ such that $G(\mathbb{A}) = P(\mathbb{A})K$ and hence $\widetilde{G}(\mathbb{A}) = \widetilde{P}(\mathbb{A})\widetilde{K}$.

5. Action of $\omega(D)$

To compute the action of $\omega(D)$, we need the following [1]:

LEMMA 5.1. *Let* C' *(resp.* C*) be the Casimir operator of* $Sp(n,\mathbb{R})$ *(resp.* $O(p,q)$*), and* $m = p+q$. *Then*

$$2(m-2)\omega(C) = 4(n+1)\omega(C') - \frac{mn}{2}(\frac{m}{2} - n - 1).$$

Specializing to our case $n=2$, $m=3$, we have $\omega(C) = 6\omega(C') + \frac{9}{4}$. Hence if we let

$$D' = (6\omega(C') + \frac{9}{4}) - 4 = 6C' - \frac{7}{4},$$

then $\omega(D) = \omega(D')$. Now we note that the map $\varphi \mapsto F(\cdot, s, \varphi)$ defines an intertwining map from $\mathcal{S}_{\mathbb{A}}$ to $I(s,\sigma)$ for $Re(s)$ sufficiently large. Therefore we have

$$F(g,s,\omega(D)\varphi) = F(g,s,\omega(D')\varphi) = F(g,s,\varphi) * D'.$$

Hence, it suffices to compute the scalar by which D' acts in the induced representation $I(s,\theta_0)$. We have the factorizations [3]

$$\chi_s = \otimes_v \chi_{s,v}, \qquad \theta_0 = \otimes_v \theta_{0,v}$$

and hence

$$I(s,\theta_0) = \bigotimes_v \left(\mathrm{Ind}_{\widetilde{P}_{1,v}}^{\widetilde{G}_v} \chi_{s,v} \otimes \theta_{0,v}\right),$$

where the local induced representations are defined similarly to the global induced representation.

Temporarily we fix the place $v = \infty$ and suppress the index v for notational convenience. We also change the notation and let $G = Sp(n,\mathbb{R})$. As usual, let B be the Borel subgroup of G with the unipotent radical N. Let

$$A^+ = \{ \left(\begin{smallmatrix} a & \\ & a^{-1} \end{smallmatrix}\right) \mid a = \operatorname{diag}(a_1,\dots,a_n), a_i > 0 \},$$
$$K = \left\{ \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right) \mid a + bi \in U(n) \right\},$$

and let $\widetilde{\mathbf{M}}$ be the centralizer of A in $\widetilde{K}$. Note that

$$\widetilde{\mathbf{M}} = \{ [\left(\begin{smallmatrix} m & \\ & m^{-1} \end{smallmatrix}\right), \epsilon] \mid m = \operatorname{diag}(\pm 1,\dots,\pm 1),\ \epsilon = \pm 1 \}.$$

The covering map $\widetilde{G} \to G$ splits over A^+ and N, and the Iwasawa decomposition of $\widetilde{G}$ is $\widetilde{G} = NA^+\widetilde{K}$. Also, since $\widetilde{G}_n$ is a connected semisimple Lie group with finite center, Harish-Chandra's general theory is applicable. In particular, every irreducible unitary representation is admissible, and is a subquotient of some (possibly non-unitary) principal series representation of the following type.

Write the Langlands decomposition $\widetilde{B} = \widetilde{\mathbf{M}}A^+N$. Let $\mathfrak{g}$, $\mathfrak{a}$ be the Lie algebras of G, A^+, resp., and let ρ be the half the sum of positive roots of $(\mathfrak{g}, \mathfrak{a})$, τ irreducible unitary representation of $\widetilde{\mathbf{M}}$ on the space V^τ, and $\nu \in \mathfrak{a}'_{\mathbb{C}}$. Then the induced representation $U(\widetilde{B}, \tau, \nu) = \mathrm{Ind}_{\widetilde{\mathbf{M}}A^+N}^{\widetilde{G}_n} \tau \otimes \exp \nu \otimes 1$ is realized on the closure of the subspace of continuous functions $F : \widetilde{G}_n \to V^\tau$ satisfying

$$F(mang) = e^{(\nu+\rho)\log a}\tau(m)F(g)$$

for all $man \in \widetilde{\mathbf{M}}A^+N$, and $\int_{\widetilde{K}} |F(k)|^2 \, dk < \infty$. It is well known that $U(\widetilde{B}, \tau, \nu)$ has infinitesimal character ν, that is $U(z) = \xi(z)(\nu)$ for all $z \in Z(\mathfrak{g}_{\mathbb{C}})$, where ξ is the Harish-Chandra homomorphism. In particular, we have

$$U(C') = <\nu, \nu> - <\rho, \rho>,$$

where $< \, , \, >$ is the Cartan-Killing form of $\mathfrak{g}$.

Coming back to our notation and still suppressing the index $v = \infty$, let $G = Sp(2, \mathbb{R})$, $G_0 = Sp(1, \mathbb{R})$. We will put the subscript 0 for the objects belonging to G_0 in the discussion above, so that $G_0 = N_0 A_0^+ \widetilde{K}_0$, etc. We have $\omega_0 = \omega_0^+ \oplus \omega_0^-$, where ω_0^+ (resp. ω_0^-) denotes the restriction of ω_0 to the space of even (resp. odd) functions. Gelbart [4] showed that they are subquotients of $U(\widetilde{B}_0, \tau_0, \nu_0)$ for $\nu_0 = \frac{1}{2}$ and for some genuine representation τ_0 of $\widetilde{\mathbf{M}}_0$.

We may regard $\widetilde{\mathbf{M}}_0$ as a subgroup of $\widetilde{\mathbf{M}}$ by the natural injective homomorphism

$$[\pm 1, \epsilon] \mapsto [m_1(1, \pm 1), \epsilon].$$

If we let $\widetilde{\mathbf{M}}' = \{[m_1(\pm 1, 1), \epsilon]\} \subset \widetilde{\mathbf{M}}$, then we have $\widetilde{\mathbf{M}} = \widetilde{\mathbf{M}}'\widetilde{\mathbf{M}}_0$ with $\widetilde{\mathbf{M}}' \cap \widetilde{\mathbf{M}}_0 = \{[1, \pm 1]\}$.

THEOREM 5.2. *$I(s, \sigma)$ is a subquotient of the double induced representation*

$$\mathrm{Ind}_{\widetilde{P}_1}^{\widetilde{G}} \left(\chi_s \otimes \mathrm{Ind}_{\widetilde{B}_0}^{\widetilde{G}_0}(\tau_0 \otimes \nu)\right) = U(\widetilde{B}, \tau, \nu)$$

for $\tau = \chi_{s|\widetilde{\mathbf{M}}'} \otimes \tau_0$ and $\nu = (s, \frac{1}{2})$.

Proof. We first note that

$$N = \left\{ n(x, y, z, b) = \begin{pmatrix} 1 & x & y & z \\ & 1 & z - bx & b \\ & & 1 & \\ & & -x & 1 \end{pmatrix} \right\} \simeq N_1 \times N_0.$$

For $t, a \in \mathbb{R}^\times$, let $\delta(t,a) = \operatorname{diag}(t, a, t^{-1}, a^{-1})$. It is enough to compute the transformation rule of Ψ in the double induced representation above under $[\delta(\epsilon_1,\epsilon_2),\epsilon]\delta(t,a)n(x,y,z,b) \in \widetilde{B} = \widetilde{\mathbf{M}}A^+N$, where $\epsilon_1, \epsilon_2 = \pm 1$, $\epsilon = \pm 1$, and $t, a > 0$. Indeed, we have

$$\begin{aligned}
&\Psi([\delta(\epsilon_1,\epsilon_2),\epsilon]\delta(t,a)n(x,y,z,b)g)\\
=&\Psi([m_1(\epsilon_1 t,1),\epsilon'][m_1(1,\left(\begin{smallmatrix}\epsilon_2 a & \\ & (\epsilon_2 a)^{-1}\end{smallmatrix}\right)),\epsilon'']m_1(1,\left(\begin{smallmatrix}1 & b\\ 0 & 1\end{smallmatrix}\right))n_1(x,y,z)g)\\
=&\chi_s([\epsilon_1 t,\epsilon'])\tau_0([\epsilon_2,\epsilon''])a^{\frac{1}{2}}\Psi(g) = \chi_s([\epsilon_1,\epsilon'])\tau_0([\epsilon_2,\epsilon''])t^s a^{\frac{1}{2}}\Psi(g)\\
=&\chi_{s|\widetilde{\mathbf{M}}'} \otimes \tau_0([\delta(\epsilon_1,\epsilon_2),\epsilon])t^s a^{\frac{1}{2}}\Psi(g),
\end{aligned}$$

where $\epsilon', \epsilon'' = \pm 1$ satisfying $\epsilon = \epsilon'\epsilon''(\epsilon_1,\epsilon_2)$.

Since $\rho = (2,1)$ and $< \, , \, >= \frac{1}{12}(\, , \,)$, where $(\, , \,)$ is the usual inner product on $\mathbb{R}^2$ we obtain $U(C') = \frac{1}{12}(s^2 - \frac{19}{4})$, and hence

$$U(D') = 6U(C') - \frac{4}{7} = \frac{1}{8}(4s^2 - 33) := P(s).$$

Returning to the global situation, we finally obtain the

THEOREM 5.3. *Let C be the Casimir element of the universal enveloping algebra of $H(\mathbb{R})$, and let $D = C - 4$. Then for all $\varphi \in \mathcal{S}(V(\mathbb{A})^2)$,*

$$\begin{aligned}
I(g,s;\omega(D)\varphi) &= \int_{H(\mathbb{Q})\backslash H(\mathbb{A})} \theta(g,h;\omega(D)\varphi)E(h,s)\,dh\\
&= \sum_{\gamma\in P_1(\mathbb{Q})\backslash G(\mathbb{Q})} F(\gamma g, s, \omega(D)pr_{K_H}(\varphi))
\end{aligned}$$

defines an Eisenstein series on $\widetilde{G}(\mathbb{A})$ attached to the maximal parabolic subgroup $\widetilde{P}_1 \simeq \widetilde{GL}(1) \times \widetilde{Sp}(1)$ of $\widetilde{G}$ and to the induced representation $I_{\widetilde{P}_1}^{\widetilde{G}}(\chi_s \otimes \theta_0)$ of $\widetilde{G}(\mathbb{A})$. Furthermore,

$$I(g,s;\omega(D)\varphi) = P(s) \sum_{\gamma\in P_1(\mathbb{Q})\backslash G(\mathbb{Q})} F(\gamma g, s, pr_{K_H}(\varphi)).$$

References

1. J. Adams, *Discrete spectrum of the reductive dual pair* $(O(p,q), Sp(2m))$, Invent. Math. **74** (1983), 449–475.
2. J. G. Arthur, *Eisenstein series and the trace formula*, Proc. Sympos. Pure Math. **33** (1979), Part II, 253–274.
3. D. Flath, *Decompositions of representations into tensor products*, Proc. Sympos. Pure Math. **33** (1979), Part II, 179–183.
4. S. Gelbart, *Weil's representation and spectrum of the metaplectic group*, Lecture Notes in Math., vol. 530, Springer-Verlag, Berlin, 1976.
5. S. Kudla and S. Rallis, *A regularized Siegel-Weil formula: the first term identity*, preprint, 1993.
6. S. Kudla, S. Rallis and D. Soudry, *On the degree 5 L-function for* $Sp(2)$, Invent. Math. **107** (1992), 483–541.
7. R. Rao, *On some explicit formulas in the theory of the Weil representation*, preprint.
8. W. Sweet, *Weil-Siegel formula for metaplectic groups*, Thesis, University of Maryland, 1990.

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