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ON REGULARITY OF SOME FINITE GROUPS IN THE THEORY OF REPRESENTATION

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1. Introduction

Investigation of the number of representations as well as of projective representations of a finite group has been important object since the early of this century. The numbers are very related to the number of conjugacy classes of G, so that this gives some informations on finite groups and on group characters. A generally well-known fact is that the number of non-equivalent irreducible representations, which we shall write as n.i.r. of G is less than or equal to the number of conjugacy classes of G, and the equality holds over an algebraically closed field of characteristic not dividing $|G|$. A remarkable result on the numbers due to Reynolds can be stated as follows.

LEMMA 1([6]). *Let F be any field of Char $F = p \geq 0$, G be a finite group, and let $f \in Z^2(G, F^*)$ be a 2-cocycle. The number of non-equivalent irreducible projective f-representations, which we shall write as n.i.p.r. of G over F is equal to the number of D_Γ-regular F-(conjugacy) classes of p'-elements of G. Here $\Gamma = F^f G$ is a twisted group algebra of G over F.*

The D_Γ-regularity condition is a generalized concept of f-regularity going back to Schur (1904). The D_Γ-regular class is a union of F-classes. If F is a complex field, F-classes coincide with the conjugacy classes. Thus Lemma 1, in this case, includes the Schur's theorem in [7], further if $f = 1$ then the number of n.i.r. of G equals that of conjugacy classes of G. In [1] and [6], it was asked whether the number of n.i.r. is equal to that of n.i.p.r. of G, and proved the following equivalent situations.

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LEMMA 2. *With the same notations, the followings are equivalent.*

1. *The number of n.i.p.r. of G equals that of n.i.r. of G.*
2. *The number of n.i.r. of Γ equals that of n.i.r. of G.*
3. *Every F-class of G is D_Γ-regular.*
4. *Every p-regular F-class of G is D_Γ-regular.*
5. *In the case that the order of 2-cocycle f is finite, $k_F(G_r(f)) = r \cdot k_F(G)$, where r is any integer divisible by the order of f, $G_r(f)$ is a generalized f-covering group and $k_F(G)$ is the number of F-conjugacy classes of G, etc.*

Lemma 1 and 2 allow us to reduce the problem to determine the number of n.i.p.r. to the determination of the number of certain classes in group theory. Further computing the number of n.i.p.r. could provide some knowledge to not-widely-known projective character theory.

The purpose of this paper is to find examples of G where all the elements in G are D_Γ-regular. In fact, we shall construct some finite groups on which all the F-classes and D_Γ-regular F-classes are calculated and the numbers are same. In case of a cocycle f having finite order, the similar constructions are made in [1]. A feature of this note is that the D_Γ-regularity makes no use of 2-cocycle explicitly, rather it uses basis elements of twisted group algebra.

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2. D_Γ mapping

Let ω be a primitive t-th root of unity in F for any positive integer t not divisible by Char F. Each automorphism α of $F(\omega)$ which leaves every element of F fixed is given by a map

$$\omega \to \omega^{m(\alpha)} \tag{2.1}$$

where $m(\alpha)$ are integers relatively prime to t. Write $I_t(F)$ for the multiplicative group consisting of those integers $m(\alpha)$, taken modulo t, for which (2.1) defines an automorphism of $F(\omega)$ over F. Indeed I_t is isomorphic to Gal$(F(\omega)/F)$, and is abelian. For example, $I_t(C) = \{1\}$, and $I_t(Q)$ is a set of all residue classes (mod t) which are relatively prime to t, where $\mathbb{C}$ is a complex field and $\mathbb{Q}$ is a rational field.

A twisted group algebra $\Gamma = F^f G$ for a finite group G over a field F is an F-algebra which has a basis $\{a_g \mid g \in G\}$ with $a_g a_{g'} = f(g,g')a_{gg'}$ where $f(g,g') \in F^*$. Following Yamazaki's approach [10], we restate the definition of twisted group algebra without using any 2-cocycle f, as follows: a twisted group algebra over F is a triple $(\Gamma, G, (\Gamma_g))$, where Γ is an F-algebra with identity 1_Γ and (Γ_g) is a family of one-dimensional F-subspaces of Γ indexed by G such that $\Gamma = \oplus_{g\in G}\Gamma_g$ and $\Gamma_g\Gamma_{g'} = \Gamma_{gg'}$ for all $g, g' \in G$. Of course Γ has dimension $|G|$, we often refer loosely to the algebra Γ as a twisted group algebra and write Γ in place of $(\Gamma, G, (\Gamma_g))$. Let E be an algebraic closure of F. Then the D_Γ mapping is defined by the following choices of integers. Firstly, choose integer n divisible by $\exp(G)$. Write $n = n_p n_{p'}$, where n_p is a p-singular part of n and $n_{p'}$ is a p-regular part of n. If $p = 0$ then $n = n_{p'}$. For this n and for each automorphism α in $\mathcal{G} = \mathrm{Gal}(E/F)$, secondly, choose integers $m(\alpha)$ such that $\omega_{n_{p'}}^{\alpha} = \omega_{n_{p'}}^{m(\alpha)}$ for any $n_{p'}$-th root of unity $\omega_{n_{p'}} \in E$, and $m(\alpha) \equiv 1 \pmod{n_p}$. Indeed, set of all $m(\alpha)$'s is $I_{n_{p'}}(F)$.

Two elements g, x in G are called *F-conjugate* if $x = z^{-1}g^{m(\alpha^{-1})}z$ for some $z \in G$, $\alpha \in \mathcal{G}$. The F-conjugacy is an equivalence relation, so G may be partitioned into F-classes. For the chosen n, $a_g^n = u(g)1_\Gamma$ for some $u(g) \in F^*$ for each $g \in G$. Thirdly, choose an element $v(g)$ in E such that $v(g)^n = u(g)$. Then D_Γ is defined by

$$(2.2) \qquad a_g D_\Gamma(\alpha, x) = v(g)^{-1}\, v(g)^{-m(\alpha^{-1})}\, a_x^{-1} a_g^{m(\alpha^{-1})} a_x$$

for $(\alpha, x) \in \mathcal{G} \times G$, $g \in G$. Certainly, D_Γ is a group action on Γ, and is independent of the particular choices of integers n and $m(\alpha)$. The action given on $a_g \in \Gamma$ can be extended by linearity to one on $\Gamma^E = E \otimes_F \Gamma$. The orbits of D_Γ composed of p'-elements of G are precisely the F-classes. An F-class L is *D_Γ-regular* provided that there exists nonzero $q(g) \in E$, $g \in L$, such that D_Γ acts as a permutation representation on the elements $q(g)a_g \in \Gamma^E$. In other words, an element $g \in G$ is D_Γ-regular if and only if

$$(2.3) \qquad a_g D_\Gamma(\alpha, x) = a_g \ \text{ for } \ (\alpha, x) \in \mathcal{G} \times G \text{ with } \ x^{-1}g^{m(\alpha^{-1})}x = g.$$

For the later use, we add an easy lemma.

LEMMA 3([6]). *Let $G = \langle g \rangle$ be a finite cyclic group. Then the twisted group algebra Γ of G over F is F-algebra isomorphic to $F[X]/(X^{|G|} - u(g))$, where $a_g^{|G|} = u(g)1_\Gamma$.*

Proof. It is not hard to show that the map $F[X] \longrightarrow \Gamma$ defined by $\Sigma\lambda_i X^i \mapsto \Sigma\lambda_i a_g^i$, for $\lambda_i \in F^*$ is obviously a surjective F-algebra homomorphism with kernel $(X^{|G|} - u(g))$.

3. Finite cyclic groups of order 2^k $(k > 2)$

Throughout the section, let $G = \langle g \rangle$ be a cyclic group of order 2^k, $(k > 2)$ and $(a_g)^{2^k} = 2^{2^{k-1}} 1_\Gamma$ where $a_g \in \Gamma_g$ and $(\Gamma, G, (\Gamma_g))$ is a twisted group algebra over $\mathbb{Q}$, the rational field. The following is the main theorem in this section.

THEOREM 1. *Let G and Γ be defined as above. Then there are $(k+1)$ F-classes of G, and these are all D_Γ-regular F-classes of G. Hence there are $(k+1)$ n.i.r. of Γ and there are the same number of n.i.r. of G.*

Note that there are 2^k conjugacy classes of G.

Proof. Choose $n = 2^k$, and choose $v(g)$ in an algebraic closure of Q as an n-th root $(u(g))^{1/n} = \left(2^{2^{k-1}}\right)^{1/n} = \sqrt{2}$. According to the remark in [6, Section 6], we can take a smaller finite extension field E of $\mathbb{Q}$ which has the following properties on E: E is a normal algebraic extension of F, E contains a primitive n-th root of unity $\omega_n = \omega$ as well as $v(g)$ for all $g \in G$, and E is a splitting field for $E \otimes_F \Gamma$. For this reason we can choose E as $E = Q(\sqrt{2}, \omega)$. For the sake of clarity, we divide the proof into some lemmas.

LEMMA 4. *The group $\mathrm{Gal}(E/F)$ is isomorphic to $Z_{2^{k-2}} \oplus Z_2$ with two generators of order 2^{k-2} and 2.*

Proof. Since $\omega^{2^{k-3}} + (\omega^{2^{k-3}})^{-1} = \sqrt{2}$, the extension field $Q(\sqrt{2}, \omega)$ is equal to $Q(\omega)$ which is a splitting field of $X^{2^k} - 1 \in Q[X]$. Further, $Q(\omega)$ is the cyclotomic extension of Q of order 2^k and $[Q(\omega) : Q] = \phi(2^k) = 2^{k-1}$ for Euler phi-function ϕ. Therefore

$$\mathrm{Gal}(Q(\omega)/Q) \cong Z_{2^{k-2}} \oplus Z_2,$$

with two generators σ and τ such that $\omega^\sigma = \omega^3$ and $\omega^\tau = \omega^{2^k-1}$ and $o(\sigma) = 2^{k-2}$, $o(\tau) = 2$.

We now consider the set of all $m(\alpha)$, $\alpha \in \mathrm{Gal}(Q(\omega)/Q)$, which we know this is equal to $I_n(Q)$, $n = 2^k$. Let $r = 2^{k-2}$. By modulo 2^k, we observe the following:

$$\begin{cases} m(1) \equiv 1,\ m(\sigma) \equiv 3,\ m(\sigma^2) \equiv 3^2, \dots,\ m(\sigma^{r-1}) \equiv 3^{r-1}, \\ m(\tau) \equiv 2^k - 1 \equiv -1,\ m(\sigma\tau) \equiv 3(2^k-1) \equiv -3, \\ m(\sigma^2\tau) \equiv -3^2, \dots,\ m(\sigma^{r-1}\tau) \equiv (2^k-1)3^{r-1} \equiv -3^{r-1}, \end{cases}$$

here we have used the fact that $m(\alpha\alpha') \equiv m(\alpha)m(\alpha') \equiv m(\alpha'\alpha)$ (mod 2^k) for any α, $\alpha' \in \mathrm{Gal}(Q(\omega)/Q)$. Then

$$I_n(Q) = \{1, 3, 3^2, \dots, 3^{r-1}, -1, -3, -3^2, \dots, -3^{r-1}\}.$$

This is congruent modulo 2^k to $\{1, 3, \dots, 2^k - 1\}$, in some order.

We now compute $a_g D_\Gamma(\alpha, x)$ for $(\alpha, x) \in \mathrm{Gal}(Q(\omega)/Q) \times G$.

LEMMA 5. *Let $r = 2^{k-2}$. The $D_\Gamma(\alpha, x)$ act like permutation representations on the independent set*

$$\{a_g, a_g^3, a_g^{3^2}, \dots, a_g^{3^{r-1}}, a_g^{-1}, a_g^{-3}, a_g^{-3^2}, \dots, a_g^{-3^{r-1}}\}.$$

Proof. Since $G = \langle g \rangle$, we have $a_{g^i} a_{g^j} = a_{g^j} a_{g^i}$ for any $0 \le i,\ j < |G|$, and this yields a simplified formula of $D_\Gamma(\alpha, x)$ (compare to (2.2)) that

$$a_g D_\Gamma(\alpha, x) = \sqrt{2}^{\alpha^{-1}} \sqrt{2}^{-m(\alpha^{-1})} a_g^{m(\alpha^{-1})}.$$

On the other hand, for the two generators σ and τ,

$$\sqrt{2}^\sigma = \omega^{3\cdot 2^{k-3}} + (\omega^{3\cdot 2^{k-3}})^{-1} = -\sqrt{2},$$
$$\sqrt{2}^\tau = (\omega^{2^{k-3}})^{-1} + \omega^{2^{k-3}} = \sqrt{2},$$

hence, for $1 \le i \le r-1$ and $0 \le j \le 1$,

$$\sigma^i\tau^j\ :\ \sqrt{2} \mapsto -\sqrt{2},\ \text{ if } i:\ \text{odd};\quad \sigma^i\tau^j\ :\ \sqrt{2} \mapsto \sqrt{2},\ \text{ if } i:\ \text{even}.$$

By adding up all the informations above, we, therefore have,

$$a_g D_\Gamma(1, x) = a_g \quad \text{where 1 is the identity of Gal}(Q(\omega)/Q)$$

$$a_g D_\Gamma(\sigma^{-1}, x) = \frac{\sqrt{2}^{\sigma}}{\sqrt{2}^{m(\sigma)}} a_g^{m(\sigma)} = -\frac{1}{2}^{(3-1)/2} a_g^3 = -\frac{1}{2} a_g^3$$

$$\cdots$$

$$a_g D_\Gamma(\tau^{-1}, x) = \sqrt{2}\sqrt{2} a_g^{-1} = 2 a_g^{-1}$$

$$\cdots$$

$$a_g D_\Gamma\left((\sigma^{r-1}\tau)^{-1}, x\right) = -2^{(3^{r-1}+1)/2} a_g^{-3^{r-1}},$$

this proves Lemma 5.

Hence, the orbit of D_Γ, which is an F-class of $g \in G$ is

$$\mathcal{Q} = \{g, g^3, g^{3^2}, \ldots, g^{3^{r-1}}, g^{-1}, g^{-3}, g^{-3^2}, \ldots, g^{-3^{r-1}}\}.$$

Moreover, all elements in $\mathcal{Q}$ are distinct so that $|\mathcal{Q}| = 2^{k-1}$, hence by reordering, $\mathcal{Q}$ can be represented as $\mathcal{C}_g = \{g, g^3, \ldots, g^{2^k-1}\}$, whence this is a D_Γ-regular F-class containing g.

LEMMA 6. *There are $(k+1)$ orbits of D_Γ. Further there are the same number of D_Γ-regular F-classes of G.*

Proof. Since G is cyclic, $(a_g)^i D_\Gamma(\alpha, x) = \left(a_g D_\Gamma(\alpha, x)\right)^i$ for all $0 \leq i < |G|$ ([6, Theorem 4 (c)]), and some calculations using Lemma 5 yield all the other orbits of D_Γ:

$$\mathcal{C}_{g^2} = \{g^2, g^6, \ldots, g^{2^k-2}\}, \quad \mathcal{C}_{g^4} = \{g^4, \ldots, g^{2^k-2^2}\}, \ldots,$$

$$\mathcal{C}_{g^{2^{k-1}}} = \{g^{2^{k-1}}\}, \quad \mathcal{C}_1 = \{1\}.$$

Of course, $|\mathcal{C}_1| = 1$, $|\mathcal{C}_g| = 2^{k-1}$, $|\mathcal{C}_{g^2}| = 2^{k-2}$, and $|\mathcal{C}_{g^{2^{k-1}}}| = 1$. Since $|\mathcal{C}_1| + \sum_{i=0}^{k-1} |\mathcal{C}_{g^{2^i}}| = 2^k$, G is partitioned by all these orbits. Further since D_Γ acts like a permutation representation, all of these orbits are D_Γ-regular F-classes. This proves Lemma 6.

Hence there are $(k+1)$ n.i.p.r. of G and there are the same number of n.i.r. of G. This completes the proof of Theorem 1.

4. Remarks and examples

(1) Let $G = \langle g \rangle$, $o(g) = 8$, and $a_g^8 = 2^4 1_\Gamma$. Then Γ is isomorphic as a $\mathbb{Q}$-algebra to $\mathbb{Q}[X]/(X^8 - 2^4)$ by Lemma 3, and an irreducible factorization in $\mathbb{Q}[X]$ is

$$X^8 - 2^4 = (X^2 - 2)(X^2 + 2)(X^2 - 2X + 2)(X^2 + 2X + 2).$$

Hence Γ is the direct sum of 4 fields, accordingly the number 4 gives the number of irreducible $\mathbb{Q}$-representations of Γ. By Lemma 1, there are also the same number of D_Γ-regular orbits.

Now for the $D_\Gamma(\alpha, x)$ mapping, we choose $n = 8$, $v(g) = \sqrt{2}$ and let ω be an 8-th root of unity, i.e., $\omega = (1 + i)/\sqrt{2}$. Take a suitable finite extension E of $\mathbb{Q}$ to be $E = \mathbb{Q}(\omega, \sqrt{2})$. Then $\mathrm{Gal}(E/\mathbb{Q})$ is generated by two elements σ and τ, where σ interchanges $\pm i$ and fixes $\pm\sqrt{2}$, and τ fixes $\pm i$ and interchanges $\pm\sqrt{2}$. Thus $m(\sigma) \equiv 7$, $m(\tau) \equiv 5 \pmod 8$, and

$$a_g\big(D_\Gamma(\sigma\tau, x)\big) = (\frac{1}{8}a_g^7)D_\Gamma(\tau, x) = \frac{1}{8}\big(a_g D_\Gamma(\tau, x)\big)^{-1} = -\frac{1}{2}a_g^3$$

for all $a_g \in \Gamma$, $x \in G$. A little more calculation like this shows that all $D_\Gamma(\alpha, x)$, $\alpha \in \mathrm{Gal}(E/\mathbb{Q})$ act on the independent set $\{a_g, -\frac{1}{2}a_g^3, -\frac{1}{4}a_g^5, -\frac{1}{8}a_g^7\}$ as permutations. Hence the orbit $\{g, g^3, g^5, g^7\}$ is D_Γ-regular. The other orbits $\{g^2, g^6\}$, $\{g^4\}$ and $\{1\}$ are more easy to check.

(2) Let $G = \langle g \rangle$, $o(g) = 8$, and $a_g^8 = 3^4 1_\Gamma$. This example will show the failure of the D_Γ-regularity. Indeed, $X^8 - 3^4 = (X^2 - 3)(X^2 + 3)(X^4 + 9)$ is the irreducible factorization in $\mathbb{Q}[X]$, thus Γ has 3 direct summands. Hence there are 3 irreducible $\mathbb{Q}$-representations of Γ, and 3 D_Γ-regular orbits.

Using the language $D_\Gamma(\alpha, x)$-regular, choose $n = 8$ and $v(g) = (3^4)^{1/8} = \sqrt{3}$. Choose $E = \mathbb{Q}\big(\omega, v(g)\big)$ where ω is an 8-th root of unity. Then E is isomorphic to $\mathbb{Q}(i, \sqrt{2}, \sqrt{3})$, thus $\mathrm{Gal}(E/\mathbb{Q})$ is generated by three elements σ_1, σ_2 and σ_3 where σ_k, $(k = 1, 2, 3)$ interchanges the element of the k-th of those pairs $\pm i$, $\pm\sqrt{2}$, $\pm\sqrt{3}$ and fixes the elements of the other two pairs. Therefore, for any $x \in G$

$$a_g D_\Gamma(\sigma_1, x) = \frac{1}{27}a_g^7, \quad a_g D_\Gamma(\sigma_2, x) = \frac{1}{9}a_g^5, \quad a_g D_\Gamma(\sigma_3, x) = -a_g.$$

The fact $a_g D_\Gamma(\sigma_3, x) = -a_g$ says that σ_3 does not act like a permutation on the orbit containing g for any basis consisting of scalar multiples of a_g, a_g^3, a_g^5, a_g^7; hence the orbit is not regular.

(3) The condition $k > 2$ in Theorem 1 is necessary. In the case $k = 2$, i.e., $a_g^4 = 2^2 1_\Gamma$, we may take $E = \mathbb{Q}(v(g), \omega_4) = \mathbb{Q}(\sqrt{2}, i)$. But for one generator $\sigma \in \mathrm{Gal}(E/\mathbb{Q})$ which send $\sqrt{2}$ to $-\sqrt{2}$, and i to i, we have $\omega_4^{m(\sigma)} = \omega_4$ so that $m(\sigma) \equiv 1 \pmod 4$. Hence $a_g D_\Gamma(\sigma, x) = -\sqrt{2}/\sqrt{2} a_g = -a_g$; this shows that σ does not act like a permutation on the orbit containing g. This is not a regular orbit.

5. Application to dihedral group

For any 2-cocycle $f \in Z^2(G, A)$, A any G-module with trivial G-action, it was proved in [3] that f can be considered as a normalized 2-cocycle; that is, $f(g_1, g_2) = 1$ whenever one of the g_i $(i = 1, 2)$ is 1. According to [9, Proposition 1], when G is a semi-direct product $N \times T$ of a normal subgroup N and a subgroup T, f can be normalized up to coboundaries on $f(N, T) = 1$ hence $f(nt, n't') = f(t, t')f(t, n)f(n', {}^t n')$ for $n, n' \in N$ and $t, t' \in T$. Such f is called a normal 2-cocycle, and thus f on G is determined uniquely by $f\mid_{N \times N}$, $f\mid_{T \times T}$ and $f\mid_{T \times N}$.

As an application, it is possible to give other groups G where all elements of G are D_Γ-regular.

THEOREM 2. *Let G be a dihedral group $\langle c, d \mid c^{2^k} = d^2 = 1, dc = c^{-1}d \rangle$, with a generator c of the type $(a_c)^{2^k} = 2^{2^{k-1}} 1_\Gamma$, where $\{a_g \mid g \in G\}$ is a basis of twisted group algebra Γ of G over $F = \mathbb{Q}$. Then there are $(k+3)$ $\mathbb{Q}$-classes of G and they are all D_Γ-regular $\mathbb{Q}$-classes of G.*

Proof. Since G is a semidirect product $\langle c \rangle \times \langle d \rangle$, the 2-cocycle f on G can be normalized and determined by restrictions $f\mid_{\langle c \rangle \times \langle c \rangle}$, $f\mid_{\langle d \rangle \times \langle d \rangle}$ and $f\mid_{\langle d \rangle \times \langle c \rangle}$. Since $f\mid_{\langle d \rangle \times \langle d \rangle}$ is a normalized 2-cocycle in $Z^2(\langle d \rangle, \mathbb{Q}^*)$, we may suppose that $f(d, d) = f\mid_{\langle d \rangle \times \langle d \rangle}(d, d) = -1$, so that $(a_d)^2 = -1_\Gamma$. Similarly, $(a_{c^i d})^2 = -1_\Gamma$ for all $1 \le i \le 2^k$.

Let $n = 2^k$, the $\exp G$. Then $(a_c)^n = 2^{2^{k-1}} 1_\Gamma$ yields that an n-th root $v(c)$ of $u(c) = 2^{2^{k-1}}$ can be chosen as $\sqrt{2}$. Since $(a_{c^i d})^n = 1_\Gamma$, we may also choose $v(c^i d)$ as a primitive n-th root of unity ω. As an algebraic normal extension E of $\mathbb{Q}$ to contain both $\sqrt{2}$ and ω, and to be a splitting field for G, we may now take E as $\mathbb{Q}(\sqrt{2},\omega)$ which is of course equal to $Q(\omega)$. Hence the same situations in Lemma 4 prevail for this case that $\mathrm{Gal}(E/\mathbb{Q})$ has two generators σ of order 2^{k-2} and τ of order 2, and $I_n(\mathbb{Q}) = \{m(\alpha) \mid \alpha \in \mathrm{Gal}(E/\mathbb{Q})\} = \{1,3,5,\ldots,2^k-1\}$.

In similar way as in Lemma 5, it is not hard to find all F-classes of G. That is, there are $(k+3)$ F-classes of G:

$$\mathcal{C}_c = \{c,\ldots,c^{2^k-1}\},\ \mathcal{C}_{c^2} = \{c^2,\ldots,c^{2^k-2}\},\ldots,\ \mathcal{C}_{c^{2^{k-1}}} = \{c^{2^{k-1}}\},$$
$$\mathcal{C}_1 = \{1\},\ \mathcal{C}_d = \{d,c^2d,\ldots,c^{2^k-2}d\},\ \ \mathcal{C}_{cd} = \{cd,c^3d,\ldots,c^{2^k-1}d\}.$$

Some computations involving $D_\Gamma(\sigma,x)$ show that D_Γ acts like a permutation representation on these classes, thus all of these are D_Γ-regular F-classes. This completes the proof.

We note that there are $4+(2^k-2)/2 = 2^{k-1}+3$ conjugacy classes of G.

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