Comm. Korean Math. Soc. 9 (1994), No. 4, pp. 767–772

BEHAVIOR OF THE PERMANENT ON POSITIVE SEMIDEFINITE HERMITIAN MATRICES WITH FIXED ROW SUMS

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1. Introduction

For an $n \times n$ complex matrix $A = [a_{ij}]$, the *permanent* of A, $\operatorname{per} A$, is defined by

$$\operatorname{per} A = \sum_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

where S_n stands for the symmetric group on the set $\{1, 2, \ldots, n\}$.

For a matrix A and for $i, j \in \{1, 2, \ldots, n\}$, let $A(i|j)$ denote the matrix obtained from A by deleting the row i and the column j, and let $s(A)$ denote the sum of all entries of A. As usual, for a complex number z, let $\overline{z}$ denote the complex conjugate of z and let $\overline{A}$ denote the matrix $[\overline{a}_{ij}]$ when $A = [a_{ij}]$.

Permanents of positive semidefinite hermitian matrices are real, in fact, nonnegative. Marcus and Minc [4] (See also Theorem 5.1 of [5]) proved that the permanent of an $n \times n$ positive semidefinite hermitian matrix with row sums $r_1, r_2, \ldots, r_n$ is greater than or equal to

$$\frac{n!}{s(A)^n} \prod_{i=1}^{n} |r_i|^2$$

provided that $s(A) \neq 0$.

In this paper we extend the theorem of Marcus and Minc with a short elementary proof, in which we make use of the following result due to Marcus and Merris [3].

Received November 26, 1993. Revised February 16, 1994.

This research was supported by the Korea Science and Engineering Foundation in 1991-1992 under grant #911-0101-001-2.

THEOREM A. *Let A be an $n \times n$ positive semidefinite hermitian matrix with row sums $r_1, r_2, \ldots, r_n$. Then*

$$\sum_{i=1}^{n} \sum_{j=1}^{n} r_i \overline{r}_j \operatorname{per} A(i|j) \leq n\, s(A) \operatorname{per} A.$$

2. The function permanent of positive semidefinite hermitian matrices

If a complex n-vector $R = (r_1, r_2, \ldots, r_n)^T$ is the row sum vector of an $n \times n$ positive semidefinite hermitian matrix A, then it is necessary that $r_1 + r_2 + \cdots + r_n$ is a nonnegative real number. For a complex n-vector $R = (r_1, r_2, \ldots, r_n)^T$ with $r_1 + r_2 + \cdots + r_n$ being equal to a positive real number s, let $J_R = [u_{ij}]$ denote the $n \times n$ matrix defined by

$$u_{ij} = \frac{r_i \overline{r}_j}{s}.$$

Then J_R is positive semidefinite hermitian and

$$\operatorname{per} J_R = \frac{n!}{s^n} \prod_{i=1}^{n} |r_i|^2.$$

Let H_R denote the set of all $n \times n$ positive semidefinite hermitian matrices with row sum vector R. Then it is not hard to show that H_R is a convex set containing J_R as a member.

THEOREM 1. *Let $R = (r_1, r_2, \ldots, r_n)$ be a complex n-vector with $r_1 + r_2 + \cdots + r_n = s$ being real positive. Then the permanent function is monotone increasing on the straight line segment from J_R to any $A \in H_R$. Moreover if, in addition, $r_1 r_2 \cdots r_n \neq 0$ and $\operatorname{per} A \neq \operatorname{per} J_R$, then the monotonicity of the permanent from J_R to A is strict.*

Proof. Given $A = [a_{ij}] \in H_R$, let $A_t = (1-t)J_R + tA$ for t, $0 \leq t \leq 1$, and let $f_A(t) = \operatorname{per} A_t$. Then $A_t \in H_R$ and hence $f_A(t)$ is a real valued function of t. Now,

$$\begin{aligned} f'_A(1) &= \sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ij} - u_{ij})\text{per}\, A_t(i|j)|_{t=1} \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\left(a_{ij} - \frac{r_i\bar{r}_j}{s}\right)\text{per}\, A(i|j) \\ &= n\,\text{per}\, A - \frac{1}{s}\sum_{i=1}^{n}\sum_{j=1}^{n} r_i\bar{r}_j\,\text{per}\, A(i|j). \end{aligned}$$

Thus it follows that $f'_A(1) \geq 0$ from Theorem A. On the other hand, since H_R is a convex set, we have $A_t \in H_R$ for any specific t, $0 \leq t \leq 1$. Therefore from the above discussion, we have

$$f'_A(t) = \frac{d}{d\theta}[(1-\theta)J_R + \theta A_t]\big|_{\theta=1} \geq 0,$$

from which it follows that the permanent function is monotone increasing from J_R to A.

For the proof of the second assertion, suppose that $r_i \neq 0$ ($i = 1, \ldots, n$) and $\text{per}\, A \neq \text{per}\, J_R$. Then, we have $\text{per} A > \text{per}\, J_R$. Now the strictness of the monotonicity of the function $f_A(t)$ follows from the fact that this function is a polynomial of t.

COROLLARY 1[MARSUS AND MINC, 5, THEOREM 5.1]. *Let A be an $n \times n$ positive semidefinite hermitian matrix with row sums $r_1, r_2, \ldots, r_n$ such that $s(A) = r_1 + r_2 + \cdots + r_n \neq 0$. Then*

$$\text{per}\, A \geq \frac{n!}{s(A)^n}\prod_{i=1}^{n}|r_i|^2.$$

Proof. By Theorem 1 we have

$$\text{per}\, A = f_A(1) \geq f_A(0) = \text{per}\, J_R = \frac{n!}{s(A)^n}\prod_{i=1}^{n}|r_i|^2.$$

REMARK. It is shown in [5, Theorem 5.1] that equality holds in Corollary 1 if and only if either A has a zero row or rank of A is 1. Therefore we have that the monotonicity of the permanent from J_R to A is strict if and only if either A has a zero row or rank of A is 1.

A real nonnegative square matrix is called *doubly stochastic* if each of its rows and columns has sum 1. It is well known that the set of all doubly stochastic matrices of order n forms a convex polyhedron in the n^2 dimensional Euclidean space. Let J_n denote the $n \times n$ matrix all of whose entries are equal to $1/n$.

COROLLARY 2[HWANG, 2, THEOREM 2]. *The permanent function is monotone increasing on the straight line segment from J_n to any positive semidefinite symmetric doubly stochastic matrix.*

Proof. Let $R = (1, 1, \ldots, 1)$, the n-tuple of 1's, in Theorem 1. Then $J_R = J_n$.

Let A be a positive semidefinitie hermitian matrix with row sum vector $R = (r_1, r_2, \ldots, r_n)$ such that $r_1 r_2 \cdots r_n \neq 0$. Let $D_R = \text{diag}(R)$ and let $\tilde{A}$ be the $n \times n$ matrix defined by $D_R \tilde{A} \overline{D}_R = A$. Then $\tilde{A}$ is also positive semidefinite hermitian. Therefore both A and $\tilde{A}$ have nonnegative entry sums. In fact, both of the entry sums of A and of $\tilde{A}$ are positive and at least one of them is greater than or equal to n as we see in the following

LEMMA. *Let $A = [a_{ij}]$ be a positive semidefinite hermitian matrix with row sum vector $R = (r_1, r_2, \ldots, r_n)$ such that $r_1 r_2 \cdots r_n \neq 0$. Then*

$$\left(\sum_{i,j=1}^{n} a_{ij} \right) \left(\sum_{i,j=1}^{n} \frac{a_{ij}}{r_i \bar{r}_j} \right) \geq n^2.$$

Proof. Let A and R be the matrix and the vector as in the statement of the Lemma. Then for all $i, j = 1, \ldots, n$,

$$\text{per } J_R(i|j) = \frac{s}{n\, r_i \bar{r}_j} \text{per } J_R$$

where $s = r_1 + r_2 + \cdots + r_n$. Since $f'_A(0) \geq 0$, by Theorem 1, we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(a_{ij} - \frac{r_i\bar{r}_j}{s}\right) \operatorname{per} J_R \geq 0,$$

from which it follows that

$$\left(\frac{s}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{a_{ij}}{r_j\bar{r}_j} - \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{n}\right) \operatorname{per} J_R \geq 0,$$

i.e.,

$$\left(\sum_{i,j=1}^{n} a_{ij}\right)\left(\sum_{i,j=1}^{n}\frac{a_{ij}}{r_i\bar{r}_j}\right) \geq n^2$$

because

$$s = \sum_{i=1}^{n} r_i = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}.$$

The inequality in the Lemma may or may not be strict. If $r_1 = \cdots = r_n$ then it is necessarily that $r_i = \bar{r}_i$ $(i = 1, \ldots, n)$ and

$$\left(\sum_{i,j=1}^{n} a_{ij}\right)\left(\sum_{i,j=1}^{n}\frac{a_{ij}}{r_i\bar{r}_j}\right) = n^2.$$

For a complex n-vector $Z = (z_1, \ldots, z_n)^T$ with $z_1 \cdots z_n \neq 0$, let Z^{-1} denote the vector $(z_1^{-1}, z_2^{-1}, \ldots, z_n^{-1})$ in the sequel.

THEOREM 2. *Let A and R be as in the Lemma. Then*

$$\|A\|\|R^{-1}\| \geq \sqrt{n}$$

with equality if and only if A is a positive multiple of the $n \times n$ matrix of ones where $\|\cdot\|$ stands for the Euclidean norm.

Proof. We see that

$$\left|\sum_{i,j=1}^{n} a_{ij}\right| \leq \sum_{i,j=1}^{n} |a_{ij}| \leq n\left(\sum_{i,j=1}^{n} |a_{ij}|^2\right)^{1/2} = n\|A\|,$$

and

$$\left|\sum_{i,j=1}^{n} \frac{a_{ij}}{r_i\bar{r}_j}\right| \le \left(\sum_{i,j=1}^{n} |a_{ij}|^2\right)^{1/2} \left(\sum_{i,j=1}^{n} \left|\frac{1}{r_i\bar{r}_j}\right|^2\right)^{1/2}$$
$$= \|A\| \left(\sum_{i=1}^{n} \left|\frac{1}{r_i}\right|^2 \sum_{j=1}^{n} \left|\frac{1}{\bar{r}_j}\right|^2\right)^{1/2}$$
$$= \|A\|\|R^{-1}\|^2.$$

Therefore from the Lemma we get that $n\|A\|^2\|R^{-1}\|^2 \ge n^2$, and the inequality of the Theorem 2 follows.

The assertion for the equality case can be proved easily.

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