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A NOTE ON ITO PROCESSES

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1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space with $\mathcal{F}$ a σ-algebra of subsets of the measure space Ω and P a probability measure on Ω. Suppose $a > 0$ and let $(\mathcal{F}_t)_{t \in [0,a]}$ be an increasing family of sub-σ- algebras of $\mathcal{F}$. If $r > 0$, let $J = [-r, 0]$ and $C(J, \mathbf{R}^n)$ the Banach space of all continuous paths $\gamma : J \to \mathbf{R}^n$ with the sup-norm $\|\gamma\|_C = \sup_{s \in J} |\gamma(s)|$ where $|\cdot|$ denotes the Euclidean norm on $\mathbf{R}^n$. Let E and F be separable real Banach spaces and $L(E, F)$ be the Banach space of all continuous linear maps $T : E \to F$ with the norm $\|T\| = \sup\{|T(x)|_F : x \in E, |x|_E \leq 1\}$.

Denote by $\mathcal{L}^2(\Omega, C(J, \mathbf{R}^n))$ the space of all $\mathcal{F}$-measurable stochastic processes $\theta : \Omega \to C(J, \mathbf{R}^n)$ such that the function $\omega \in \Omega \mapsto \|\theta(\omega)\|_C \in \mathbf{R}^n$ is of class $\mathcal{L}^2$. Then $\mathcal{L}^2(\Omega, C(J, \mathbf{R}^n))$ is complete when endowed with the semi-norm

$$\|\theta\|_{\mathcal{L}^2(\Omega, C)} = [\int_\Omega \|\theta(\omega)\|_C^2 \, dP(\omega)]^{\frac{1}{2}}.$$

One may also look at the space $C([0,a], \mathcal{L}^2(\Omega, C(J, \mathbf{R}^n)))$ of all $\mathcal{L}^2$-continuous $C(J, \mathbf{R}^n)$-valued processes $y : [0,a] \to \mathcal{L}^2(\Omega, C(J, \mathbf{R}^n))$; again this is complete under the semi-norm

$$\|y\|_{C([0,a], \mathcal{L}^2(\Omega, C))} = \sup_{t \in [0,a]} \|y(t)\|_{\mathcal{L}^2(\Omega, C)}.$$

Denote by $C_A([0,a], \mathcal{L}^2(\Omega, C(J, \mathbf{R}^n)))$ the set of all $(\mathcal{F}_t)$-adapted $C([0,a], \mathcal{L}^2(\Omega, C(J, \mathbf{R}^n)))$-valued processes. For a given initial process $x(0)$, $y(0) \in \mathcal{L}^2(\Omega, C(J, \mathbf{R}^n))$ we may seek solutions x, $y \in \mathcal{L}^2(\Omega, C([-r, a], \mathbf{R}^n))$ of the stochastic functional differential equations

$$x(\omega)(t) = x(\omega)(0) + (\omega) \int_0^t g_1(u, x_u) \, dz_1(\cdot)(u),$$

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$$y'(\omega)(t) = y'(\omega)(0) + (\omega) \int_0^t g_2(u, y_u, y'_u)\, dz_2(\cdot)(u)$$

under the following conditions.

Condition (A). The noise processes z_1, $z_2 : \Omega \to C([0,a], \mathbf{R}^m)$ are expressible in the forms

$$z_i(\omega)(t) = \lambda_i(t) + z^i_m(t) \ (i = 1, 2)$$

where $\lambda_i : [0,a] \to \mathbf{R}^m$ are Lipschitz functions and $z^i_m : \Omega \to C([0,a], \mathbf{R}^m)$ are separable martingales adapted to $(\mathcal{F}_t)_{t\in[0,a]}$ and is such that there are constants $K_i > 0 (i = 1, 2)$ with

$$|E(z^i_m(\cdot)(t_2) - z^i_m(\cdot)(t_1)|\mathcal{F}_{t_1})| \le K_i(t_2 - t_1),$$

$$E(|z^i_m(\cdot)(t_2) - z^i_m(\cdot)(t_1)|^2|\mathcal{F}_{t_1}) \le K_i(t_2 - t_1)$$

a.s. whenever t_1, $t_2 \in [0,a]$ and $t_1 \le t_2$.

Condition (B). The coefficient processes g_1, g_2 are continuous and also uniformly Lipschitz in the sense that there exist constants L_1, $L_2 > 0$ with

$$\|g_1(t, \xi_2) - g_1(t, \xi_1)\|_{\mathcal{L}^2} \le L_1 \|\xi_2 - \xi_1\|_{\mathcal{L}^2},$$

$$\|g_2(t, \xi_2, \xi'_2) - g_2(t, \xi_1, \xi'_1)\|_{\mathcal{L}^2} \le L_2 \|\xi_2 - \xi_1\|_{\mathcal{L}^2}$$

for all $t \in [0,a]$ and all ξ_1, $\xi_2 \in \mathcal{L}^2(\Omega, C(J, \mathbf{R}^n))$.

Condition (C). For each process $\theta \in C_A([0,a], \mathcal{L}^2(\Omega, C(J, \mathbf{R}^n))$ the process

$$t \in [0,a] \mapsto g_1(t, \theta(t)) \in \mathcal{L}^2(\Omega, L(\mathbf{R}^m, \mathbf{R}^n)),$$

$$t \in [0,a] \mapsto g_2(t, \theta(t), \theta'(t)) \in \mathcal{L}^2(\Omega, L(\mathbf{R}^m, \mathbf{R}^n))$$

is also adapted to $(\mathcal{F}_t)_{t\in[0,a]}$.

In [2], it is well known that if Ito processes x and y are well defined, then x and y have same transition functions under the Wasserstein metrics when time runs infinite. In this note, we prove the linear property of process y. And as the application of [2], we derive the inequality on Ito processes x and y under the $\mathcal{L}^2$-norm, similar to Lipschitz condition.

2. The Main Results

We begin with:

LEMMA 1. *Suppose that $z_i(\omega)(t)\,(i=1,2)$ are processes satisfying Condition* (A). *Let $g:[0,a]\to\mathcal{L}^2(\Omega,L(\mathbf{R}^m,\mathbf{R}^n))$ be an $\mathcal{L}^2$-continuous process adapted to $(\mathcal{F}_t)_{t\in[0,a]}$. Then there are constants $M_i>0\,(i=1,2)$ such that*

$$E(\sup_{t\in[o,a]}|\int_0^t g(u)\,dz_i(\cdot)(u)|^2)\le M_i\int_0^t E(\|g(u)\|^2)\,du.$$

Proof. Write, by the Condition (A)

$$\int_0^t g(u)\,dz(\cdot)(u)=\int_0^t g(u)\,d\lambda_i(u)+\int_0^t g(u)\,dz_m^i(\cdot)(u).$$

The first integral on the right-hand side is a Riemann-Stieltjes integral for a.s.ω. Since z_m^i are martingale with a.s. sample path continuous, so is the second integral on the right-hand side of previous integral representation. We therefore have, by the martingale inequality ([4], [5])

$$\begin{aligned}E(\sup_{t\in[0,a]}|\int_0^t g(u)\,dz_m^i(\cdot)(u)|^2)&\le 4E(|\int_0^a g(u)\,dz_m^i(\cdot)(u)|^2)\\&\le 4C_i\int_0^a E(\|g(u)(\cdot)\|^2)\,du\end{aligned}$$

where $C_i=2K_ia^{1/2}+K_i^{1/2}$.

If $r_i>0\,(i=1,2)$ is the Lipschitz constant for λ_i, then since

$$|\int_0^t g(u)\,d\lambda_i(u)|^2\le r_i^2a\int_0^t|g(u)|^2\,du$$

we have

$$E(\sup_{t\in[0,a]}|\int_0^t g(u)(\cdot)\,d\lambda_i(u)|^2)\le r_i^2a\int_0^a E(|g(u)(\cdot)|^2)\,du.$$

Therefore the result follows from the fact that

$$E(\sup_{t\in[0,a]} |\int_0^t g(u)\,dz_i(\cdot)(u)|^2) \le 2E(\sup_{t\in[0,a]} |\int_0^t g(u)(\cdot)\,d\lambda_i(u)|^2) + 2E(\sup_{t\in[0,a]} |\int_0^t g(u)\,dz_m^i(\cdot)(u)|^2).$$

We recall ([3], [4], [5]) that for any process $x(0)\in\mathcal{L}^2(\Omega,C(J,\mathbf{R}^n);\mathcal{F}_{t_1})$, there exists only solution $x\in\mathcal{L}^2(\Omega,C([t_1-r,t_1],\mathbf{R}^n))$ such that

$$x(\omega)(t) = x(\omega)(0) + (\omega)\int_{t_1}^t g_1(u,x_u)\,dz_1(\cdot)(u).$$

Take $y_1 : [-r,a]\times\Omega \to \mathbf{R}^n$ to be

$$y_1'(t,\omega) = y'(\omega)(0)\ a.s.$$

and

$$y_{m+1}'(t,\omega) = y'(\omega)(0) + (\omega)\int_{t_1}^t g_2(u, y_{m\,u}, y'_{m\,u})\,dz_2(\cdot)(u)\ a.s.$$

Using the inductions, simple calculations show that $\{y_m'\}_{m=1}^\infty$ converges to some $y'\in\mathcal{L}^2(\Omega,C([t_1-r,t_1],\mathbf{R}^n))$ and y' must satisfy the stochastic functional differential equation

$$y'(\omega)(t) = y'(\omega)(0) + (\omega)\int_{t_1}^t g_2(u,y_u,y_u')\,dz_2(\cdot)(u).$$

They give a family of maps

$$T_t^{t_1} : \mathcal{L}^2(\Omega,C(J,\mathbf{R}^n):\mathcal{F}_{t_1}) \to \mathcal{L}^2(\Omega,C(J,\mathbf{R}^n):\mathcal{F}_t),$$
$$x(0)\mapsto x_t$$

and

$$S_t^{t_1} : \mathcal{L}^2(\Omega,C(J,\mathbf{R}^n):\mathcal{F}_{t_1}) \to \mathcal{L}^2(\Omega,C(J,\mathbf{R}^n):\mathcal{F}_t),$$
$$y(0)\mapsto y_t.$$

When $t_1=0$, we define T_t, S_t to be $T_t=T_t^0$, $S_t=S_t^0$.

We now meet:

THEOREM 2. *Suppose that there exist* $\eta_j \in C(J,\mathbf{R}^n)$, $j = 1,2,\ldots,k$ *and* $\{\Omega_j\}_{j=1}^k \subset \mathcal{F}_{t_1}$ *of* Ω *such that* $y(0) = \sum_{j=1}^k \eta_j I_{\Omega_j}$, *where* I_{Ω_j} *is the indicator function of* Ω_j. *Then*

$$S_t^{t_1}(y(0))(\omega) = \sum_{j=1}^k S_t^{t_1}(\eta_j)(\omega) I_{\Omega_j}(\omega).$$

Proof. Let $1 \le j \le k$. Solving the stochastic functional differential equation at $\eta_j \in C(J,\mathbf{R}^n)$, we get a solution y_{η_j} satisfying

$$y'_{\eta_j}(t) = \eta'_j(0) + \int_{t_1}^t g_2(u, y_{\eta_j u}, y'_{\eta_j u})\, dz_2(\cdot)(u).$$

Since I_{Ω_j} is $\mathcal{F}_{t_1}$-measurable, then the process $g_2(u, y_{\eta_j u}, y'_{\eta_j u}) I_{\Omega_j}$ is adapted to $(\mathcal{F}_t)_{t \ge t_1}$. Therefore since

$$y'_{\eta_j}(t) I_{\Omega_j} = \eta'_j(0) I_{\Omega_j} + \int_{t_1}^t g_2(u, y_{\eta_j u}, y'_{\eta_j u}) I_{\Omega_j}\, dz_2(\cdot)(u),$$

it follows that by the property of stochastic integral ([3], [5]),

$$\begin{aligned}\sum_{j=1}^k y'_{\eta_j}(t) I_{\Omega_j} &= \sum_{j=1}^k \eta'_j(0) I_{\Omega_j} + \sum_{j=1}^k \int_{t_1}^t g_2(u, y_{\eta_j u}, y'_{\eta_j u}) I_{\Omega_j}\, dz_2(\cdot)(u) \\ &= y'(0) + \int_{t_1}^t g_2(u, \sum_{j=1}^k y_{\eta_j u} I_{\Omega_j}, \sum_{j=1}^k y'_{\eta_j u} I_{\Omega_j})\, dz_2(\cdot)(u).\end{aligned}$$

Therefore by uniqueness of solutions to stochastic functional differential equation, we obtain

$$y(t) = \sum_{j=1}^k y_{\eta_j}(t) I_{\Omega_j}.$$

This implies that

$$S_t^{t_1}(y(0)) = y_t = \sum_{j=1}^k y_{\eta_j t} I_{\Omega_j} = \sum_{j=1}^k S_t^{t_1}(\eta_j) I_{\Omega_j}.$$

We are now ready to prove the main theorem. We recall ([1]) that coefficient process g_2 is a function of two variables only under mild condition.

Condition (D). Neither g_1 or g_2 is negative and there exists constant $L > 0$ such that

$$\|g_1(t,\xi_1) - g_2(t,\xi_2)\|_{\mathcal{L}^2} \le L\|\xi_1 - \xi_2\|_{\mathcal{L}^2}. \tag{1}$$

whenever $(t,\xi_1) \in \mathcal{L}^2(\Omega, C(J,\mathbf{R}^n))$ and $(t,\xi_2) \in \mathcal{L}^2(\Omega, C(J,\mathbf{R}^{\frac{n}{2}}))$.

We conclude with:

THEOREM 3. *Let $y'(0) \ge 0$ and $z_2(t) \ge z_1(t)$ for all t. If g_1 and g_2 are processes satisfying condition (D), then there exists a value $s \in [0,t]$ such that*

$$\|T^1_{2+s}(x(0)) - S^1_{2+s}(y(0))\|_{\mathcal{L}^2} \le \sqrt{2}\|x(0) - y(0)\|_{\mathcal{L}^2} e^{M_i L^2(2+s)} \tag{2}$$

where M_i is the constant of Lemma.

Proof. For the y-process, we have

$$\begin{aligned} y_t - y(0) &= (1+s)y'(0) + \int_1^{2+s}\int_s^{1+s} g_2(t, y_t, y'_t)\,dv\,dz_2(\cdot)(t-1) \\ &= (1+s)y'(0) + \int_1^{2+s} g_2(t, y_t, y'_t)\,dz_2(\cdot)(t-1). \end{aligned}$$

Writing $z_3(t) = z_2(t-1)$, since

$$\begin{aligned} &\|T^1_{2+s}(x(0)) - S^1_{2+s}(y(0))\|^2_{\mathcal{L}^2} \\ &= \int_\Omega \sup_{v\in J} |T^1_{2+s}(x(0)(\omega)(v) - S^1_{2+s}(y(0))(\omega)(v)|^2 dP(\omega), \end{aligned}$$

we have

$$\begin{aligned} &\|T^1_{2+s}(x(0)) - S^1_{2+s}(y(0))\|^2_{\mathcal{L}^2} \\ &\le 2\int_\Omega \sup_{v\in J} |x(0)(\omega)(v) - y(0)(\omega)(v) - (1+s)y'(0)(\omega)(v)|^2 dP(\omega) \\ &\quad + 2\int_\Omega \sup_{v\in[-2-s,0]} |(\omega)\int_1^{2+s+v} \{g_1(t, T^1_t(x(0)))\,dz_1(\cdot)(t) \\ &\quad - g_2(t, S^1_t(y(0)))\,dz_3(\cdot)(t)\}|^2 dP(\omega). \end{aligned}$$

Hence it follows from $y'(0) \geq 0$ that for $i = 1, 3$,

$$\begin{aligned}
&\|T^1_{2+s}(x(0)) - S^1_{2+s}(y(0))\|^2_{\mathcal{L}^2} \\
&\leq 2 \int_\Omega \sup_{v \in J} |x(0)(\omega)(v) - y(0)(\omega)(v)|^2 \, dP(\omega) \\
&\quad + 2 \int_\Omega \sup_{v\in[-2-s,0]} (\omega) \int_1^{2+s+v} |\{g_1(t, T^1_t(x(0)) \\
&\quad - g_2(t, S^1_t(y(0))\} \, dz_i(\cdot)(t)|^2 dP(\omega).
\end{aligned}$$

We therefore have, by Lemma 1 and Condition (D),

$$\begin{aligned}
&\|T^1_{2+s}(x(0)) - S^1_{2+s}(y(0))\|^2_{\mathcal{L}^2} \\
&\quad \leq 2\|x(0) - y(0)\|^2_{\mathcal{L}^2} + 2M_i L^2 \int_1^{2+s} \|T^1_t(x(0)) - S^1_t(y(0))\|^2_{\mathcal{L}^2} dt.
\end{aligned}$$

Using Gronwall's Lemma ([5]), the result follows.

REMARK. If we focus on the processes $x(t)$ and $y'(t)$, then we obtain the inequality similar to the Theorem, under the mild conditions: If g_1 and g_2 are processes satisfying the inequality (1) only, then the inequality (2) becomes to

$$\|x(t) - y'(t)\|_{\mathcal{L}^2} \leq \sqrt{2}\|x(0) - y'(0)\|_{\mathcal{L}^2} e^{M_i L^2 t}.$$

References

1. W. Choi, *Second order Ito processes based on the martingale problems*, Ph. D. Thesis, Sung Kyun Kwan Univ. 1991.
2. ———, *Relations between the Ito processes based on the Wasserstein function*, Comm. Korean Math. Soc. **4** (1993), 793–797.
3. N. Ikeda and S. Watanabe, *Stochastic differential equations and diffusion processes*, North-Holland, Amsterdam, 1989.
4. S. E. A. Mohammed, *Stochastic functional differential equations*, Pitman, Boston, 1984.
5. L. C. G. Rogers and David Williams, *Diffusions, Markov processes and martingales*, Wiley, New York, 1987.

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