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LIMITING PROPERTIES FOR A MARKOV PROCESS GENERATED BY NONDECREASING CONCAVE FUNCTIONS ON $\mathbb{R}_n^+$

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1. Introduction

Suppose $\{X_n\}$ is a Markov process taking values in some arbitrary space (S, φ) with n-step transition probability

$$P^{(n)}(x, B) = \text{Prob}(X_n \in B | X_0 = x), \quad x \in S, \ B \in \varphi.$$

We shall call a Markov process with transition probabilities $P^{(n)}(x, B)$ ϕ-irreducible for some non-trivial σ-finite measure ϕ on φ if whenever $\phi(B) > 0$,

$$\sum_{n=1}^{\infty} 2^{-n} P^{(n)}(x, B) > 0, \quad \text{for every } x \in S.$$

A non-trivial σ-finite measure π on φ is called invariant for $\{X_n\}$ if

$$\int P(x, B)\pi(dx) = \pi(B), \quad B \in \varphi.$$

If the unique invariant measure π is finite, then we shall call $\{X_n\}$ positive recurrent.

We call $\{X_n\}$ geometric ergodic if it is positive recurrent and there exists positive $\rho < 1$ such that $\|P^{(n)}(x, \cdot) - \pi(\cdot)\| = \mathcal{O}(\rho^n)$ $(n \to \infty)$ for π-a.s. $x \in S$, where $\|\cdot\|$ denotes the total variation norm and $\mathcal{O}$ stands for the usual "big $\mathcal{O}$".

When using a Markov process as a model, it is often of great importance to know whether the model is positive recurrent, or whether the

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model is geometrically ergodic, and characterize functions which holds functional central limit theorem. There are extensive literature on these subjects (see Bhattacharya and Lee (1988a, 1988b), Lee (1989, 1993, 1994), Tong (1990), Tweedie (1975, 1983) etc.).

Let $\{X_n\}$ be a ϕ-irreducible Markov process on (S, φ) with transition probabilities $P^{(n)}(x, \cdot)$. Call a set $B \in \varphi$ small if $\phi(B) > 0$ and for every $A \in \varphi$ with $\phi\ (A) > 0$, there exists j such that

$$\inf_{x \in B} \sum_{n=1}^{j} P^{(n)}(x, A) > 0.$$

For an irreducible, aperiodic Markov process $\{X_n\}$ with state space (S, φ), φ is countably generated, following theorem has proved by Nummelin and Touminen (1982).

THEOREM 1.1. *Assume that there exist a nonnegative measurable function h on S, a small set $B \in \varphi$, and real numbers $r > 1$, $\varepsilon > 0$ such that*

$$\int P(x, dy) h(y) \leq \frac{1}{r} h(x) - \varepsilon, \quad x \in B^c,$$

$$\sup_{x \in B} \int_{B^c} P(x, dy) h(y) < \infty.$$

Then $\{X_n\}$ is geometrically ergodic.

A function $f : \mathbb{R}_n^+ \to \mathbb{R}_n^+$ is called concave if for $\alpha \in (0, 1)$,

$$f(\alpha x + (1 - \alpha) y) \geq \alpha f(x) + (1 - \alpha) f(y).$$

Function f is called nondecreasing if for $x < y$, $f(x) \leq f(y)$. Here we define $x \leq y$ for $x, y \in \mathbb{R}_n^+$ by $x^{(i)} \leq y^{(i)}$ for all $1 \leq i \leq n$, where $x^{(i)}$ is the i-th coordinate of x, and $x < y$ if $x \leq y$ but $x \neq y$. Let $\mathbb{R}_n^+ = \{x \in \mathbb{R}_n \,|\, x \geq 0\}$.

Throughout this paper we assume the followings:
A is an arbitrary index set. For a in A,

$$f_a = (f_a^{(1)}, f_a^{(2)}, \ldots, f_a^{(n)}) : \mathbb{R}_n^+ \to \mathbb{R}_n^+$$

is a nondecreasing concave function such that $f_a^{(i)}$ has continuous first partial derivatives for each $x^{(j)}$, $1 \le i \le n$, $1 \le j \le n$.

$\{\alpha_n\}$ denotes a sequence of independent, identically distributed random elements on some probability space $(\Omega, \mathcal{T}, P)$, taking values on A. The sets $\{\omega : f_{\alpha_{i(\omega)}}(y) \le x\}$ are in $\mathcal{T}$ for all $x, y \in \mathbb{R}_n^+$.

Take X_0 as an arbitrary random variable taking values on $\mathbb{R}_n^+$, but independent of the sequence α_n.

Let

$$X_n = f_{\alpha_n}(X_{n-1}), \quad n = 1, 2, 3, \ldots, \tag{1.1}$$

Denote $X_n(y)$ is X_n with $X_0 = y$, i.e., $X_n(y) = f_{\alpha_n}(f_{\alpha_{n-1}} \cdots (f_{\alpha_1}(y)))$.

In this paper, we consider the Markov process $\{X_n : n \ge 0\}$ which is generated by (1.1) and give some sufficient conditions for irreducibility, geometric ergodicity and characterize functions which holds functional central limit theorem.

2. Irreducibility

Let $g(y) = E[f_{\alpha_i}(y)]$.

Then it is easy to show that $g(y)$ is nondecreasing and concave function from $\mathbb{R}_n^+$ to $\mathbb{R}_n^+$.

We make the following assumptions on g:

(A_1) There exists $y_0 > 0$ such that

(1) $g(y_0) = y_0$ (2) for $y > y_0$, $g(y) < y$ (3) for $y < y_0$, $g(y) > y$.

Following lemma is the multidimensional extensions of Yahav's results.

LEMMA 2.1. *Under the assumption* (A_1), *we have*

(1) $P(X_n(0) \le x)$ *is nonincreasing in* n *and hence converges.*

(2) *For every* $y > y_0$, $X_n(y)$ *converges with probability 1 as* $n \to \infty$ *and*

$$\lim_{n \to \infty} E(X_n(y)) \le y_0. \tag{2.1}$$

Proof. Let $\tilde{X}_n(y) = f_{\alpha_1}(f_{\alpha_2}(\cdots f_{\alpha_n}(y)))$.

(1) Since $\tilde{X}_n(0)$ is nondecreasing in n and $\tilde{X}_n(0)$ and $X_n(0)$ have the same distribution, the conclusion follows.

(2) Let β_n be the σ-field generated by $\alpha_1, \alpha_2, \ldots, \alpha_n$. Then conditional Jensen's inequality shows that

$$E[\tilde{X}_{n+1}^{(i)}(y)|\beta_n] \le f_{\alpha_1}^{(i)}(f_{\alpha_2}(\cdots(Ef_{\alpha_{n+1}}(y))) \\ = f_{\alpha_1}^{(i)}(f_{\alpha_2}(\cdots(g(y))) \quad \text{a.s.} \tag{2.2}$$

Since $f_{\alpha_1}^{(i)} f_{\alpha_2} \cdots f_{\alpha_n}$ is nondecreasing, (A_1) ensures that for $y > y_0$,

$$E[\tilde{X}_{n+1}^{(i)}(y)|\beta_n] \le \tilde{X}_n^{(i)}(y) \quad \text{a.s.}$$

By supermartingale convergence theorem, $\tilde{X}_n^{(i)}(y)$ converges a.s., and hence $X_n(y)$ converges a.s., since $X_n(y)$ and $\tilde{X}_n(y)$ have the same distribution.

On the other hand, from (2.2), we have

$$E(\tilde{X}_{n+1}(y)) \le g(g(\cdots(g(y)))) = g^{(n-1)}(y) \to y_0 \text{ as } n \to \infty,$$

which follows from the fact that $g(\cdot)$ is nondecreasing concave function and (A_1).

LEMMA 2.2. *Assume* (A_1). *Then for* $y > y_0$, $P(X_n(y) \le y_0 + \delta) > 0$ *for some* n.

Proof. Suppose $P(X_n(y) \le y_0 + \delta) = 0$ for all n. Then

$$E(X_n(y)) = \int_{[X_n(y) > y_0 + \delta]} X_n(y) dP > y_0 + \delta, \quad \text{for every } n.$$

This contradicts to (2.1).

For the next lemma, we need the following additional assumption:
(A_2) $P(f_{\alpha_1}(y_0) > y_0) > 0$.

LEMMA 2.3. *Let the assumptions* (A_1), (A_2) *hold. Then every* $\delta > 0$ *sufficiently small,*

$$P(X_n(0) \ge y_0 + \delta \quad \text{for some } n) = 1. \tag{2.3}$$

Proof. The proof is essentially the same as that of lemma 3 in Lee (1989).

THEOREM 2.4. *Let the assumptions* (A_1), (A_2) *hold. Then* $F_0(x) = F_y(x)$ *for all* y, $0 \le y \le y_0 + \delta$, *where* δ *is chosen to satisfy* (2.3), *and* $F_y(x) = \lim_{n\to\infty} P(X_n(y) \le x)$.

Proof. Define $\tau = \inf\{n : X_n(0) \ge y_0 + \delta\}$. Then by (2.3), $P(\tau < \infty) = 1$. Now

$$
\begin{aligned}
(2.4)\qquad & P(X_n(0) \le x,\ \tau \le n) \\
&= \sum_{i=1}^{n} P(X_n(0) \le x | \tau = i) P(\tau = i) \\
&= \sum_{i=1}^{n} \{\int_{y \ge y_0+\delta} P(X_{n-i}(y) \le x) dQ_1(y|\tau = i)\} P(\tau = i) \\
&\le \sum_{i=1}^{n} P(X_{n-i}(y_0 + \delta) \le x) P(\tau = i),
\end{aligned}
$$

where $Q_1(y|\tau = i) = P(X_\tau(0) \le y,\ \tau = i)$. The existence of limits of the first and the last term of (2.4) ensures by lemma 2.1, and letting $n \to \infty$, we have,

$$F_0(x) \le F_{y_0+\delta}(x).$$

Therefore the conclusion follows from the relation

$$P(X_n(0) \le x) \ge P(X_n(y) \le x) \ge P(X_n(y_0 + \delta) \le x),$$

for any y, $0 \le y \le y_0 + \delta$.

Now let ψ be the corresponding measure with respect to the distribution function F_0 on $\mathbb{R}_n^+$, i.e., $\psi([0, x]) = F_0(x)$. Then ψ becomes a nontrivial finite measure.

THEOREM 2.5. *Under* (A_1), (A_2), $\{X_n\}$ *is* ψ*-irreducible.*

Proof. Let $\psi(A) > 0$ for some $A \in B(\mathbb{R}_n^+)$. Let $\delta > 0$ be chosen to satisfy (2.3). For $y \le y_0+\delta$, $F_y(x) = F_0(x)$ and hence $P(X_n(y) \in A) > 0$ for sufficiently large n. Now for $y > y_0 + \delta$,

$$
\begin{aligned}
&P(X_n(y) \in A) \\
&\quad \ge P(X_{n-n_1}(X_{n_1}(y)) \in A | X_{n_1}(y) \le y_0 + \delta) \cdot P(X_{n_1}(y) \le y_0 + \delta) \\
&\quad > 0,
\end{aligned}
$$

for sufficiently large n. Here n_1 is chosen to satisfy $P(X_{n_1}(y) \leq y_0+\delta) > 0$, which is possible by lemma 2.2.

Now assume y is neither $y \leq y_0 + \delta$ nor $y > y_0 + \delta$. Then choose y' such that $y' > y$ and $y' > y_0 + \delta$ and then choose n_2 such that $P(X_{n_2}(y') \leq y_0 + \delta) > 0$.

$$\begin{aligned}
&P(X_n(y) \in A) \\
&\quad \geq P(X_{n-n_2}(X_{n_2}(y)) \in A | X_{n_2}(y) \leq y_0 + \delta) \cdot P(X_{n_2}(y) \leq y_0 + \delta) \\
&\quad \geq P(X_{n-n_2}(y_0 + \delta) \in A) \cdot P(X_{n_2}(y') \leq y_0 + \delta) > 0,
\end{aligned}$$

for sufficiently large n.

3. Geometric ergodicity and central limit theorem

In this section we make the additional assumption on g:

(A$_3$) There exists $x_0 > 0$ such that $\sum_{i=1}^{n} \frac{\partial g^{(i)}}{\partial x^{(j)}}(x_0) < 1$, $\forall j$.

THEOREM 3.1. *If we assume* (A_1), (A_2), (A_3), *then* $\{X_n\}$ *which is generated by* (1.1) *is geometrically ergodic.*

Proof. Let us first define a nonnegative measurable function h on $\mathbb{R}_n^+$ as $h(y) = \sum_{i=1}^{n} y^{(i)}$. Then

$$\begin{aligned}
\int P(x, dy) h(y) &= \sum_{i=1}^{n} \int P(x, dy) y^{(i)} \\
&= \sum_{i=1}^{n} g^{(i)}(x).
\end{aligned}$$

Since $g^{(i)}(\cdot)$ is a nondecreasing concave function from $\mathbb{R}_n^+$ to R^+, for $x > 0$,

$$\begin{aligned}
g^{(i)}(x) &\leq \nabla g^{(i)}(x_0)(x - x_0) + g^{(i)}(x_0) \\
&= \sum_{j=1}^{n} g_{ij}(x_0) \cdot x^{(j)} + b
\end{aligned}$$

where $g_{ij}(x_0) = \dfrac{\partial g^{(i)}}{\partial x^{(j)}}(x_0)$ and $b = \nabla g^{(i)}(x_0) \cdot x_0 + g^{(i)}(x_0)$. Now,

$$\begin{aligned}\int P(x, dy)h(y) &\le \sum_{i=1}^{n} \Big[\sum_{j=1}^{n} g_{ij}(x_0) \cdot x^{(j)} + b \Big] \\ &= \theta \sum_{j=1}^{n} x^{(j)} + nb,\end{aligned}$$

where $0 < \theta = \sum_{i=1}^{n} g_{ij}(x_0) < 1$.

For given $\varepsilon > 0$, we may choose θ', $1 > \theta' > \theta$ which holds

$$\theta \sum_{j=1}^{n} x^{(j)} + nb < \theta' \sum_{j=1}^{n} x^{(j)} - \varepsilon, \tag{3.1}$$

for x whose $\sum_{j=1}^{n} x^{(j)}$ is sufficiently large.

It is known that every relatively compact set with positive invariant measure is a small set. Moreover we may choose k in B such that $\psi(B) > 0$ and (3.1) holds for $x \in B^c$, where $B = \{x \in \mathbb{R}_n^+ : \sum x^{(i)} \le k\}$. Now,

$$\sup_{x \in B} \int_{B^c} P(x, dy)h(y) \le \sup_{x \in B} \sum_{i=1}^{n} g^{(i)}(x) < \infty,$$

since $g^{(i)}(x)$ is continuous and B is a compact set in $\mathbb{R}_n^+$. Hence by theorem 1.1, $\{X_n\}$ is geometrically ergodic.

In order to state the functional central limit theorem, fix $u \in L^2(\mathbb{R}_n^+, \pi)$. For each positive integer n, write

$$Y_n(t) = n^{-1/2} \sum_{j=0}^{[nt]} (u(X_j) - \int u d\pi), \quad 0 \le t < \infty, \tag{3.2}$$

where $[nt]$ is the integer part of nt.

Let T be the transition operator on $L^2(\mathbb{R}_n^+, \pi)$ such that

$$(Tu)(x) = \int u(y)P(x, dy), \quad u \in L^2(\mathbb{R}_n^+, \pi).$$

We let $B(\mathbb{R}_n^+)$ be the linear space of all real-valued bounded measurable functions on $\mathbb{R}_n^+$.

THEOREM 3.2. *Let the assumptions* (A_1), (A_2), (A_3) *hold, and let* π *be the invariant probability measure. Whatever the initial distribution,* $Y_n(t)$ *in* (3.2) *converges in distribution to a Brownian motion for* $u \in B(\mathbb{R}_n^+)$.

Proof. Suppose $u \in B(\mathbb{R}_n^+)$ with for some $M > 0$, $u(x) \le M$, $\forall x \in \mathbb{R}_n^+$. Write

$$\overline{u} = \int u d\pi.$$

First assume that the distribution of X_0, initial distribution, is π. Then $\{X_n\}$ with $X_0 \sim \pi$ becomes a stationary ergodic Markov process. Moreover,

$$\begin{aligned} \|T^n(u - \overline{u})\|_2^2 &= \int \Big(\int u(y)p^n(x, dy) \\ &\qquad - \int u(y)\pi(dy)\Big)^2 \pi(dx) \\ &\le \int (M\|p^n(x,\cdot) - \pi(\cdot)\|)^2 \pi(dx) \\ &\le 2M^2 \int \pi(dx) \|p^n(x,\cdot) - \pi(\cdot)\| \\ &\le 2M^2 k\rho^n \end{aligned} \tag{3.3}$$

for some $0 < k < \infty$ and for some $0 < \rho < 1$. The last inequality in (3.3) follows from the fact that if $\{X_n\}$ is geometrically ergodic, then there exists $0 < \rho < 1$ such that

$$\int \pi(dx) \|P^n(x,\cdot) - \pi(\cdot)\| = \mathcal{O}(\rho^n)$$

as $n \to \infty$ (Nummelin and Touminen (1982)).

From (3.3), we have $\sum_{n=0}^{\infty} \|T^n(u-\overline{u})\|_2 < \infty$, which implies that $u-\overline{u}$ belongs to the range of $T-I$, I is an identity operator, and hence functional central limit theorem holds for $u \in B(\mathbb{R}_n^+)$ (Gordin and Lifsic (1978), Bhattacharya and Lee (1988a)).

Now we consider the process $\{X_n(x)\}$. Let $\{X_n\}$ denote the process with initial distribution π. Write

$$S_{mm'}(x) = n^{-1/2} \sum_{j=m}^{m'} (u(X_j(x)) - \overline{u}),$$

$$S_{mm'} = n^{-1/2} \sum_{j=m}^{m'} (u(X_j) - \overline{u}).$$

Then for all $r \in R$,

$$P(S_{n_0,n}(x) > r) = \int P(S_{0,n-n_0}(y) > r) P^{(n_0)}(x,dy), \text{ and}$$

$$\sup_{n>n_0} \left| \int P(S_{0,n-n_0}(y) > r) P^{(n_0)}(x,dy) - \int P(S_{0,n-n_0}(y) > r)\pi(dy) \right| \to 0$$

as $n \to \infty$ geometric ergodicity.

Also, $S_{0,n}(x) = S_{0,n_0-1}(x) + S_{n_0,n}(x)$ and for every n_0,

$$S_{0,n_0-1}(x) \to 0 \quad \text{a.s.} \quad n \to \infty.$$

Therefore, given $\varepsilon > 0$, one may choose $n(\varepsilon)$ such that

$$|P(S_{0,n}(x) > r) - P(S_{0,n} > r)| < \varepsilon, \quad \forall n \geq n(\varepsilon).$$

Hence the distribution of $S_{0,n}(x)$ converges in the weak-star topology to the appropriate Gaussian law, and hence the convergence of the finite dimensional distribution of $Y_n(t)$ to those of a Brownian motion when the initial distribution is x follows.

Now modifying the process used in Bhattacharya and Lee (1988a), one may show that the distribution of $Y_n(t)$, $n = 1, 2, \ldots$ form a precompact. Hence by Prohorov's theorem (Billingsley (1968)), Y_n with $X_0 \equiv x$, x is arbitrary, converges in distribution to a Brownian motion.

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