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ON TRIANGULAR STOCHASTIC MATRICES

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1. Introduction

Loewner ([13]) showed the semigroup of the real non-singular totally positive matrices is generated by its infinitesimal elements, that is, the set of all Jacobi matrices with non-negative off-diagonal elements. In general, a semigroup is not completely recreated from the set of its infinitesimal elements. We extend Loewner's work and show that the semigroup of all invertible upper (or lower) triangular stochastic matrices is generated by the set of its infinitesimal elements.

Let G be a Lie group and let S be a subsemigroup of G containing e. We assume S is locally closed, i.e., that there is a neighborhood U of e such that $S \cap U$ is closed in U. We define

$$L(S) = \{X \in L(G) : \exp(tX) \in S \text{ for all } t \geq 0\}.$$

Then $L(S)$ is the set of subtangent vectors of S at e, that is, $L(S)$ corresponds to the set of infinitesimal elements of the semigroup S. The set $L(S)$ is a cone or wedge, a closed convex set closed under addition and multiplication by positive scalars. The set $L(S) \cap -L(S)$ is the largest subspace contained in $L(S)$, is a Lie algebra and $L(S)$ is invariant under the adjoint action of the analytic subgroup corresponding to this subalgebra. Cones (or wedges) with this property are called *Lie wedges* (see [6]). Computing tangent cones of given semigroups may be considered as a generalization of taking derivative. We find tangent cone of the semigroup of triangular stochastic matrices and we present some basic techniques for calculating the tangent cones.

For each Lie wedge W, there is a local subsemigroup of G for which the given cone (or wedge) is the set of tangent vectors, but in general

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there need not exist a globally defined semigroup for which this is true (see [6]). We say that a semigroup S is *infinitesimally generated* if S is contained in the analytic group generated by $\exp(L(S))$ and the semigroup generated by $\exp(L(S))$ is dense in S. Computing the infinitesimally generated semigroups may be considered as a generalization of finding integrand. However, it is generally quite difficult to compute the infinitesimally generated semigroups. We find semigroups generated by the exponential image of the set of upper (or lower) triangular intensity matrices, which is the tangent cone of the semigroup of invertible upper (or lower) triangular stochastic matrices.

A number of problems (see [1]) in control theory lead to the question of characterizing the semigroup generated by $\exp(K)$ in terms of K when K is the cone in the Lie algebra of all n by n matrices. In this point of view, we prove that the set of reachable matrices obtained using the control variables of triangular intensity matrices coincides with the semigroup of triangular stochastic matrices by translating the terminology of Lie semigroup theory into that of control theory.

2. Basic theorems

Let G be a Lie group, let $L(G)$ be its Lie algebra, and let $\exp : L(G) \to G$ denote the exponential mapping. The mappings $t \longmapsto \exp(tX) : \mathbb{R} \to G$ for $X \in L(G)$ are 1-parameter subgroups (1-parameter subgroups are always assumed to be continuous), and in this way a one-to-one correspondence is established between $L(G)$ and the set of all 1-parameter subgroups of G. Here $\mathbb{R}$ denotes the set of all real numbers and hereafter we shall use this notation.

We consider a basic example. Let V denote a Banach space, $gl(V)$ the space of continuous linear transformations from V into V, and $GL(V)$ the space of all invertible continuous linear transformations from V into V.

For $G = GL(V)$ and $L(G) = gl(V)$, it is well known that the exponential map exp: $gl(V) \to GL(V)$ coincides with the usual matrix exponential, that is, $\exp(tX) = I + tX + \frac{1}{2!}(tX)^2 + \cdots$ for $X \in gl(V)$. If S is a closed subgroup of $GL(V)$, then its Lie algebra arises as the set $\{X \in gl(V) : \exp(tX) \in S \text{ for all } t \geq 0\}$. This motivates the following definition.

DEFINITION 2.1. For $S \subseteq GL(V)$, let $L(S) = \{X \in gl(V) : \exp(tX) \in S$ for all $t \geq 0\}$. $L(S)$ is called the *tangent set* of S.

The following Proposition is standard. For a proof, see [14].

PROPOSITION 2.2. *For $A,\ B \in gl(V)$, one has*

$$\exp(A+B) = \lim_{n\to\infty} \left(\exp\left(\frac{A}{n}\right)\exp\left(\frac{B}{n}\right)\right)^n.$$

Notation: Hereafter we shall denote the set of nonnegative real numbers with $\mathbb{R}^+$.

DEFINITION 2.3. A subset W of a real topological vector space V is called a *wedge* or a *cone* if it satisfies the following conditions:
(1) $W + W \subseteq W$, (2) $\mathbb{R}^+ W \subseteq W$, and (3) W is closed in V.

PROPOSITION 2.4. *If S is a closed semigroup of $GL(V)$, then $L(S)$ is a closed wedge.*

Proof. Let $A,\ B \in L(S)$. Then $\exp(tA) \in S$ and $\exp(tB) \in S$ for all $t \geq 0$. Thanks to the Proposition 2.2, $\exp(t(A+B)) = \lim_{n\to\infty} (\exp(tA/n) \exp(tB/n))^n$. Clearly, $\exp(tA/n)$, $\exp(tB/n) \in S$. Since S is a closed semigroup, $\exp(t(A+B)) \in S$ for all $t \geq 0$. That is, $A+B \in L(S)$. Also $rA \in L(S)$ for $r \geq 0$. Let B_n be a sequence in $L(S)$ which converges to B. Then $\exp(tB_n) \in S$ for all $t \geq 0$. Since the exponential map is continuous, $\exp(tB_n)$ converges to $\exp(tB)$. Therefore, $\exp(tB) \in \overline{S}$. Since S is closed, $\exp(tB) \in S$. That is, $B \in L(S)$. Thus $L(S)$ is a wedge.

PROPOSITION 2.5. *For $S_1,\ S_2 \subseteq GL(V)$, $L(S_1 \cap S_2) = L(S_1) \cap L(S_2)$.*

Proof. Straightforward.

THEOREM 2.6. *Let W be a (closed) wedge in $gl(V)$ which is also a multiplicative subsemigroup. Then $S = (I+W) \cap GL(V)$ is a subsemigroup, and $L(S) = W$.*

Proof. If $X,\ Y \in W$, then $(I+X)(I+Y) = I+X+Y+XY \in I+W$. So $I+W$ is a subsemigroup. If $X \in W$, then $\exp(tX) = I + (tX + \frac{1}{2!}t^2X^2 + \cdots) \in I+W$ for all $t \geq 0$. Thus $W \subseteq L(S)$. Conversely, if $\exp(tA) \in I+W$ for all $t \geq 0$, then $\frac{d}{dt}e^{tA}|_{t=0} = \lim_{t\to 0^+} \frac{e^{tA}-I}{t} \in W$ and $\frac{d}{dt}e^{tA}|_{t=0} = A\ e^{tA}|_{t=0} = A$. Hence $A \in W$. Thus $L(S) \subseteq W$.

DEFINITION 2.7. Suppose $L(G)$ is any Lie algebra over a field k. For any $X \in L(G)$, $ad\ X$ is defined as the endomorphism of $L(G)$ given by $(ad\ X)(Y) = [X, Y]$ where $Y \in L(G)$. And $ad : X \longmapsto ad\ X$ is called the *adjoint representation* of $L(G)$.

DEFINITION 2.8. A wedge W in a Lie algebra $L(G)$ is called a *Lie wedge* if it satisfies the condition,

$$\exp(ad\ X)W = W \quad \text{for all} \quad X \in W \cap -W.$$

DEFINITION 2.9. Let $i_g : G \to G$ be an automorphism defined by $i_g(x) = gxg^{-1}$ for each $g \in G$. Then Ad_g is an automorphism of $L(G)$ defined by $Ad_g = di_g$ for each $g \in G$. And $Ad : y \longmapsto Ad_y$ is called the *adjoint representation* of G.

THEOREM 2.10. *Let G_i be a Lie group with Lie algebra $L(G_i)$ $(i = 1, 2)$, and let $\pi : G_1 \to G_2$ be an analytic homomorphism. Then the following diagram is commutative.*

$$\begin{array}{ccc} G_1 & \xrightarrow{\pi} & G_2 \\ \uparrow{\scriptstyle \exp} & & \uparrow{\scriptstyle \exp} \\ L(G_1) & \xrightarrow{d\pi} & L(G_2). \end{array}$$

This theorem is standard. For a proof, see [15].

THEOREM 2.11. *Let G be a Lie group with Lie algebra $L(G)$. Then the differential of the adjoint representation of G is the adjoint representation of $L(G)$ and $Ad_{\exp X} = e^{ad\ X}$ for $X \in L(G)$.*

This theorem is standard. For a proof, see [14].

THEOREM 2.12. *If S is a closed semigroup of $GL(V)$, then $L(S)$ is a Lie wedge.*

Proof. Let $Y \in L(S)$ and $X \in L(S) \cap -L(S)$. Then $\exp(tY) \in S$ for all $t \geq 0$, $\exp(tX) \in S$ for all $t \geq 0$, and $\exp(-tX) \in S$ for all

$t \geq 0$. Let $i_{\exp X} : GL(V) \to GL(V)$ be an automorphism defined by $i_{\exp X}(Y) = (\exp X)Y(\exp -X)$. By Theorem (2.10), the following diagram is commutative.

$$\begin{array}{ccc} GL(V) & \xrightarrow{i_{\exp X}} & GL(V) \\ \uparrow \exp & & \uparrow \exp \\ gl(V) & \xrightarrow{di_{\exp X} = Ad_{\exp X}} & gl(V). \end{array}$$

That is, $\exp(Ad_{\exp X}(tY)) = i_{\exp X}(\exp(tY)) = \exp X \exp(tY) \exp(-X)$. According to the Theorem (2.11), $Ad_{\exp X}(tY) = e^{ad\, X}(tY) = te^{ad\, X}Y$. Therefore, $\exp(te^{ad\, X}Y) = \exp(Ad_{\exp X}(tY)) = \exp X \exp(tY) \exp(-X)$. Thus, $\exp(te^{ad\, X}Y) \in S$ for all $t \geq 0$, that is, $e^{ad\, X}Y \in L(S)$. We showed $e^{ad\, X}L(S) \subseteq L(S)$. The other direction of the proof is straightforward.

3. Tangent cone of triangular stochastic matrices and its infinitesimally generated semigroups

Given a closed semigroup S, computing $W = L(S)$ may be thought of the generalization of "taking the derivative". In the following the tangent cones of some kinds of semigroups are obtained using Proposition 2.4, Proposition 2.5, and Theorem 2.6.

In the following we consider matrices with entries from $\mathbb{R}$. In the case V is an n-dimensional vector space over $\mathbb{R}$ with a fixed basis, we identify $gl(V)$ with the set of all $n \times n$ matrices over $\mathbb{R}$.

DEFINITION 3.1. A matrix $A = \|a_{ij}\|$ $(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$ such that $a_{ij} \geq 0$ for $i \neq j$ and $\sum_{j=1}^{n} a_{ij} = 0$ for $i = 1, 2, \ldots, m$ is called an *intensity matrix*. An intensity matrix B is called an *extreme intensity matrix* if B has only one nonzero off-diagonal element which is equal to 1. An extreme intensity matrix $B = \|b_{ij}\|$ is denoted by E_{mk} $(m \neq k)$ if $b_{mm} = -1$ and $b_{mk} = 1$. E_{mk} is called an *extreme Jacobi intensity matrix* if $|m - k| = 1$.

LEMMA 3.2. *Let $V = \mathbb{R}^n$ and S be the semigroup of all invertible matrices with nonnegative entries. Then $L(S) = W$, where W is the set of all matrices which are nonnegative off the diagonal.*

Proof. Let E_{ij} be an extreme intensity matrix as denoted in Definition 3.1. Then $\exp(tE_{ij}) \in S$ for $t \geq 0$. Thus $E_{ij} \in L(S)$. Let E_k be the matrix whose elements are 0 except that the kth diagonal element is equal to 1. Then $\exp(tE_k) \in S$ and $\exp(t(-E_k)) \in S$. Since $L(S)$ is a wedge, $\sum_{1 \leq i \neq j \leq n} \alpha_{ij} E_{ij} + \sum_{k=1}^{n} \beta_k E_k - \sum_{k=1}^{n} \gamma_k E_k \in L(S)$ for α_{ij}, β_k, $\gamma_k \geq 0$. Therefore, $W \subseteq L(S)$.

Conversely suppose $X \in L(S)$. Then $X = \frac{d}{dt} e^{tX}|_{t=0} = \lim_{t \to 0^+} \frac{e^{tX} - I}{t} \in W$. Therefore, $L(S) \subseteq W$.

DEFINITION 3.3. A matrix $A = \|a_{ik}\|$ $(i = 1, 2, \ldots, m; k = 1, 2, \ldots, n)$ is called a *stochastic matrix* if $a_{ik} \geq 0$ and $\sum_{k=1}^{n} a_{ik} = 1$ for $i = 1, 2, \ldots, m$.

LEMMA 3.4. *The set of all $n \times n$ stochastic matrices forms a semigroup.*

Proof. Let $A = \|a_{ij}\|$ and $B = \|b_{kl}\|$ be $n \times n$ stochastic matrices. Suppose $AB = C$. Then $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$ for i=1, 2, ..., n and j=1, 2, ..., n. $\sum_{j=1}^{n} c_{ij} = \sum_{j=1}^{n} \sum_{k=1}^{n} a_{ik} b_{kj} = \sum_{k=1}^{n} \sum_{j=1}^{n} a_{ik} b_{kj} = \sum_{k=1}^{n} (a_{ik} \sum_{j=1}^{n} b_{kj}) = \sum_{k=1}^{n} a_{ik} = 1$ for i= 1, 2, ..., n. This completes the proof.

LEMMA 3.5. *Let $V = \mathbb{R}^n$ and S be the semigroup of all invertible matrices with each row sum equal to 1. Then $L(S) = W$, where W is the set of all matrices with each row sum equal to 0.*

Proof. Clearly, W is a wedge in $gl(V)$. Suppose $AB = C$. Then

$$\sum_{j=1}^{n} c_{ij} = \sum_{j=1}^{n} (\sum_{k=1}^{n} a_{ik} b_{kj}) = \sum_{k=1}^{n} \sum_{j=1}^{n} a_{ik} b_{kj} = \sum_{k=1}^{n} (a_{ik} \sum_{j=1}^{n} b_{kj}) = \sum_{k=1}^{n} a_{ik} 0 = 0.$$

Therefore, W is a multiplicative subsemigroup. By Theorem 2.6, $L((I+W) \cap GL(V)) = W$. Hence $L(S) = W$ from $(I+W) \cap GL(V) = S$.

THEOREM 3.6. *Let $V = \mathbb{R}^n$, S be the semigroup of all invertible stochastic matrices, W be the set of all intensity matrices. Then $L(S) = W$.*

Proof. Immediate from Proposition 2.5, Lemma 3.2, and Lemma 3.5.

LEMMA 3.7. *Let S be the semigroup of all invertible $n \times n$ upper triangular matrices and W be the set of all $n \times n$ upper triangular matrices. Then $L(S) = W$.*

Proof. Let $w \in W$. Then $\exp(tw) = I + tw + \frac{t^2}{2!}w^2 + \cdots$. Thus, $\exp(tw) \in S$ for all $t \geq 0$. Therefore, $W \subseteq L(S)$. Let $A \in L(S)$. Then $\exp(tA) \in S \subseteq W$ for all $t \geq 0$.

$$A = \frac{d}{dt}e^{tA}|_{t=0} = \lim_{t \to 0+} \frac{e^{tA} - I}{t} \in W.$$

Thus, $L(S) \subseteq W$.

THEOREM 3.8. *Let S be the semigroup of all $n \times n$ invertible upper triangular stochastic matrices and W be the set of all $n \times n$ upper triangular intensity matrices. Then $L(S) = W$.*

Proof. Immediate from Proposition 2.5, Theorem 3.6, and Lemma 3.7.

For each Lie wedge W, there is a local subsemigroup of Lie group G for which the given wedge is the set of subtangent vectors, but in general there need not exist a globally defined semigroup for which this is true (see [6]). This motivates the following definition:

DEFINITION 3.9. A semigroup S is said to be *infinitesimally generated* if S is contained in the analytic group generated by $\exp(L(S))$ and the semigroup generated by the latter set is dense in S.

Given $L(S)$, computing the infinitesimally generated semigroups may be considered as a generalization of finding integrand. Here we find semigroups infinitesimally generated by the exponential image of the set of upper (or lower) triangular intensity matrices which is the tangent cone of the semigroup of all upper (lower) triangular stochastic matrices.

LEMMA 3.10. *Let*

$$A = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & a_{pp} & a_{pp+1} & \dots & a_{pn} \\ 0 & 0 & \dots & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 1 \end{pmatrix}$$

be an $n \times n$ nonsingular stochastic matrix. Then A can be represented as $A = \exp(t_{p,p+1}E_{p,p+1})\exp(t_{p,p+2}E_{p,p+2})\cdots\exp(t_{pn}E_{pn})$, where E_{ij} is an extreme intensity matrix as denoted in Definition 3.1

Proof. Since A is stochastic, $a_{pp} + a_{p,p+1} + \cdots + a_{pn} = 1$. Since A is upper triangular and nonsingular, determinant of $A = a_{pp} > 0$. Let

$$x_{p+i} = \frac{a_{pp} + a_{p,p+i+1} + \cdots + a_{pn}}{a_{pp} + a_{p,p+i} + \cdots + a_{pn}}$$

for $i = 1, 2, \dots, n$. Then $0 < x_{p+i} \leq 1$ for $i = 1, 2, \dots, n$ since $a_{pp} > 0$. For $i = 1$, $x_{p+1} = a_{pp} + a_{p,p+2} + \cdots + a_{pn}$. Thus, $a_{p,p+1} = 1 - x_{p+1}$. Also

$$x_{p+2} = \frac{a_{pp} + a_{p,p+3} + \cdots + a_{pn}}{a_{pp} + a_{p,p+2} + \cdots + a_{pn}} = \frac{a_{pp} + a_{p,p+3} + \cdots + a_{pn}}{x_{p+1}}.$$

Thus, $a_{p,p+2} = x_{p+1} - x_{p+1}x_{p+2} = x_{p+1}(1 - x_{p+2})$. Inductively, $x_{p+1}x_{p+2}\cdots x_{p+k-1} = a_{pp} + a_{p,p+k} + \cdots + a_{pn}$ for $k = 2, \dots, n-p$ and $x_{p+1}x_{p+2}\dots x_{p+k-1}x_{p+k} = a_{pp} + a_{p,p+k+1} + \cdots + a_{pn}$. Therefore, $a_{p,p+k} = x_{p+1}\dots x_{p+k-1}(1 - x_{p+k})$ for $k = 2, \dots, n-p$. We have $1 = a_{pp} + a_{p,p+1} + a_{p,p+2}\cdots + a_{pn} = a_{pp} + (1 - x_{p+1}) + x_{p+1}(1 - x_{p+2}) + \cdots + x_{p+1}\dots x_{n-1}(1 - x_n) = a_{pp} + 1 - x_{p+1}\dots x_n$. Therefore, $a_{pp} = x_{p+1}x_{p+2}\dots x_n$. Let

$$A_{x_{p+j}} = \begin{pmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & x_{p+j} & \dots & 1 - x_{p+j} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{pmatrix}$$

for $j = 1, 2, \ldots, n-p$. Then $A = A_{x_{p+1}} A_{x_{p+2}} \cdots A_{x_{n-p}}$. Since $0 < x_{p+j} \leq 1$, $A_{x_{p+j}} = \exp(t_{p,p+j} E_{p,p+j})$ for some $t_{p,p+j} \geq 0$. Thus, $A = \exp(t_{p,p+1} E_{p,p+1}) \exp(t_{p,p+2} E_{p,p+2}) \cdots \exp(t_{pn} E_{pn})$.

THEOREM 3.11. *If B is an $n \times n$ nonsingular upper triangular stochastic matrix, then it can be represented as $B = C_{n-1} C_{n-2} \cdots C_1$, where $C_p = \exp(t_{p,p+1} E_{p,p+1}) \cdots \exp(t_{pn} E_{pn})$ for $t_{ij} \geq 0$.*

Proof. Let $B_1, \ldots, B_n$ be the rows of $B = (B_1, \ldots B_n)$ and I_j be the jth row of $n \times n$ identity matrix. Then $B = C_{n-1} C_{n-2} \cdots C_1$ where C_p is $n \times n$ matrix such that $C_p = (I_1,\ I_2, \ldots,\ I_{p-1},\ B_p,\ I_{p+1}, \ldots,\ I_n)$. According to the Lemma 3.10, $C_p = \exp(t_{p,p+1} E_{p,p+1}) \cdots \exp(t_{pn} E_{pn})$.

THEOREM 3.12. *If B is an $n \times n$ nonsingular lower triangular stochastic matrix, then it can be represented as $B = C_2 C_3 \cdots C_n$, where $C_p = \exp(s_{p1} E_{p1}) \exp(s_{p2} E_{p2}) \cdots \exp(s_{p,p-1} E_{p,p-1})$ for $p = 2, \ldots, n$ and $s_{ij} \geq 0$.*

Proof. The proof is analogous to that of Theorem 3.11.

THEOREM 3.13. *The semigroup S of $n \times n$ nonsingular upper (or lower) triangular stochastic matrices is infinitesimally generated by the exponential image of the cone W of upper (or lower) triangular intensity matrices.*

Proof. By Theorem 3.8, $\exp(W) \subseteq S$ and $L(S) = W$. By Theorem 3.11, $S \subseteq \{\exp\ W\}_{SG}$, where $\{\exp\ W\}_{SG}$ denotes the semigroup generated by $\exp\ W$. The proof for the case of the semigroup of lower triangular stochastic matrices is similar to the upper triangular case.

4. Control theoretic version of the main results

Consider a real Lie algebra L in the set of $n \times n$ matrices. Let K be a conical subset of L. Let $\{\exp\ K\}_{SG}$ indicate the smallest semigroup which contains $\exp K$. A number of problems (see [1]) in control lead to the question of characterizing $\{\exp\ K\}_{SG}$ in terms of K. The connection between a Lie algebra and its corresponding Lie group suggests analogous relationships between cones in the algebra and semigroups in the corresponding group. Typically, the relationship between a cone in the Lie algebra and the semigroup which the exponential maps it into is very difficult to describe. In this view, for the case of S={triangular stochastic matrices}, we found its tangent cone and represented S in terms of its tangent cone in section 3. These results can be restated in more concrete form of control theory as follows.

The cone $K = L(S)$, which is the set of upper triangular intensity matrices, corresponds to the control variables and S, which is the semigroup of invertible upper triangular stochastic matrices, corresponds to the semigroup of reachable matrices in a control system. With this translation Theorem 3.8 and Theorem 3.13 can be summarized as

THEOREM 4.1. *Suppose that an $n \times n$ matrix $X(t)$ evolves in time according to*

$$\dot{X}(t) = A(t)X(t)\ ;\ X(0) = I.$$

If $A(t)$ is in the cone of upper (or lower) triangular intensity matrices, then $X(t)$ will be a nonsingular upper (or lower) triangular stochastic matrices. Furthermore, the set consisting of $X(t)$ for all $t \geq 0$, that is, the set of reachable matrices, coincides with the semigroup of all invertible upper (or lower) triangular stochastic matrices.

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