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REPRESENTATION OF THE GENERALIZED FUNCTIONS OF GELFAND AND SHILOV

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§0. Introduction

I. M. Gelfand and G. E. Shilov [GS] introduced the Gelfand-Shilov spaces of type S, generalized type S and type W of test functions to investigate the uniqueness of the solutions of the Cauchy problems of partial differential equations. Using the heat kernel method Matsuzawa gave structure theorems for distributions, hyperfunctions and generalized functions in the dual space $(S_r^s)'$ of the Gelfand-Shilov space of type S in [M1, M2 and DM], respectively. Also, we gave structure theorems for ultradistributions, Fourier hyperfunctions in [CK, KCK], respectively.

In this paper we characterize the generalized functions in the dual space $(S_{M_p}^{N_p})'$ of the Gelfand-Shilov space of generalized type S as the initial values of the solutions of the heat equations by applying the heat kernel method of Matsuzawa[M] as follows:

If M_p and N_p satisfy (M.1) and (M.2), and if M_p^2 satisfies (M.0)$'$ and N_p^2 satisfies (M.0) and (M.3), then an element $u \in (S_{M_p}^{N_p})'$ defines a solution $U(x,t)$ of the heat equation $(\partial/\partial t - \Delta)U(x,t) = 0$ such that

$$|U(x,t)| \le C_\varepsilon \exp[M(\varepsilon|x|) + \overline{N}(\varepsilon/t)]$$

for any $\varepsilon > 0$, where $\overline{N}(x)$ is the associated function of $N_p^2/p!$, and $U(x,t) \to u$ as $t \to 0^+$ in the following sense:

$$\lim_{t\to 0^+} \int U(x,t)\varphi(x)\,dx = u(\varphi)$$

for $\varphi \in S_{M_p}^{N_p}$ (see the next section for the definitions).

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Conversely any C^∞-solution $U(x,t)$ of the heat equation satisfying the inequality $|U(x,t)| \le C_\varepsilon \exp[M(\varepsilon|x|) + \overline{N}(\varepsilon/t)]$ defines a unique element $u \in (S_{M_p}^{N_p})'$ such that $u_y(E(x-y,t)) = U(x,t)$ where $E(x,t)$ is the n-dimensional heat kernel.

We use the following multi-index notations: $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n}$, $\partial_j = \partial/\partial x_j$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ where $\mathbb{N}_0$ is the set of nonnegative integers.

1. Preliminaries

Let M_p, $p = 0,1,2,\ldots$, be a sequence of positive numbers such that $M_0 = 1$. We impose the following conditions on M_p:

(M.0) For any $A > 0$ there exists a constant $C > 0$ such that

$$p! \le CA^p M_p, \quad p = 0,1,2,\ldots.$$

(M.0)$'$ There exist constants C and A such that

$$p! \le CA^p M_p, \quad p = 0,1,2,\ldots.$$

(M.1) $M_p^2 \le M_{p-1}M_{p+1}, \quad p = 1,2,\ldots.$

(M.2) There are constants C and H such that

$$M_{p+q} \le CH^{p+q}M_pM_q, \quad p,q = 0,1,2,\ldots.$$

(M.3) (Strong non-quasianalyticity) There is a constant A such that

$$\sum_{q=p+1}^{\infty} M_{q-1}/M_q \le Ap\frac{M_p}{M_{p+1}}, \quad p = 0,1,2,\ldots.$$

For each defining sequence M_p we define for $t > 0$

$$M(t) = \sup_p \log \frac{t^p}{M_p},$$

$$\overline{M}(t) = \sup_p \log \frac{p!t^p}{M_p^2}.$$

We call $M(t)$ the associated function of M_p.

We are now in a position to introduce the Gelfand-Shilov spaces of generalized type S.

DEFINITION 1.1. Let M_p and N_p, $p = 0, 1, 2, \ldots$, be sequences of positive numbers. Then the spaces S_{M_p}, S^{N_p} and $S_{M_p}^{N_p}$ consist of all infinitely differentiable functions $\varphi(x)$ on $\mathbb{R}^n$ satisfying the inequalities;

$$(1.1)' \qquad S_{M_p} : \|x^\alpha \partial^\beta \varphi(x)\|_\infty \le C_{|\beta|} A^{|\alpha|} M_{|\alpha|},$$

$$(1.2)' \qquad S^{N_p} : \|x^\alpha \partial^\beta \varphi(x)\|_\infty \le C_{|\alpha|} B^{|\beta|} N_{|\beta|},$$

$$(1.3)' \qquad S_{M_p}^{N_p} : \|x^\alpha \partial^\beta \varphi(x)\|_\infty \le C A^{|\alpha|} B^{|\beta|} M_{|\alpha|} N_{|\beta|}$$

for some positive constants A and B.

REMARK 1.2. In particular, if $M_p = p!^r$ and $N_p = p!^s$ then we denote the spaces S_{M_p}, S^{N_p} and $S_{M_p}^{N_p}$ by S_r, S^s and S_r^s, respectively. We will call these spaces S_r, S^s and S_r^s the Gelfand–Shilov spaces of type S.

§2. Main Theorem

As a main theorem in this section, we characterize the generalized functions u in $(S_{M_p}^{N_p})'$ as the initial values of smooth solutions of the heat equation $(\partial/\partial t - \Delta)U = 0$ in $\mathbb{R}_+^{n+1} = \{(x,t) : x \in \mathbb{R}^n,\ t > 0\}$. Here, the topology of $S_{M_p}^{N_p}$ is given in a usual manner: The space $S_{M_p}^{N_p}$ is the inductive limit of the spaces

$$S_{M_p,A}^{N_p,B} = \{\varphi \in C^\infty(\mathbb{R}^n) : |x^\alpha \partial^\beta \varphi(x)| \le C A^{|\alpha|} B^{|\beta|} M_{|\alpha|} N_{|\beta|}\}$$

with the norm

$$\|\varphi\| = \sup_{\substack{x \in \mathbb{R}^n \\ \alpha,\beta \in \mathbb{N}_0^n}} \frac{|x^\alpha \partial^\beta \varphi(x)|}{A^{|\alpha|} B^{|\beta|} M_{|\alpha|} N_{|\beta|}}.$$

Also we denote by $(S_{M_p}^{N_p})'$ the strong dual of the space $S_{M_p}^{N_p}$.

The following theorems are obtained by the standard method, so we omit the proofs.

THEOREM 2.1. *Let $\{\varphi_j\}$ be a sequence in $S_{M_p}^{N_p}$. Then we have*

$$(2.1) \qquad \varphi_j \to 0 \quad \text{in } S_{M_p}^{N_p}$$

as $j \to \infty$ *if and only if*

$$\sup_{\substack{x\in\mathbb{R}^n \\ \beta\in\mathbb{N}_0^n}} \frac{|\partial^\beta \varphi_j(x)| \exp M(a|x|)}{B^{|\beta|} N_{|\beta|}} \to 0 \quad \text{as} \quad j \to \infty \tag{2.2}$$

for some $a, B > 0$, *where* M *is the associated function of* M_p.

THEOREM 2.2. *Let* $u \in (S_{M_p}^{N_p})'$. *Then for any positive numbers* a *and* B *there exists a positive constant* C *depending on* a *and* B *such that*

$$|u(\varphi)| \leq C \sup_{\substack{x\in\mathbb{R}^n \\ \beta\in\mathbb{N}_0^n}} \frac{|\partial^\beta \varphi(x)| \exp M(a|x|)}{B^{|\beta|} N_{|B|}}. \tag{2.3}$$

We now denote by $E(x,t)$ the n-dimensional heat kernel

$$E(x,t) = \begin{cases} (4\pi t)^{-n/2} \exp(-|x|^2/4t), & t > 0 \\ 0 \quad , & t \leq 0 \end{cases}$$

PROPOSITION 2.3 [M]. *Let* $E(x,t)$ *be the* n*-dimensional heat kernel. Then* $E(\cdot\,,t)$ *is an entire function of order 2 for* $t > 0$ *and we have the following properties on* E:

(i) $\int_{\mathbb{R}^n} E(x,t)\,dx = 1, \quad t > 0.$

(ii) *There are positive constants* C *and* a' *such that*

$$|\partial^\beta E(x,t)| \leq C^{|\beta|} t^{-(n+|\beta|)/2} \beta!^{1/2} \exp[-a'|x|^2/4t]$$

where a' *can be taken as close as desired to 1 and* $0 < a' < 1$.

THEOREM 2.4. *Let*

$$\varphi_t(x) = \int_{\mathbb{R}^n} E(x-y,t)\varphi(y)\,dy, \quad t > 0$$

for $\varphi \in S_{M_p}^{N_p}$ *where* M_p^2 *and* N_p *satisfy* (M.0)′ *and* (M.2), *respectively. Then we have*

$$\varphi_t(x) \in S_{M_p}^{N_p} \tag{2.4}$$

for sufficiently small $t > 0$ and

$$\varphi_t(x) \to \varphi(x) \quad \text{in } S_{M_p}^{N_p} \tag{2.5}$$

as $t \to 0^+$.

Proof. Let $\varphi \in S_{M_p}^{N_p}$. Then by Theorem 2.2 and Proposition 2.3, it suffices to show that for some positive constants a and B

$$\|\varphi\|_{a,B} = \sup_{\substack{x\in\mathbb{R}^n \\ \beta\in\mathbb{N}_0^n}} \frac{|\partial^\beta\varphi_t(x) - \partial^\beta\varphi(x)| \exp M(a|x|)}{B^{|\beta|}N_{|\beta|}} \to 0 \tag{2.6}$$

as $t \to 0^+$, where M is the associated function of M_p.

Let θ be a positive number such that $0 < \theta < 1$. Then

$$\begin{aligned} |\partial^\beta\varphi_t(x) - \partial^\beta\varphi(x)| &= \left|\int_{\mathbb{R}^n} E(x-y,t)(\partial^\beta\varphi(y) - \partial^\beta\varphi(x))\,dy\right| \\ &\le \int_{\mathbb{R}^n} E(y,t)|\nabla(\partial^\beta\varphi)(x-\theta y)|\,|y|\,dy. \end{aligned}$$

Let $M(x)$ be the associated function of M_p. Then we have

$$\begin{aligned} |\nabla(\partial^\beta\varphi)(x-\theta y)| &\le nCB^{|\beta|+1}N_{|\beta|+1}\exp[-M(a|x-\theta y|)] \\ &\le C_1B_1^{|\beta|}N_{|\beta|}\exp[-M(\frac{a}{2}|x|) + M(a|y|)] \end{aligned}$$

for some positive constants a, B and C_1.

Since M_p^2 satisfies (M.0)$'$ we have

$$\begin{aligned} \|\varphi_t - \varphi\|_{a,B} &\le C_1\int_{\mathbb{R}^n} E(y,t)\exp M(a|y|)|y|\,dy \\ &\le C_1\int_{\mathbb{R}^n} E(y,t)\exp(a_1|y|^2)|y|\,dy \\ &= C_1\int_{|y|\le\delta} E(y,t)\exp(a_1|y|^2)|y|\,dy \\ &\quad + C_1\int_{|y|\ge\delta} E(y,t)\exp(a_1|y|^2)|y|\,dy \\ &\equiv C_{1,t} + C_{2,t} \end{aligned}$$

for some positive constant a_1.

For any $\varepsilon > 0$, taking δ so small that $C_1 \delta \exp(a_1 \delta^2) < \varepsilon$ we have $C_{1,t} \leq \varepsilon$.

Also, for any small t such that $0 < t < t_0(\varepsilon)$ we have

$$\begin{aligned} C_{2,t} &\leq C_1 (4\pi t)^{n/2} \exp(-\delta^2/8t) \int_{|y| \geq \delta} E(y/2, t)\, dy \\ &\leq C_2 (4\pi t)^{-n/2} \exp(-\delta^2/8t) \end{aligned}$$

which tends to zero as $t \to 0^+$. Therefore we have

$$\varphi_t \to \varphi \text{ in } S_{M_p}^{N_p}$$

as $t \to 0^+$.

We are now in a position to state and prove the main theorem in this paper.

THEOREM 2.5. *Suppose that M_p and N_p satisfy* (M.1), (M.2), *and that M_p^2 satisfies* (M.0)$'$ *and N_p^2 satisfies* (M.0) *and* (M.3). *Let $u \in (S_{M_p}^{N_p})'$. Then the function $U(x,t) = u_y(E(x-y,t))$ is well defined in $\mathbb{R}_+^{n+1}$ and satisfies the following conditions:*

$$(\partial/\partial t - \Delta) U(x,t) = 0 \quad in \quad \mathbb{R}^{n+1}; \tag{2.7}$$

for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|U(x,t)| \leq C_\varepsilon t^{-n/2} \exp[M(\varepsilon|x|) + \overline{N}(\varepsilon/t)] \tag{2.8}$$

where $M(x)$ is the associated function of M_p and $\overline{N}(x)$ is the associated function of $N_p^2/p!$.

Also, $U(x,t) \to u$ as $t \to 0^+$ in the following sense

$$u(\varphi) = \lim_{t \to 0+} \int U(x,t) \varphi(x)\, dx, \quad \varphi \in S_{M_p}^{N_p}. \tag{2.9}$$

Conversely, every C^∞-function $U(x,t)$ defined in $\mathbb{R}_+^{n+1}$ satisfying the condition (2.7) *and* (2.8) *can be expressed in the form*

$$U(x,t) = u_y(E(x-y,t)), \tag{2.10}$$

with a unique element $u \in (S_{M_p}^{N_p})'$.

Proof. Let $u \in (S_{M_p}^{N_p})'$. Then we have obviously $U(x,t) \in C^\infty(\mathbb{R}^{n+1})$ and $U(x,t)$ satisfies the condition (2.7). Also $u \in (S_{M_p}^{N_p})'$ means that for any $a, B > 0$ there exists a positive constant C such that

$$|U(x,t)| \le C \sup_{\substack{y\in\mathbb{R}^n \\ \beta\in\mathbb{N}_0^n}} \frac{|\partial_y^\beta E(x-y,t)| \exp M(a|x|)}{B^{|\beta|} N_{|\beta|}}$$

where $M(x)$ is the associated function of M_p.

By Proposition 2.3.(ii) we have, for $0 < t < T$

(2.11)

$$\begin{aligned} &|U(x,t)| \\ &\le C \sup_{\substack{y\in\mathbb{R}^n \\ \beta\in\mathbb{N}_0^n}} \left[\frac{\beta! t^{-|\beta|} C^{2|\beta|}/B^{2|\beta|}}{N_{|\beta|}^2}\right]^{1/2} \sup_{y\in\mathbb{R}^n} \exp[M(a|x|) - \frac{|x-y|^2}{8t}] \\ &\le C \exp \overline{N}(\frac{C_1^2}{B^2 t}) \sup_{y\in\mathbb{R}^n} \exp[M(a|x|) - \frac{|x-y|^2}{8t}]. \end{aligned}$$

By the condition $M_p \supset p!^{1/2}$ we have

$$|U(x,t)| \le C \exp[\overline{N}(C_1^2/B^2 t) + M(2a|x|)].$$

For every $\varepsilon > 0$, taking a and B so that $a < \varepsilon/2$ and $B \ge c_1/\sqrt{\varepsilon}$ we have the estimate of the form (2.8). Now by Theorem 2.4 we can easily see that

$$\varphi_t(x) \to \varphi(x) \quad \text{in} \quad S_{M_p}^{N_p} \tag{2.12}$$

as $t \to 0^+$ and

$$\int U(x,t)\varphi(x)\,dx = u_y(\varphi_t(y)) \tag{2.13}$$

by taking the limit of the Riemann sum of left-hand side. Thus it follows from (2.12) and (2.13) that

$$U(x,t) \to u$$

as $t \to 0^+$ in the following sense:

$$\lim_{t\to 0+} \int U(x,t)\varphi(x)\,dx = u(\varphi), \quad \varphi \in S_{M_p}^{N_p}.$$

To prove the converse we need to following

LEMMA 2.6 [CK2]. *Let M_p be a defining sequence such that M_p satisfies (M.1), (M.2) and M_p^2 satisfies (M.0) and (M.3). Then there exist a differential operator $P(d/dt) = \sum_{k=0}^{\infty} a_k(d/dt)^k$ and a function $v(t) \in C^\infty(\mathbb{R})$ satisfying the following properties:*

(i) *For any $L > 0$ there is a constant $C = C(L)$ such that*

$$|a_k| \le CL^k/M_k^2, \quad k = 0, 1, 2, \ldots.$$

(ii) $\operatorname{supp} v(t) \subset [0,2]$.

(iii) $|v(t)| \le C\exp[-\overline{M}(L/t)]$, $t > 0$
for some $L, C > 0$, where $\overline{M}$ is the associated function of $M_p^2/p!$.

(iv) $P(d/dt)v(t) = \delta + w(t)$.

Here $w(t)$ belongs to $C^\infty(\mathbb{R})$ with $\operatorname{supp} w(t) \subset [1,2]$ *and δ is the Dirac measure.*

We now prove the converse.

Proof of the converse of Theorem 2.5. Let $U(x,t)$ be a C^∞-function in $\mathbb{R}_+^{n+1}$ satisfying (2.7) and (2.8) and let

$$\begin{aligned} \widetilde{U}(x,t) &= \int_0^\infty U(x,t+s)v(s)\,dx, \\ h(x,t) &= \int_0^\infty U(x,t+s)w(s)\,ds \end{aligned} \tag{2.14}$$

where v and w are the functions in Lemma 2.6. Then it is easy to see that $\widetilde{U}(x,t)$, $h(x,t) \in C^\infty(\mathbb{R}^{n+1}_+)$ and by Lemma 2.6 (ii) and (iii) we have

(2.15)
$$\widetilde{U}(x,t)| \leq C_\varepsilon \exp M(\varepsilon|x|) \int_0^2 \exp[\overline{N}(\frac{\varepsilon}{t+s}) - \overline{N}(\frac{L}{s})]\, ds$$
$$\leq C'_\varepsilon \exp M(\varepsilon|x|)$$

for all small $\varepsilon > 0$ where C_ε depends on ε.

Similarly we have

$$|h(x,t)| \leq C_\varepsilon \exp M(\varepsilon|x|) \tag{2.16}$$

where ε is arbitrary and C_ε depends on ε.

Moreover $\widetilde{U}(x,t)$ and $h(x,t)$ satisfy the heat equation in $\mathbb{R}^{n+1}_+$ and if we define

$$g_0(x) = \lim_{t\to 0} \widetilde{U}(x,t),$$
$$h_0(x) = \lim_{t\to 0} h(x,t) \quad \text{in} \quad \mathbb{R}^n.$$

Then by the uniqueness of the solutions of heat equation we have

$$\widetilde{U}(x,t) = \int E(x-y,t) g_0(y)\, dy \tag{2.17}$$

Operating $P(-d/dt)$ to both sides of (2.17) we have

$$U(x,t) + h(x,t) = P(-d/dt) \int E(x-y,t) g_0(y)\, dy$$
$$= \int E(x-y,t) P(-\Delta) g_0(y)\, dy.$$

From where we have formally

$$U \equiv U(x,0) = P(-\Delta) g_0(x) + h_0(x) \tag{1.18}$$

By the estimates

$$|g_0(x)|, |h_0(x)| \leq C_\varepsilon \exp M(\varepsilon|x|)$$

we can easily see that $P(-\Delta)g_0(x)$ and $h_0(x)$ belong to $(S^{N_p}_{M_p})'$. This completes the proof.

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