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# ANALYTIC OPERATOR-VALUED FUNCTION SPACE INTEGRAL REPRESENTED AS THE BOCHNER INTEGRAL: AN $\mathcal{L}(L_2)$ THEORY

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## 1. Notations and Lemmas

In [1], Cameron and Storvick introduced the analytic operator-valued function space integral. Johnson and Lapidus proved that this integral can be expressed in terms of an integral of operator-valued functions [6]. In this paper, we find some operator-valued Bochner integrable functions and prove that the analytic operator-valued function space integral of a certain function is represented as the Bochner integral of operator-valued functions on some conditions.

Throughout this paper, we adopt the following notations. Let $N$ be a fixed natural number and let $\mathbb{R}^N$ be the $N$-dimensional Euclidean space. Let $\mathbb{C}, \mathbb{C}_+$ and $\mathbb{C}_+^\sim$ be the set of all complex numbers, all complex numbers with positive real part and all non-zero complex numbers with non-negative real part, respectively. Let $C^N[a,b]$ be the space of all $\mathbb{R}^N$-valued continuous functions on $[a,b]$ and $C_0^N[a,b]$ will denote those $x$ in $C^N[a,b]$ such that $x(a)=0$. $C_0^N[a,b]$ will be referred to as "Wiener space" and integration over $C_0^N[a,b]$ will always be with respect to Wiener measure $m_w$. Let $\mathcal{L}(L_2(\mathbb{R}^N))$ be the space of all bounded linear operators from $L_2(\mathbb{R}^N)$ into itself.

DEFINITION 1.1. Let $F$ be a function from $C^N[a,b]$ into $\mathbb{C}$. Given $\lambda > 0$, $\psi$ in $L_2(\mathbb{R}^N)$ and $\xi$ in $\mathbb{R}^N$, let

$$(1.1) \qquad [I_\lambda(F)\psi](\xi) = \int_{C_0^N[a,b]} F(\lambda^{-\frac{1}{2}}x+\xi)\psi(\lambda^{-\frac{1}{2}}x(b)+\xi)\,dm_w(x).$$

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If $I_\lambda(F)\psi$ is in $L_2(\mathbb{R}^N)$ and if the correspondence $\psi \to I_\lambda(F)\psi$ gives an element of $\mathcal{L}(L_2(\mathbb{R}^N))$, we say that the operator-valued function space integral $I_\lambda(F)$ exists. Suppose that $I_\lambda(F)$exists for all $\lambda > 0$ and there exists an $\mathcal{L}(L_2(\mathbb{R}^N))$-valued function which is analytic in $\mathbb{C}_+$ and agrees with $I_\lambda(F)$ for $\lambda > 0$, then this $\mathcal{L}(L_2(\mathbb{R}^N))$-valued function is denoted by $I_\lambda^{an}(F)$ and is called the analytic operator-valued function space integral of $F$. For a non-zero real number $q$, suppose there is an operator $I_{-iq}^{an}(F)$ in $\mathcal{L}(L_2(\mathbb{R}^N))$ such that for every $\psi$ in $L_2(\mathbb{R}^N)$,

$$\|I_\lambda^{an}(F)\psi - I_{-iq}^{an}(F)\psi\|_2 \to 0 \quad \text{as} \quad \lambda \to -iq \tag{1.2}$$

through $\mathbb{C}_+$, then $I_{-iq}^{an}(F)$ is called the analytic operator-valued Feynman integral of $F$.

DEFINITION 1.2. For $\lambda$ in $\tilde{\mathbb{C}}_+$, $\psi$ in $L_2(\mathbb{R}^N)$, $\xi$ in $\mathbb{R}^N$ and a positive real number $s$, define $C_{\lambda/s}\psi$ by

$$[C_{\lambda/s}\psi](\xi) = \left(\frac{\lambda}{2\pi s}\right)^{\frac{N}{2}} \int_{\mathbb{R}^N} \psi(u)\, \exp\left(-\frac{\lambda\|u-\xi\|^2}{2s}\right) dm_l(u), \tag{1.3}$$

where $m_l$ is the Lebesgue measure on $\mathbb{R}^N$. When $N$ is odd, we always choose $\lambda^{1/2}$ with non-negative real part. When $\mathrm{Re}\lambda = 0$, the integral in (1.3) should be interpreted in the mean just as in the theory of the $L_2$-Fourier transform.

REMARK. $C_{\lambda/s}$ is a bounded linear operator from $L_2(\mathbb{R}^N)$ into itself and $\|C_{\lambda/s}\| = 1$. $C_{\lambda/s}$ is analytic in $\mathbb{C}_+$ and it is strongly continuous in $\tilde{\mathbb{C}}_+$ as a function of $\lambda$ [7].

We state the following lemma without proof. For the proof, see [3].

LEMMA 1.3. *Let $(Y, \mathcal{B}, \nu)$ be a finite measure space and let $(Z, \mathcal{C}, \tau)$ be a $\sigma$-finite measure space. And let $H$ be a $\mathbb{C}$-valued measurable function on $Y \times Z$ such that $H(y, \cdot)$ is in $L_p(Z, \tau)$ for $\nu - a.e.\, y, 1 \le p < +\infty$ and*

$$\int_Y \|H(y, \cdot)\|_p \, d\nu(y) < +\infty. \tag{1.4}$$

*Then $[H(y, \cdot)]$ is Bochner integrable with respect to $\nu$ and*

$$(B)-\int_Y [H(y, \cdot)]\, d\nu(y) = \left[\int_Y H(y, \cdot)\, d\nu(y)\right], \tag{1.5}$$

*where* $[H(y,\cdot)]$ *is the equivalence class of* $H(y,\cdot)$ *in the* $L_p(Z,\tau)$*-sense and the integral in the left hand side of* (1.5) *is Bochner integral.*

The following examples show that if $H$ is not measurable, then Lemma 1.3 does not hold.

EXAMPLE 1.4. Choose a subset $B$ of $[0,1]\times[0,1]$ with properties,

(i) for each $t_0$ in $[0,1]$, the set $\{s \mid (s,t_0)\in B\}$ is countable

(ii) for each $s_0$ in $[0,t]$, the set $\{t \mid (s_0,t)\notin B\}$ is countable.

Let $1\le p\le +\infty$ be given and let $f$ be a function from $[0,1]$ into $L_p([0,1],m_l)$ such that $f(t)=[\chi_B(\cdot,t)]$. Then $f$ is Bochner integrable on $[0,1]$, but

$$(1.6)\qquad (B)-\int_{[0,1]} f(t)\,dm_l(t)=0$$

and

$$(1.7)\qquad \Big(\int_{[0,1]}\chi_B(\cdot,t)\,dm_l(t)\Big)(s)=1\quad \text{for all } s \text{ in } [0,1].$$

Hence, if $H$ is not measurable, Lemma 1.3 does not hold.

EXAMPLE 1.5. Let $H$ be a function defined on $[0,1]\times[0,1]$ by $H(x,y)=\chi_{[0,x]}(y)$. Then $H$ is Lebesgue measurable on $[0,1]\times[0,1]$ and $[H(x,\cdot)]$ is in $L_\infty([0,1],m_l)$ for all $x$ in $[0,1]$. But $[H(x,\cdot)]$ is not strongly measurable. Hence Lemma 1.3 is not true for $p=+\infty$.

LEMMA 1.6. *Let* $(Y,\mathcal{B},\mu)$ *be a complex measure space and let* $\langle r_n\rangle$ *be a sequence of* $\mathbb{C}$*-valued measurable functions in* $L_1(Y,\mu)$ *such that*

$$(1.8)\qquad \sum_{n=1}^{\infty}\|r_n\|_1\equiv\sum_{n=1}^{\infty}\int_Y |r_n(s)|\,d|\mu|(s)<+\infty.$$

*Let* $g(s)=\langle r_n(s)\rangle$ *on* $Y$. *Then we have*

(a) $g$ *is a* $l^1$*-valued function on* $Y$ *which is well-defined,*

(b) $g$ *is Bochner integrable with respect to* $\mu$,

(c)

$$(B)-\int_Y g(s)\,d\mu(s)=\Big\langle\int_Y r_n(s)\,d\mu(s)\Big\rangle.$$

*Proof.* (a) Since $\int_Y \sum_{n=1}^{\infty} |r_n(s)|\, d|\mu|(s) = \sum_{n=1}^{\infty} \int_Y |r_n(s)|\, d|\mu|(s) < +\infty$, $\sum_{n=1}^{\infty} r_n(s)$ converges absolutely for $|\mu|$-a.e. $s$. Hence $g$ is well-defined.

(b) Let $z = \langle b_n \rangle$ be in $(l^1)^* = l^\infty$. Then $(z \circ g)(s) = \sum_{n=1}^{\infty} b_n\, r_n(s)$ is measurable. Since $l^1$ is a separable metric space, $g$ is strongly measurable. And since $\int_Y \|g(s)\|_{l^1}\, d|\mu|(s) = \sum_{n=1}^{\infty} \int_Y |r_n(s)|\, d|\mu|(s) < +\infty$, $g$ is Bochner integrable with respect to $\mu$.

(c) For $E$ in $\mathcal{B}$, we define $y_E$ by $y_E = \langle \int_E r_n(s)\, d\mu(s) \rangle$. Then for $z = \langle b_n \rangle$ in $l^\infty$, $z(y_E) = \sum_{k=1}^{\infty} b_k \int_E r_k(s)\, d\mu(s) = \int_E (z \circ g)(s)\, d\mu(s)$. Hence by [4], we have the result.

## 2. Main Result

In 1986, Johnson and Lapidus proved that the operator-valued function space integral on $L_2(\mathbb{R}^N)$ can be expressed in terms of the strong integral [6]. In this section, we will show that this integral can be represented as the Bochner integral on some conditions.

Let $\mu$ be a complex Borel measure on $[a, b]$ with $\mu\{a\} = \mu\{b\} = 0$ and let $\langle r_n \rangle$ be a sequence of $\mathbb{C}$-valued Borel measurable functions in $L_1([a, b], \mu)$ such that $\langle \|r_n\|_1 \rangle$ is in $l^1$. And let $\langle \theta_n \rangle$ be a bounded sequence in $L_\infty(\mathbb{R}^N, m_l)$. Let $\theta(s, \xi) = \sum_{n=1}^{\infty} r_n(s)\, \theta_n(\xi)$ for $(s, \xi)$ in $(a, b) \times \mathbb{R}^N$. Then $\theta$ is well-defined for $\mu \times m_l - a.e.\,(s, \xi)$. Let $\theta(s)$ be the multiplication operator from $L_\infty(\mathbb{R}^N)$ into itself defined by $[\theta(s)]\psi = M_{\theta(s,\cdot)}(\psi) \equiv \theta(s, \cdot)\psi(\cdot)$.

THEOREM 2.1. *$f(s) = \theta(s, \cdot)$ is Bochner integrable with respect to $\mu$.*

*Proof.* Let $T_{\langle \theta_n \rangle}$ be a $L_\infty(\mathbb{R}^N)$-valued function on $l^1$ given by $T_{\langle \theta_n \rangle}(\langle a_n \rangle) = \sum_{n=1}^{\infty} a_n\, \theta_n$. Then $T_{\langle \theta_n \rangle}$ is a bounded linear operator and $f = T_{\langle \theta_n \rangle} \circ g$ where $g$ is given in Lemma 1.6. Since $g$ is Bochner integrable, $f$ is also Bochner integrable.

THEOREM 2.2. *Let $\lambda$ be in $\mathbb{C}_+$. Then $f(s) \equiv C_{\lambda/s-a} \circ \theta(s) \circ C_{\lambda/(b-s)}$ is Bochner integrable with respect to $\mu$.*

*Proof.* Obviously, $f$ is uniformly measurable. And $\int_{(a,b)} \|f(s)\|\, d|\mu|(s) \leq (\sup \|\theta_n\|_\infty) \sum_{n=1}^{\infty} \|r_n\|_1 < +\infty$. Hence $f$ is Bochner integrable.

THEOREM 2.3. *Let* $\lambda$ *be in* $\mathbb{C}_+$. *Then the function* $f(s_1, s_2, \ldots, s_n; \lambda) \equiv C_{\lambda/(s_1-a)} \circ \theta(s_1) \circ C_{\lambda/(s_2-s_1)} \circ \theta(s_2) \circ \cdots \circ \theta(s_n) \circ C_{\lambda/(b-s_n)}$ *is Bochner integrable on* $\Delta_n = \{(s_1, \ldots, s_n) \mid a < s_1 < s_2 < \cdots < s_n < b\}$ *with respect to* $\prod_{i=1}^n \mu$. *Moreover,*

$$\begin{aligned}
&(B)-\int_{\Delta_n} f(s_1, \ldots, s_n; \lambda)\, d\prod_{i=1}^n \mu(s_i) \\
&= (B)-\int_{(a,b)} C_{\lambda/(s_1-a)} \circ \theta(s_1) \circ \Bigg((B)-\int_{(s_1,b)} C_{\lambda/(s_2-s_1)} \circ \theta(s_2) \circ \\
&\Bigg((B)-\int_{(s_2,b)} \cdots \Bigg((B)-\int_{(s_n,b)} C_{\lambda/(s_n-s_{n-1})} \circ \theta(s_n) \circ \\
&C_{\lambda/(b-s_n)}\, d\mu(s_n)\Bigg) d\mu(s_{n-1})\Bigg) \cdots d\mu(s_2)\Bigg)\, d\mu(s_1).
\end{aligned}$$

*Proof.* It can be proved by the same method as in the proof of Theorem 2.2.

By the dominated convergence theorem, Fubini's theorem and Morera's theorem, we have the following theorem.

THEOREM 2.4. *In Theorem 2.3,* $(B)-\int_{\Delta_n} f(s_1, \ldots, s_n; \lambda) d\prod_{i=1}^n \mu(s_i)$ *is analytic in* $\mathbb{C}_+$ *and strongly continuous in* $\mathbb{C}_+^\sim$ *as a function of* $\lambda$.

Let $\eta$ be a complex Borel measure on $(a, b)$ and $\mu+\nu$ be the decomposition of $\eta$ into continuous part $\mu$ and discrete part $\nu$ with $\nu = \sum_{p=1}^\infty w_p\, \delta_{\tau_p}$ where $\delta_{\tau_p}$ is the Dirac measure with total mass one concentrated at $\tau_p$. Let $f(z) = \sum_{n=1}^\infty a_n z^n$ be an analytic function whose radius of convergence is strictly greater than $\sum_{n=1}^\infty \|r_n\|_1 \|\theta_n\|_\infty$. For $y$ in $C^N[a, b]$, let $F(y) = f(\int_{(a,b)} \theta(s, y(s))\, d\eta(s))$. Then, by [6], the operator-valued function space integral $I_\lambda(F)$ exists on $\mathbb{C}_+^\sim$ and for $\psi$ in $L_2(\mathbb{R}^N)$,

$$\begin{aligned}
(2.1)\qquad & I_\lambda(F)\psi \\
&= \sum_{n=0}^\infty n!\, a_n \sum_{h=0}^\infty \sum_{q_0+\cdots+q_h=n,\, q_h\neq 0} \Bigg\{ \frac{w_1^{q_1} \cdots w_h^{q_h}}{q_1! \cdots q_h!} \sum_{j_1+\cdots+j_{h+1}}
\end{aligned}$$

$$\left[(S)-\int_{\Delta_{q_0;j_1,\dots,j_{h+1}}}(L_0\circ L_1\circ\cdots\circ L_h)\psi\, d\prod_{i=1}^{q_0}\mu(s_i)\right]\Bigg\},$$

where for each $h$, $\sigma$ is the permutation on $\{1,2,\dots,h\}$ such that

$$\tau_{\sigma(1)}<\tau_{\sigma(2)}<\cdots<\tau_{\sigma(h)},$$

and the integral in the right hand side of (2.1) is the strong integral,

$$\begin{aligned}&\Delta_{q_0;j_1,\dots,j_{h+1}}\\ &=\big\{(s_1,\dots,s_{q_0})\,|\,a<s_1<\cdots<s_{j_1}<\tau_{\sigma(1)}<s_{j_1+1}<\cdots\\ &\quad<s_{j_1+j_2}<\tau_{\sigma(2)}<\cdots<s_{j_1+\cdots+j_h}<\tau_{\sigma(h)}<\cdots\\ &\quad<s_{j_1+\cdots+j_{h+1}}=s_{q_0}<b\big\}\end{aligned}$$

and for $(s_1,\dots,s_{q_0})\in\Delta_{q_0;j_1,\dots,j_{h+1}}$, $r\in\{0,1,2,\dots,h\}$,

$$\begin{aligned}L_r&=\theta(\tau_{\sigma(r)})^{q_{\sigma(r)}}\circ C_{\lambda/(s_{j_1+\cdots+j_r+1}-\tau_{\sigma(r)})}\circ\theta(s_{j_1+\cdots+j_r+1})\\ &\quad\circ C_{\lambda/(s_{j_1+\cdots+j_r+2}-s_{j_1+\cdots+j_r+1})}\circ\cdots\\ &\quad\circ\theta(s_{j_1+\cdots+j_{r+1}})\circ C_{\lambda/(\tau_{\sigma(r+1)}-s_{j_1+\cdots+j_{r+1}})}.\end{aligned}$$

From the above lemmas and theorems, we have the following theorem.

THEOREM 2.5. *The perturbation series $I_\lambda(F)$ exists in $\mathbb{C}_+^{\sim}$ and for $\psi$ in $L_2(\mathbb{R}^N)$, $\xi$ in $\mathbb{R}^N$ and $\lambda$ in $\mathbb{C}_+^{\sim}$,*

$$\begin{aligned}&[I_\lambda(F)\psi](\xi)\\ &=\sum_{n=1}^{\infty}n!\,a_n\sum_{h=0}^{\infty}\sum_{q_0+\cdots+q_h=n,q_h\neq0}\Bigg\{\frac{w_1^{q_1}\cdots w_h^{q_h}}{q_1!\cdots q_h!}\\ &\qquad\sum_{j_1+\cdots+j_{h+1}=q_0}(L)-\int_{\Delta_{q_0;j_1,\dots,j_{h+1}}}[(L_0\circ L_1\circ\cdots\circ L_h)\psi](\xi)\,d\prod_{i=1}^{q_0}\mu(s_i)\Bigg\}\\ &=\sum_{n=0}^{\infty}n!\,a_n\sum_{h=0}^{\infty}\sum_{q_0+\cdots+q_h=n,q_h\neq0}\Bigg\{\frac{w_1^{q_1}\cdots w_h^{q_h}}{q_1!\cdots q_h!}\end{aligned}$$

$$\sum_{j_1+\cdots+j_{h+1}=q_0}\left[(B)-\int_{\Delta_{q_0;j_1,\ldots,j_{h+1}}}(L_0\circ\cdots\circ L_h)\psi\, d\prod_{i=1}^{q_0}\mu(s_i)\right]\Bigg\}(\xi)$$

$$=\sum_{n=1}^{\infty}n!\,a_n\sum_{h=0}^{\infty}\sum_{q_0+\cdots+q_h=n,q_h\neq 0}\left\{\frac{w_1^{q_1}\cdots w_h^{q_h}}{q_1!\cdots q_h!}\right.$$

$$\sum_{j_1+\cdots+j_{h+1}=q_0}\left[(S)-\int_{\Delta_{q_0;j_1,\ldots,j_{h+1}}}L_0\circ L_1\circ\cdots\circ L_h\, d\prod_{i=1}^{q_0}\mu(s_i)\psi\right]\Bigg\}(\xi)$$

$$=\sum_{n=1}^{\infty}n!\,a_n\sum_{h=0}^{\infty}\sum_{q_0+\cdots+q_h=n,q_h\neq 0}\left\{\frac{w_1^{q_1}\cdots w_h^{q_h}}{q_1!\cdots q_h!}\right.$$

$$\sum_{j_1+\cdots+j_{h+1}=q_0}\left[(B)-\int_{\Delta_{q_0;j_1,\ldots,j_{h+1}}}L_0\circ L_1\circ\cdots\circ L_h\, d\prod_{i=1}^{q_0}\mu(s_i)\psi\right]\Bigg\}(\xi),$$

*where the integral in the right hand side of the above first equality is the Lebesgue integral.*

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