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A TOPOLOGICAL PROOF OF THE PERRON-FROBENIUS THEOREM

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1. Fixed points and eigenvectors

In this article we prove a version of the Perron-Frobenius Theorem in linear algebra using the Brouwer's Fixed Point Theorem in topology. We will mostly concentrate on the qualitative aspect of the Perron-Frobenius Theorem rather than quantitative formulas, which would be enough for theoretical investigations in ergodic theory. By the nature of the method of the proof, we do not expect to obtain a numerical estimate. But we may regard it worthwhile to see why a certain type of result should be true from a topological and geometrical viewpoint. However, a geometric argument alone would give us a sharp numerical bounds on the size of the eigenvalue as shown in Section 2. Eigenvectors of a matrix A will be fixed points of a certain mapping defined in terms of A. We shall modify an existing proof of Frobenius Theorem and that will do the trick for Perron-Frobenius Theorem.

Let $A = (a_{ij})$ be an $n \times n$ matrix with positive coefficients a_{ij}. (We may merely assume that $a_{ij} \geq 0$ and none of the column vectors is a zero vector.) Then Frobenius Theorem states that there exist a positive eigenvalue λ and a corresponding eigenvector $\vec{v} = (v_i)$ with $v_i > 0$ for every i. One of its proofs is to use Brouwer's Fixed Point Theorem as follows: Let S be the subset of the unit sphere centered at the origin in $\mathbb{R}^n$ given by

$$S = \{\vec{v} : \|\vec{v}\| = 1, v_i \geq 0 \text{ for every } i\}$$

where $\|\vec{v}\| = \sqrt{\sum_i {v_i}^2}$ is the usual Euclidean norm. Define a continuous function $f : S \to S$ by

$$f(\vec{v}) = A\vec{v}/\|A\vec{v}\|.$$

Note that $\|A\vec{v}\| > 0$ on S. Then it has a fixed point $\vec{v}_0$ since S is homeomorphic to the closed unit ball in $\mathbb{R}^{n-1}$. Hence $A\vec{v}_0/\|A\vec{v}_0\| = \vec{v}_0$

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and by putting $\lambda = \|A\vec{v}_0\|$ we have $A\vec{v}_0 = \lambda\vec{v}_0$. At first there seem to be no way of knowing the value of λ without finding the eigenvector $\vec{v}_0$. For a reference, see [1, Corollary 10.3].

2. Derivation of numerical estimates by topological considerations

Let $A = (a_{ij})$, $a_{ij} \geq 0$, be a matrix none of whose columns is a zero vector. We have an estimate on the size of the eigenvalue λ since

$$\lambda = \|A\vec{v}_0\| \leq \|A\|\,\|\vec{v}_0\| = \|A\|$$

where $\|A\|$ is the operator norm with respect to the Euclidean norm on $\mathbb{R}^n$. It is a well-known fact that $\|A\|$ is equal to the square root of the maximum of eigenvalues of the matrix AA^T, which does not give us much numerical information in this case.

To find better estimates for λ, we try other norms. First, we use the 1-norm $\|\vec{v}\|_1 = \sum_i |v_i|$ on $\mathbb{R}^n$. We put

$$S_1 \equiv \{\vec{v} : \|\vec{v}\|_1 = 1, v_i \geq 0 \text{ for every } i\}$$

and

$$f : S_1 \to S_1, \quad f(\vec{v}) = A\vec{v}/\|A\vec{v}\|_1.$$

Then $\lambda = \|A\vec{v}_0\|_1 \leq \|A\|_1\,\|\vec{v}_0\|_1 = \|A\|_1$ where $\|A\|_1$ is the operator norm with respect to the 1-norm on $\mathbb{R}^n$, which is equal to the maximum of the 1-norms of the column vectors of A. To illustrate the geometric idea involved here, we note that for $\|\vec{v}_0\| = 1$, the vector $A\vec{v}_0$ is a convex linear combination K of the column vectors of A. Hence its 1-norm $\|A\vec{v}_0\|_1$, which is the distance from the origin, is between $\min_{w\in K}\|w\|_1$ and $\max_{w\in K}\|w\|_1$. Note that the minimum and the maximum occur at the vertex points since K is a convex set with *flat* faces in a hyperplane. Since the vertices of K are given by the column vectors of A, we see that the value $\|A\vec{v}_0\|_1$ is between the minimum and the maximum of the 1-norms of the column vectors. In other words,

$$\min_i(\sum_{j=1}^n a_{ij}) \leq \lambda \leq \max_i(\sum_{j=1}^n a_{ij}),$$

which is the usual numerical estimate in the original Perron-Frobenius Theorem.

Next, we try the max-norm $\|\vec{v}\|_\infty = \max_i |v_i|$ on $\mathbb{R}^n$. By proceeding as before, we derive that $\lambda = \|A\vec{v}_0\|_1 \le \|A\|_\infty \|\vec{v}_0\|_1 = \|A\|_\infty$ where $\|A\|_\infty$ is the operator norm with respect to the max-norm on $\mathbb{R}^n$, which is equal to the maximum of the 1-norms of the row vectors of A. Therefore we obtain an upper bound $\lambda \le \max_j(\sum_{i=1}^n a_{ij})$, which is expected since A and its transpose have the same eigenvalues. So there is nothing new in using this norm.

3. Aperiodic matrices and the uniqueness of the positive eigenvalue

A matrix $A = (a_{ij})$ with $a_{ij} \ge 0$ is said to be *aperiodic* if there exists $m > 0$ such that $(A^m)_{ij} > 0$ for every i, j where A^m is the m-th power of A. We will show that there is a unique positive eigenvalue for an aperiodic matrix and that the corresponding eigenvector has positive coefficients and it is unique up to scalar multiplication. In fact we will prove that there is a unique positive eigenvector with the property $\|\vec{v}\|_1 = 1$.

To prove the claim, we suppose that there exist two eigenvalues. We first observe that if $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors for $A = (a_{ij})$ with eigenvalues $\lambda_1 > 0$ and $\lambda_2 > 0$, respectively, then they are eigenvectors with the eigenvalues $\lambda_1^m > 0$ and $\lambda_2^m > 0$ for the powers A^m, $m = 1, 2, 3, \ldots$. Thus we may assume that $a_{ij} > 0$ and show that it would lead to a contradiction. Geometrically this is equivalent to the fact that all the column vectors of A are pointing in the positive direction, i.e., they lie in the convex subset of R^n given by $x_1 > 0, x_2 > 0, \ldots, x_n > 0$. As in the previous section we define S_1 and $f : S_1 \to S_1$. If $\vec{v} = (v_1, \ldots, v_n)$, $v_i \ge 0$ for every i, is an eigenvector of A for a positive eigenvalue, then we observe that $v_i > 0$ for every i. Hence if there is a fixed point of f then it exists in the interior of S_1 with respect to the relative topology on the hyperplane $x_1 + x_2 + \cdots + x_n = 1$. Note that the mapping f sends a line segment to another line segment so that the image of S_1 under f is a polygon whose vertices are given by the images of the column vectors of A.

Also note that there exist n points $B_1, \ldots, B_n$ in the interior of S_1 with the property that the set of all convex linear combinations K of

$B_1, \dots, B_n$ contains $f(S_1)$ since the image of f is totally inside the interior of S_1. For example, we may choose a sufficiently small $\epsilon > 0$ so that all the coefficients of B_i are equal to ϵ except the i-th coefficient which is equal to $1 - \epsilon$. Hence there exists a positive constant $\rho < 1$ such that $\text{length}(f(K)) \le \rho \cdot \text{length}(K)$ and the diameter of $f^m(S_1)$ monotonically decreases to 0 as $m \to \infty$.

Since $S_1 \supset f(S_1) \supset f^2(S_1) \supset \cdots$, we see that $\bigcap_{m=1}^{\infty} f^m(S_1)$ consists of a single point. In other words, there exists a unique fixed point for the mapping $f : S_1 \to S_1$ since the set of all fixed points of f is included in $\bigcap_{m=1}^{\infty} f^m(S_1)$. Hence there exists a unique positive eigenvalue.

But there might exist an eigenvector $\vec{w}$ with the positive eigenvalue while some of its coefficients are negative. To show that the obtained positive eigenvalue is simple, we consider a two dimensional subspace J spanned by two eigenvectors $\vec{v}_0$ and $\vec{w}$ where $\vec{v}_0 \in S_1$. Since $S_1 \cap J$ is a one-dimensional line segment on S_1, there are infinitely many eigenvectors on $S_1 \cap J$ whose coefficients are all positive, which contradicts the previous observation.

4. Stochastic matrices

Recall that a square matrix A is called a *stochastic matrix* if all of its coefficients are nonnegative and the sum of all the coefficients of the column vectors of A are all equal to 1, that is, $\sum_i a_{ij} = 1$ for every j. If we consider its transpose A^T instead of A, then the sum of all the coefficients of the row vectors should be all equal to 1. In this case we multiply row vectors from the left. Note that if A is a stochastic matrix then its powers A^m are also stochastic matrices.

Since some of the coefficients of A may be zeros, one cannot guarantee that all of the coefficients of the eigenvector $\vec{v}_0$ are positive. Instead they are nonnegative. For example, choose the identity matrix for A. Stochastic matrices are used to define a Markov process which is widely used in ergodic theory, probability theory, coding theory, symbolic dynamics, etc. The value a_{ij} represents the probability of the transition from the j-th state to the i-th state. The existence of an eigenvector $\vec{v} = (p_1, p_2, \dots, p_n)$ with $p_i \ge 0$ and $\sum_i p_i = 1$ implies that there is a steady state solution of the probabilistic transition problem. In other words, sufficiently long time later the probability that we have the state i is approximately equal to p_i.

The probabilistic interpretation of aperiodicity is that if transition probabilities a_{ij} are given by an aperiodic stochastic matrix $A = (a_{ij})$, then any arbitrary state j can be changed into any other state i passing through sufficiently many other intermediate states. It will be shown that for an aperiodic stochastic matrix A we can choose $\vec{v}$ so that $v_i > 0$ for every i and this is unique up to scalar multiplication. If we normalize it so that its 1-norm is equal to 1, it is called the *invariant probability vector*. Hence a long time later we expect that the probability that we stay at the state i is $p_i > 0$ for every i and that whatever the initial probabilistic distribution is for the states $1, 2, \ldots, n$ it would evolve into the probabilistic distribution given by the unique normalized eigenvector.

It is easy to see that the transpose A^T has an eigenvalue 1 since the vector $u = (1, 1, \ldots, 1) \in \mathbb{R}^n$ obviously satisfies $A^T u = u$. Hence A also has an eigenvalue 1. The existence of the corresponding eigenvector with positive coefficients is another matter and the topological method using the Brouwer's Fixed Point Theorem shows why this should be true.

For the full statement of the quoted results, see [2] and P.16 and P.42 in [3].

For stochastic matrices the proof of the Perron-Frobenius Theorem is relatively simple. Define S_1 as in Section 2. Then for any $\vec{v} \in S$, we have

$$\|A\vec{v}\|_1 = \sum_i (Av)_i = \sum_i \Big(\sum_j a_{ij} v_j\Big) = \sum_j v_j \Big(\sum_i a_{ij}\Big) = \sum_j v_j = \|\vec{v}\|_1 = 1.$$

Hence we have a continuous function $f : S_1 \to S_1$ defined by $f(\vec{v}) = A\vec{v}$. By Brouwer's Fixed Point Theorem, we have $f(\vec{v}_0) = \vec{v}_0$ for some $\vec{v}_0$, hence $A\vec{v}_0 = \vec{v}_0$. Note that the coefficients of v_0 are nonnegative. We may use the result from the previous sections to obtain the existence and uniqueness of the normalized positive eigenvector for an aperiodic stochastic matrix.

5. An algorithm for finding the eigenvector

In proving the uniqueness of the fixed point, we observe that $f^m(\vec{v})$ converges to the fixed point $\vec{v}_0$ for any initial point $\vec{v} \in S_1$. Hence $A^m \vec{e}_i = f^m(\vec{e}_i)$, $m = 1, 2, 3, \ldots$, converges to $\vec{v}_0$ as $m \to \infty$ where $\vec{e}_i = (0, \ldots, 1, \ldots, 0)$ is the i-th standard basis vector for $\mathbb{R}^n$. Therefore we see that the limit of the sequence of the vectors given by the i-th columns of the matrices A^m, $m = 1, 2, 3, \ldots$, is the unique eigenvector $\vec{v}_0$ for any i, hence we conclude that the sequence A^m, $m = 1, 2, 3, \ldots$, converges to the matrix whose columns are all identical with $\vec{v}_0$. By calculating the constant $\rho > 0$ in Section 3, we can estimate the speed of convergence.

6. Irreducible matrices

Without much additional work, we can show that all the previous results hold true for an irreducible matrix which is a generalization of an aperiodic matrix. A square matrix $A = (a_{ij})$, $a_{ij} \geq 0$ for every i, j, is said to be *irreducible* if for any (i, j) there exists a positive integer m such that $(A^m)_{ij} > 0$. Note that aperiodic matrices are irreducible by definition. Most of the results of Perron and Frobenius hold true not only for aperiodic matrices but also for irreducible matrices. The proofs are similar.

It is known that if a stochastic matrix $A = (a_{ij})$ has a simple eigenvalue 1 and if its corresponding eigenvector has positive coefficients, then A should be irreducible. We prove the fact as follows: Suppose that A is not irreducible. Then there exists (i, j) for which $(A^m)_{ij} = 0$ for every $m = 1, 2, 3, \ldots$. Define f as in Section 4. Then $(A^m)_{ij}$ is the i-th component of the vector $f^m(\vec{e}_j)$ for every m. Let L be the linear subspace spanned by the vectors $f^m(\vec{e}_j)$. Note that L is perpendicular to the vector $\vec{e}_i$ and that $L \cap S_1$ is contained in $x_i = 0$, $x_1 + \cdots + x_{i-1} + x_{i+1} + \cdots + x_n = 1$, $x_1 \geq 0, \ldots, x_{i-1} \geq 0$, $x_{i+1} \geq 0, \ldots, x_n \geq 0$ for some i. Since $L \cap S_1$ is invariant under f, if we restrict f to $L \cap S_1$, then f has a fixed point there. In other words, A has an eigenvalue 1 and a corresponding eigenvector with its i-th coefficient equal to zero, which is a contradiction.

References

1. J. R. Munkres, *Topology; a first course*, Prentice-Hall, Englewood Cliffs, N.J., 1975.
2. F. R. Gantmacher, *Matrix Theory*, Chelsea Publ. Co., New York, 1959.
3. P. Walters, *An Introduction to Ergodic Theory, 2nd Ed.*, Springer-Verlag, 1982.

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