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ON THE COMPACT METHODS FOR ABSTRACT NONLINEAR FUNCTIONAL EVOLUTION EQUATIONS

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1. Introduction

Let X be a real Banach space. We consider the existence of solutions of the abstract nonlinear functional evolution equation:

$$\text{(E)}\qquad \begin{aligned} &\frac{du(t)}{dt} + A(t)u(t) + F(u)(t) \ni h(t), \\ &u(s) = x_0 \in \overline{D(A(s))}, \quad 0 \le s \le t \le T, \end{aligned}$$

where $u : [s,T] \to X$ is an unknown function, $\{A(t) : 0 \le t \le T\}$ is a given family of nonlinear (possibly multivalued) operators in X, and $F : C([s,T];X) \to L^\infty([s,T];X)$ and $h : [s,T] \to X$ are given functions.

The existence problems for this kind have been studied by many authors, for example, Aizicovici [1], Crandall and Evans [4], Crandall and Pazy [5], Evans [6], Kato [7], Kobayasi, Kobayashi and Oharu [8], Oharu and Takahashi [9], and Pavel [10, 12]. In particular, using the method of steps as in the theory of differential equations with delay, Pavel [12] established an existence result of integral solutions of the problem (E) with $h(t) = 0$ in the case that $\{A(t)\}$ generates a compact evolution operator and F is a superposition operator, i.e., $F(u)(t) = f(t, u(t))$ $(= f(t)u(t))$ for every $u \in C([s,T];X)$ and $t \in [s,T]$, where $f : [s,T] \times X \to X$ is a given function. We also note that Pavel's result was an extension of those of Pazy [14] and Vrabie [15], in which t-independent operator $A = A(t)$ generates a compact semigroup.

In this paper, we show that compactness arguments can be employed to conclude the local solvability of the problem (E) in the case that the resolvent operator $J_\lambda(t) = (I + \lambda A(t))^{-1}$ of $A(t)$ for $s \le t \le T$

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is compact. Our approach is also based upon compactness arguments involving Schauder's fixed point theorem as in quasi-autonomous case that A is independent of t [15, 16].

The plan of this paper is as follows. Section 2 is devoted to several conditions, some background material and auxiliary results. In Section 3, we study the main compactness results. The local existence theorem and the continuation of solutions are presented in Section 4.

2. Preliminaries

Let X be a real Banach space and let X^* be the dual space of X. We recall that the duality mapping J in X is defined by

$$J(x) = \{x^* \in X^* : (x, x^*) = \|x\|^2 = \|x^*\|^2\}$$

where (,) denotes the usual pairing of X and X^*. We define

$$\langle y, x\rangle_s = \sup\{(y, x^*) : x^* \in J(x)\},\ \langle y, x\rangle_+ = \langle y, x\rangle_s \|x\|^{-1}$$

for $y,\ x \in X,\ x \neq 0$.

For each $T \geq 0$, let $\{A(t) : 0 \leq t \leq T\}$ be a family of nonlinear (possibly multivalued) operators $A(t) : D(A(t)) \subset X \to 2^X$ satisfying the hypotheses, which have been considered by Pavel [10, 11],

(C1) $R(I + \lambda A(t)) = X$ for all $h > 0$ and $t \in [0, T]$, where I is the identity on X.

(C2) There exist a continuous function $f : [0, T] \to X$ and a continuous, nondecreasing, bounded (on bounded subsets) function $L : R_+ \to R_+$ such that

$$\begin{aligned}&\langle y_1 - y_2, x_1 - x_2\rangle_s \\ \geq\ & - \|f(t) - f(s)\| \|x_1 - x_2\| L(\max\{\|x_1\|, \|x_2\|\})\end{aligned}$$

for all $0 \leq s \leq t \leq T$, $[x_1, y_1] \in A(t)$ and $[x_2, y_2] \in A(s)$.

(C3) If $t_n \uparrow t$, $x_n \in D(A(t_n))$ and $x_n \to x$, then $x \in \overline{D(A(t))}$ (the closure of $D(A(t))$, $(t_n,\ t \in [0, T])$.

Recall that if $\{A(t)\}$ satisfies the conditions (C1), (C2) and (C3), then $\{A(t)\}$ generates an operator $U(t,s)$ via the following formula:

$$(2.1)\quad \lim_{n\to\infty}\prod_{j=1}^{n}\left(I+\frac{t-s}{n}A(s+j\frac{t-s}{n})\right)^{-1}x_0^n \equiv U(t,s)x_0 \in \overline{D(A(t))}$$

for every $0 \le s \le t \le T$, $x_0 \in \overline{D(A(s))}$ and $x_0^n \in D(A(s))$ with $x_0^n \to x_0$, and $U(t,s)$ is an evolution operator in the following sense:

(i) $U(t,s):\overline{D(A(s))} \to \overline{D(A(t))}$, $U(s,s)=I$, $0\le s\le t\le T$,
(ii) $U(t,s)U(s,r)=U(t,r)$, $0\le r\le s\le t\le T$,
(iii) the function $(t,s,x)\to U(t,s)x$ is continous,
(iv) $\|U(t,s)x-U(t,s)y\|\le\|x-y\|$, $0\le s\le t$, $x,\ y\in\overline{D(A(s))}$.

The function $t\to U(t,s)x_0$ is an integral solution of the problem $u'(t)+A(t)u(t)\ni 0$, $u(s)=x_0$, $0\le s\le t\le T$. We also note that (C1) and (C2) imply that $\overline{D(A(t))}$ is independent of t. For these facts, see [10] or [11, Ch. 1].

Now we mention the relation between the evolution operator $U(t,s)$ and the resolvent operator $J_\lambda(t)$. To this end, set

$$\overline{D(A(t))}=\overline{D(A(0))}=\overline{D},\ t\in[0,T].$$

Let $U(t,s)$ be the evolution operator generated by $\{A(t)\}$ via the formula (2.1) and let $J_\lambda(t)=(I+\lambda A(t))^{-1}$ be the operator from X into $D(A(t))$ for each $\lambda>0$. It is known that the following inequalities:

$$(2.2)\quad \begin{aligned}&\|J_\lambda(s)x-x\|\\ &\le\frac{2}{t-s}\int_s^t\|U(\tau,s)x-x\|d\tau+\frac{\lambda}{t-s}\|U(t,s)x-x\|\\ &\quad+\frac{\lambda C}{t-s}\int_s^t\|f(\tau)-f(s)\|d\tau\end{aligned}$$

and

(2.3)

$$\begin{aligned}&\|U(t,s)x-x\|\le(2+\frac{t-s}{\lambda})\|J_\lambda(s)x-x\|\\ &\quad+(t-s)L(\|J_\lambda(s)x\|)\sup\{\|f(t)-f(s)\|:t,\ s\in[0,T],|t-s|\le T\}\end{aligned}$$

hold for every $x \in \overline{D}$, $0 \le s \le t \le T$ and $\lambda > 0$, where $C = C(\|x\|, s)$ is a positive constant with property that $C = C(\|x\|, s)$ is bounded on x-bounded set, and L and f are as in (C2). For proof of this result, known as the extension of Brézis's inequalities [3], we refer to [11, Ch. 1] and [13].

In what follows the notion of integral solution is necessary. Let $g \in L^1([0,T]; X)$ and $s \in [0,T]$. Consider the problem:

$$\begin{aligned} &\frac{du(t)}{dt} + A(t)u(t) \ni g(t), \\ &u(s) = x_0 \in \overline{D(A(s))}, \quad s \le t \le T. \end{aligned} \tag{2.4}$$

Recall that a function $u : [s,T] \to X$ is said to be an integral solution of the problem (2.4) on $[s,T]$ if u is continuous on $[s,T]$, $u(t) \in \overline{D(A(t))}$ for every $t \in [s,T]$, $u(s) = x_0$ and the following inequality is satisfied:

$$\begin{aligned} &\|u(t) - x\| \\ &\le \|u(t_0) - x\| + \int_{t_0}^{t} (\langle y + g(\tau), u(\tau) - x\rangle_+ + C\|f(\tau) - f(r)\|)d\tau \end{aligned} \tag{2.5}$$

for every $s \le t_0 \le t \le T$, $r \in [s,T]$, $x \in D(A(r))$ and $y \in A(r)x$ with $C = L(\max\{\|u\|, \|x\|\})$, where $\|u\| = \sup_{s \le t \le T} \|u(t)\|$, and L and f are as in (C2).

When $A(t) = A$ is independent of t, it is just the notion of integral solution in the sense of Benilan [2]. It is known [10, 11] that (2.4) has a unique integral solution. Moreover, if v is the integral solution of problem:

$$\begin{aligned} &\frac{dv(t)}{dt} + A(t)v(t) \ni g_1(t), \\ &v(s) = y_0 \in \overline{D(A(s))}, \quad s \le t \le T_1, \end{aligned} \tag{2.6}$$

then

$$\|u(t) - v(t)\| \le \|u(t_0) - v(t_0)\| + \int_{t_0}^{t} \|g(\tau) - g_1(\tau)\| d\tau \tag{2.7}$$

for every $s \le t_0 \le t \le T_0$, $T_0 = \min\{T, T_1\}$.

From now we denote $\overline{D} = \overline{D(A(t))} = \overline{D(A(0))}$ $(t \in [0,T])$.

We prepare a lemma which will be used in Section 3.

LEMMA 1. *Assume that* $\{A(t)\}$ *satisfies the conditions* (C1), (C2) *and* (C3). *Let* $g \in L^1([s,T];X)$ *and* $x_0 \in \overline{D}$ *and let* u *be the unique integral solution of* (2.4) *on* $[s,T]$. *Then for all* $t \in (s,T]$, $l \in [s,T)$, $h > 0$ *with* $t-h \in [s,T]$ *and* $l+h \in [s,T]$,

$$\|U(t,t-h)u(t-h) - u(t)\| \le \int_{t-h}^{t} \|g(\tau)\| d\tau \tag{2.8}$$

and

$$\|U(l+h,l)u(l) - u(l+h)\| \le \int_{l}^{l+h} \|g(\tau)\| d\tau \tag{2.9}$$

hold. Furthermore, the following inequalities hold:

$$\begin{aligned} \|u(l) - u(l+h)\| \le \|U(s+h,s)x_0 - x_0\| &+ \int_s^{s+h} \|g(\tau)\| d\tau \\ &+ \int_s^l \|g(\tau+h) - g(\tau)\| d\tau \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} \|u(t) - u(t-h)\| \le \|U(s+h,s)x_0 - x_0\| &+ \int_s^{s+h} \|g(\tau)\| d\tau \\ &+ \int_{s+h}^t \|g(\tau-h) - g(\tau)\| d\tau. \end{aligned} \tag{2.11}$$

Proof. We prove only (2.8) since (2.9) follows in a similar way. Let $v^h : [t-h,t] \to \overline{D}$ be the function defined by

$$v^h(\tau) = U(\tau, t-h)u(t-h), \quad t-h \le \tau \le t.$$

Then v^h is the unique integral solution of the problem:

$$\begin{aligned} &\frac{dv^h(\tau)}{d\tau} + A(\tau)v^h(\tau) \ni 0, \\ &v^h(t-h) = u(t-h), \quad t-h \le \tau \le t. \end{aligned}$$

Thus, applying (2.7) to v^h and to u on $[t-h,t]$, we obtain

$$\|U(t,t-h)u(t-h)-u(t)\| \le \int_{t-h}^{t} \|g(\tau)\|d\tau,$$

which proves (2.8).

Now we prove (2.10). (2.11) follows in the same way. Let $w:[s,T-h]\to X$ be the function defined by

$$w(l)=u(l+h), \quad s\le l\le T-h.$$

Then w is the integral solution of problem:

$$\frac{dw(l)}{dl}+A(l)w(l)\ni g(l+h),$$
$$w(s)=u(s+h), \quad s\le l\le T-h.$$

From (2.7) we deduce

$$\|u(l)-w(l)\| \le \|u(s)-w(s)\| + \int_s^l \|g(\tau+h)-g(\tau)\|d\tau.$$

Since $\|u(s)-w(s)\| = \|x_0-u(s+h)\|$, we obtain

$$\begin{aligned}\|u(l)-u(l+h)\| \le \|U(s+h,s)x_0-x_0\| + \|u(s+h)-U(s+h,s)x_0\| \\ + \int_s^l \|g(\tau+h)-g(\tau)\|d\tau.\end{aligned}$$

Applying (2.9) to majorize $\|u(s+h)-U(s+h,s)x_0\|$, we have (2.10), which completes the proof.

Finally we recall that an operator $T:C\subset X\to X$ is called compact if it is continuous and maps bounded subsets in C into relatively compact subsets in X

The following well-known result will be useful in Section 3. For proof, see [17, p. 49].

LEMMA 2. *If* $\{T_i\}_{i\in\Lambda}$ *is a generalized sequence of compact operators from* $C \subset X$ *into* X, *and if* $\lim_i T_i = T$ *uniformly on bounded subsets in* C, *then* $T : C \subset X \to X$ *is also compact.*

3. Compactness results

In this section, we study some compactness results which are useful in Section 4.

A subset G in $L^1([s,T];X)$ is called uniformly integrable if given $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that

$$\int_E \|g(t)\|dt < \epsilon$$

for every measurable subset E in $[s,T]$ with Lebesgue measure $m(E) < \delta(\epsilon)$ and uniformly for $g \in G$.

A subset G in $L^1([s,T];X)$ is called equiintegrable if it is uniformly integrable and, in addition

$$\lim_{h\downarrow 0} \int_s^{T-h} \|g(t+h) - g(t)\|dt = 0$$

holds uniformly for $g \in G$.

REMARK 1. We may easily verify, by using Hölder's inequality, that each bounded subset of $L^p([s,T];X)$ with $1 < p \leq +\infty$ is uniformly integrable. We also note that, since $[s,T]$ is compact, each uniformly integrable subset is bounded in $L^1([s,T];X)$, and that any relatively compact subset of $L^1([s,T];X)$ is equiintegrable, but the converse statement is no longer true unless X is finite dimensional. For these facts, see [17].

From now on, for a fixed $x_0 \in \overline{D}$, we denote the integral solution u of the problem (2.4) by u^g in order to exhibit the dependence of u on $g \in L^1([s,T];X)$.

Now we give a lemma.

LEMMA 3. *Assume that $\{A(t)\}$ satisfies the conditions (C1), (C2) and (C3). Let G be a uniformly integrable subset in $L^1([s,T];X)$ and let $x_0 \in \overline{D}$ and $M(G) = \{u^g : u^g$ is the integral solution of (2.4) and $g \in G\}$. If the set $M(G)(t) = \{u^g(t) : u^g \in M(G)\}$ is relatively compact in X for each $t \in [s,T)$, then the set $M(G)$ is equicontinuous on $[s,T]$.*

Proof. Let us recall that $U(t,s)x_0 = u^0(t)$ is the unique integral solution of the problem:

$$\frac{du(t)}{dt} + A(t)u(t) \ni 0,$$
$$u(s) = x_0, \quad s \le t \le T.$$

Thus by (2.7), we have

$$\begin{aligned} \|u^g(t) - x_0\| &\le \|u^g(t) - U(t,s)x_0\| + \|U(t,s)x_0 - x_0\| \\ &\le \|U(t,s)x_0 - x_0\| + \int_s^t \|g(\tau)\|d\tau \end{aligned}$$

for each $g \in G$ and $t \in [s,T]$. Since $U(t,s)$ is continuous at (s,s) and G is uniformly integrable, this inequality shows that $M(G)$ is equicontinuous at $t = s$. Now let $t \in (s,T]$, let $\epsilon > 0$ be arbitrary, and choose $\lambda > 0$ such that $t - \lambda \in [s,T)$ and, in addition

$$\int_E \|g(\tau)\|d\tau \le \frac{\epsilon}{16} \tag{3.1}$$

for each measurable subset E in $[s,T]$ with $m(E) < 2\lambda$ and uniformly for $g \in G$. Since $M(G)(t-\lambda)$ is relatively compact in X and precompact in X, for $\epsilon > 0$, there exists a finite family $\{g_1, g_2, \ldots, g_{n(\epsilon)}\}$ in G with the property that given $g \in G$, there exists an $i \in \{1,2,\ldots,n(\epsilon)\}$ such that

$$\|u^g(t-\lambda) - u^i(t-\lambda)\| \le \frac{\epsilon}{4}, \tag{3.2}$$

where we denoted $u^i = u^{g_i}$.

On the other hand, the family $\{u^1, u^2, \dots, u^{n(\epsilon)}\}$ is equicontinuous at t since it is finite. Consequently there exists $\delta(\epsilon) \in (0, \lambda)$ such that

$$\|u^i(t+h) - u^i(t)\| \le \frac{\epsilon}{4} \tag{3.3}$$

for each $i \in \{1, 2, \dots, n(\epsilon)\}$, $h \in R$ with $|h| < \delta(\epsilon)$ and $t + h \in [s, T]$. Now, using (2.7), we obtain

$$\begin{aligned}
&\|u^g(t+h) - u^g(t)\| \\
&\le \|u^g(t+h) - u^i(t+h)\| + \|u^i(t+h) - u^i(t)\| + \|u^i(t) - u^g(t)\| \\
&\le 2\|u^g(t-\lambda) - u^i(t-\lambda)\| + \int_{t-\lambda}^{t+h} \|g(\tau) - g_i(\tau)\| d\tau \\
&\quad + \|u^i(t+h) - u^i(t)\| + \int_{t-\lambda}^{t} \|g(\tau) - g_i(\tau)\| d\tau
\end{aligned}$$

for each $g \in G$ and $h \in R$ with $t + h \in [s, T]$. Thus, if $|h| < \delta(\epsilon)$ and $t + h \in [s, T]$, by (3.1), (3.2) and (3.3),

$$\|u^g(t+h) - u^g(t)\| \le \epsilon$$

uniformly for $g \in G$. Therefore $M(G)$ is equicontinuous on $[s, T]$. This completes the proof.

THEOREM 1. *Assume that* $\{A(t)\}$ *satisfies the conditions* (C1), (C2) *and* (C3). *Let the resolvent operator* $J_\lambda(t) = (I + \lambda A(t))^{-1}$ *from* X *into* $D(A(t))$ *be compact for each* $t \in [0, T]$ *and* $\lambda > 0$. *Let* G *be a uniformly integrable subset of* $L^1([s, T]; X)$ *and* $x_0 \in \overline{D}$. *Then the following conditions are equivalent:*

(i) *The set* $M(G) = \{u^g : u^g$ *is the integral solution of* (2.4) *and* $g \in G\}$ *is relatively compact in* $C([s, T]; X)$,

(ii) *The set* $M(G)$ *is equicontinuous from the right at each* $t \in [s, T)$.

Proof. Obviously (i) implies (ii). To prove the converse, we first prove that (ii) together with the compactness of $J_\lambda(t)$ implies that $M(G)(t)$ is relatively compact in X for each $t \in [s, T)$. First of all, using (2.2), we have

$$\begin{aligned} &\|J_\lambda(s)u^g(t) - u^g(t)\| \\ (3.4) \quad &\leq \frac{2}{t-s}\int_s^t \|U(\tau, s)u^g(t) - u^g(t)\| d\tau \\ &+ \frac{\lambda}{t-s}\|U(t,s)u^g(t) - u^g(t)\| + \frac{\lambda C}{t-s}\int_s^t \|f(\tau) - f(s)\| d\tau \end{aligned}$$

for each $g \in G$, $t \in [s, T)$ and $\lambda > 0$ with $t + \lambda \in [s, T]$, where C is the constant in (2.2). At this point, we observe that $U(\tau, t)u^g(t)$ is the unique integral solution of the problem:

$$\begin{aligned} &\frac{dv^\epsilon(\tau)}{d\tau} + A(\tau)v^\epsilon(\tau) \ni 0, \\ &v^\epsilon(t) = u^g(t), \quad t \leq \tau \leq t + \epsilon. \end{aligned}$$

Then by (2.9) in Lemma 1,

$$\|U(t+\epsilon, t)u^g(t) - u^g(t+\epsilon)\| \leq \int_t^{t+\epsilon} \|g(\tau)\| d\tau.$$

We also observe that

$$(3.5) \quad \|u^g(t) - U(t+\epsilon, t)u^g(t)\| \leq \int_t^{t+\epsilon} \|g(\tau)\| d\tau + \|u^g(t+\epsilon) - u^g(t)\|.$$

From (3.4) and (3.5) we obtain, for $\lambda = \epsilon$,
(3.6)

$$\begin{aligned} \|J_\lambda(t)u^g(t) - u^g(t)\| &\leq \frac{2}{\lambda}\int_t^{t+\lambda} \|U(\tau, t)u^g(t) - U(t+\lambda, t)u^g(t)\| d\tau \\ &+ \frac{2}{\lambda}\int_t^{t+\lambda}\left(\int_t^{t+\lambda} \|g(\tau)\| d\tau\right) d\tau + \frac{2}{\lambda}\int_t^{t+\lambda} \|u^g(t+\lambda) - u^g(t)\| d\tau \\ &+ \int_t^{t+\lambda} \|g(\tau)\| d\tau + \|u^g(t+\lambda) - u^g(t)\| + C\int_t^{t+\lambda} \|f(\tau) - f(t)\| d\tau \end{aligned}$$

for $g \in G$, $t \in [s, T)$ and $t + \lambda \in [s, T]$.

On the other hand, since G is uniformly integrable, $M(G)$ is equicontinuous from the right at each $t \in [s,T)$ and f is continuous, there exists a nondecreasing function $\mu_t : R_+ \to R_+$ depending on t with $\lim_{h\downarrow 0} \mu_t(h) = 0$ and satisfying the following:

$$\|U(\tau,t)u^g(t) - U(t+h,t)u^g(t)\| \le \mu_t(h),$$

$$\int_t^{t+h} \|g(\tau)\|d\tau \le \mu_t(h), \quad \|u^g(t+h) - u^g(t)\| \le \mu_t(h), \tag{3.7}$$

$$C\int_t^{t+h} \|f(\tau)-f(t)\|d\tau \le \mu_t(h).$$

for each $g \in G,\ t \in [s,T),\ h \in [0,\lambda]$ with $t+\lambda \in [s,T]$ and $\tau \in [t,t+h]$. Thus from (3.6) and (3.7) we deduce that

$$\|J_\lambda(t)u^g(t) - u^g(t)\| \le 9\mu_t(\lambda).$$

Therefore, $\lim_{\lambda\downarrow 0} J_\lambda(t) = I$ uniformly on $M(G)(t)$. Since $J_\lambda(t)$ is compact for each $\lambda > 0$ and $t \in [s,T]$, it follows from Lemma 2 that I is a compact operator from $M(G)(t)$ into itself. On the other hand, since G is bounded in $L^1([s,T];X)$, $M(G)(t)$ is bounded in X for each $t \in [s,T)$ (see Remark 1 and (2.7)). Therefore $M(G)(t)$ is relatively compact in X for each $t \in [s,T)$.

Next, for $h > 0$ with $T-h \in [s,T)$, let us define the operator $T_h : M(G)(T) \to X$ by

$$T_h u^g(T) = u^g(T-h)$$

for each $u^g(T) \in M(G)(T)$. Since $T_h(M(G)(T)) = M(G)(T-h)$ and $M(G)(T-h)$ is relatively compact, T_h is a compact operator. In addition, since $M(G)$ is equicontinuous at T by Lemma 3, it follows that $\lim_{h\downarrow 0} T_h = I$ uniformly on $M(G)(T)$. Again by Lemma 2, I is a compact operator from $M(G)(T)$ into itself. Therefore $M(G)(T)$ is relatively compact in X.

Consequently $M(G)(t)$ is relatively compact in X for each $t \in [s,T]$. Also by Lemma 3, M(G) is equicontinuous on $[s,T]$. From Ascoli-Arzelà's theorem we conclude that $M(G)$ is relatively compact in $C([s,T];X)$. This completes the proof.

As an important consequence of Theorem 1, we have the following result.

THEOREM 2. *Assume that* $\{A(t)\}$ *satisfies the conditions* (C1), (C2) *and* (C3). *Let the resolvent operator* $J_\lambda(t) = (I + \lambda A(t))^{-1}$ *from* X *into* $D(A(t))$ *be compact for each* $t \in [0, T]$ *and* $\lambda > 0$. *Let* G *be a equiintegrable subset in* $L^1([s,T];X)$ *and* $x_0 \in \overline{D}$. *Then the set* $M(G) = \{u^g : u^g$ *is the integral solution of* (2.4) *and* $g \in G\}$ *is relatively compact in* $C([s,T];X)$.

Proof. From (2.10) in Lemma 1 it follows that

$$\|u^g(t+h) - u^g(t)\| \le \|U(s+h,s)x_0 - x_0\| + \int_s^{s+h} \|g(\tau)\| d\tau + \int_s^{T-h} \|g(\tau+h) - g(\tau)\| d\tau$$

for each $g \in G$, $t \in [s,T)$ and $h > 0$ with $t + h \in [s,T]$. Since G is equiintegrable and $U(t,s)$ is continuous at (s,s), this inequality shows that $M(G)$ is equicontinuous from the right at each $t \in [s,T)$. Therefore by Theorem 1, $M(G)$ is relatively compact in $C([s,T];X)$. This complete the proof.

The following lemma gives a simple sufficient condition in order that a subset G of $L^1([s,T];X)$ be equiintegrable.

LEMMA 4. *If* G *is bounded in* $W^{1,1}([s.T];X)$, *then* G *is equiintegrable.*

Proof. Since $W^{1,1}([s,T];X)$ is continuously embedded in $L^p([s,T];X)$ for each $p \in [1, +\infty)$, each bounded subset in $W^{1,1}([s,T];X)$ is uniformly integrable by Remark 1. Thus, to complete the proof, it suffices to check that each bounded subset in $W^{1,1}([s,T];X)$ satisfies the second condition in the definition of equiintegrability. To this end, let G be a bounded subset in $W^{1,1}([s,T];X)$ and let us define the set $K = \{f_g : g \in G\}$, where for each $g \in G$, f_g is defined by

$$f_g(t) = \int_s^t \|\frac{dg}{d\tau}(\tau)\| d\tau$$

for each $t \in [s,T]$. Obviously K is bounded in $W^{1,1}([s,T];R)$. Since $W^{1,1}([s,T];R)$ is compactly embedded in $L^1([s,T];R)$, K is relatively compact in $L^1([s,T];R)$. Thus by Remark 1, K is equiintegrable in $L^1([s,T];R)$. Now we observe that

$$\lim_{h\downarrow 0}\int_s^{T-h}\|g(\tau+h)-g(\tau)\|d\tau = \lim_{h\downarrow 0}\int_s^{T-h}\|\int_\tau^{\tau+h}\frac{dg}{dt}(t)dt\|d\tau$$

$$\leq \lim_{h\downarrow 0}\int_s^{T-h}\int_\tau^{\tau+h}\|\frac{dg}{dt}(t)\|dtd\tau = \lim_{h\downarrow 0}\int_s^{T-h}|f_g(\tau+h)-f_g(\tau)|d\tau = 0$$

uniformly for $g \in G$. This completes the proof.

A useful consequence of Theorem 2 with Lemma 4 is the following:

COROLLARY 1. *Assume that* $\{A(t)\}$ *satisfies the conditions* (C1),(C2) *and* (C3). *Let the resolvent operator* $J_\lambda(t) = (I+\lambda A(t))^{-1}$ *from* X *into* $D(A(t))$ *be compact for each* $t \in [0,T]$ *and* $\lambda > 0$. *Let* G *be a bounded subset in* $W^{1,1}([s,T];X)$ *and* $x_0 \in \overline{D}$. *Then the set* $M(G) = \{u^g : u^g$ *is the integral solution of* (2.4) *and* $g \in G\}$ *is relatively compact in* $C([s,T];X)$.

REMARK 2. The above results in the case that $A(t) = A$ is independent of t were given in [17].

4. Existence and continuation of the solutions

We begin this section with the following definition.

DEFINITION 1. A function $u : [s,T] \to X$ is called an integral solution of the problem (E) on $[s,T]$ if u is an integral solution of (2.4) on $[s,T]$ with $g = -F(u) + h$.

At this point, we give the following hypotheses to F which are necessary for our main results.

(H1) U is a given open subset of X.
(H2) The mapping $F : C([s,T];U) \to L^\infty([s,T];X)$ is continuous and for every bounded subset $B \subset C([s,T];U)$, $F(B)$ is equiintegrable.

The following is our main existence result.

THEOREM 3. *Assume that* $\{A(t)\}$ *satisfies the conditions* (C1), (C2) *and* (C3) *and that* (H1) *and* (H2) *are satisfied. Let the resolvent operator* $J_\lambda(t) = (I+\lambda A(t))^{-1}$ *from* X *into* $D(A(t))$ *be compact for each* $t \in [0,T]$ *and* $\lambda > 0$. *Then for each* $x_0 \in \overline{D} \cap U$ *and* $h \in L^1([s,T];X)$, *there exists* $T_0 \in (s,T]$ *such that the problem* (E) *has at least one integral solution* u *on* $[s,T_0]$.

Proof. Since U is open in X, F is continuous, $U(t,s)$ is continuous at (s,s) and $h \in L^1([s,T];X)$, we can choose $T_0 > s \geq 0$, $\rho > 0$ and $M > 0$ such that

$$B(x_0,\rho) = \{y \in X : \|y - x_0\| \leq \rho\} \subset U,$$

$$\|F(u)(t)\| \leq M \quad \text{a.e. on } [s,T_0] \tag{4.1}$$

for all $u \in C([s,T_0];U)$ with $u(t) \in B(x_0,\rho)$ for $s \leq t \leq T_0$, and

$$MT_0 + \int_s^{s+T_0} \|h(\tau)\| d\tau + \sup_{0 \leq t-s \leq T_0} \|U(t,s)x_0 - x_0\| \leq \rho. \tag{4.2}$$

Now let us define the set

$$K = \{u \in C([s,T_0];U) : u(0) = x_0,\ u(t) \in B(x_0,\rho) \ \text{ for } \ s \leq t \leq T_0\}.$$

Obviously K is nonempty, bounded, convex and closed in $C([s,T_0];X)$. For each $v \in K$, consider the following initial value problem:

$$\begin{aligned} &\frac{du(t)}{dt} + A(t)u(t) + F(v)(t) \ni h(t), \\ &u(s) = x_0, \qquad s \leq t \leq T_0. \end{aligned} \tag{4.3}$$

Then, from Pavel's existence and uniqueness theorem [10, 11] it follows that for each $v \in K$, the problem (4.3) has a unique integral solution u. Therefore we can define the operator $Q : K \to C([s,T_0];X)$ by

$$Qv = u, \tag{4.4}$$

where u and v satisfy (4.3). Now let us observe that the problem (E) has at least one integral solution if and only if the operator Q has at least one fixed point. Thus it suffices to show that Q is a continuous operator from K into K and $Q(K)$ is relatively compact, i.e., that Q satisfies the hypotheses of the well-known Schauder's fixed point theorem. To this end, let us apply (2.7) to Qv and to the integral solution w of the following problem:

$$\frac{dw(t)}{dt} + A(t)w(t) \ni 0,$$
$$w(s) = x_0, \quad s \leq t \leq T_0.$$

Then we obtain

$$\begin{aligned} \|(Qv)(t) - w(t)\| &\leq \int_s^t \|F(v)(\tau)\|d\tau + \int_s^t \|h(\tau)\|d\tau \\ &\leq MT_0 + \int_s^t \|h(\tau)\|d\tau. \end{aligned} \tag{4.5}$$

Taking into account that $w(t) = U(t,s)x_0$, from (4.5) we easily deduce

$$\|(Qv)(t) - x_0\| \leq MT_0 + \int_s^{s+T_0} \|h(\tau)\|d\tau + \|U(t,s)x_0 - x_0\|, \tag{4.6}$$

and from (4.2) we obtain

$$\|(Qv)(t) - x_0\| \leq \rho, \quad s \leq t \leq T_0. \tag{4.7}$$

This (4.7) implies that $Q(K) \subset K$.

Now, from the definition of Q and (2.7) it follows that

$$\|(Qv)(t) - (Qw)(t)\| \leq \int_s^t \|F(v)(\tau) - F(w)(\tau)\|d\tau. \tag{4.8}$$

Using the continuity of the mapping $F : C([s,T_0];U) \to L^\infty([s,T_0];X)$ and (4.1), we obtain that Q is continuous on K.

Finally, by (4.1) and the definition of K, we conclude that the set $G = \{g_u : g_u \in L^\infty([s,T_0];X),\ g_u(t) = -F(u)(t) \text{ for } s \leq t \leq T_0,\ u \in K\}$ is bounded in $L^\infty([s,T_0];X)$. By Remark 1, the set G is uniformly integrable, and so $G + h = \{g_u + h : g_u \in G\}$ is equiintegrable by (H2). Thus by Theorem 2, $Q(K)$ is relatively compact in $C([s,T_0];X)$. Therefore, from Schauder's fixed point theorem it follows that Q has at least one fixed point $u \in K$ which is an integral solution of (E) on $[s,T_0]$. This completes the proof.

REMARK 3. It is known that the evolution operator $U(t,s)$ generated by $\{A(t)\}$ via the formula (2.1) is compact for every $0 \le s < t \le T$ if and only if the following two conditions below hold:

(I) For each $t \in [0,T]$ and $\lambda > 0$, the resolvent operator $J_\lambda(t) = (I + \lambda A(t))^{-1}$ of $A(t)$ from X into $D(A(t))$ is compact.

(II) For each $t_0 \in (s,T]$, the functions $\{t \to U(t,s)x : x \in Y\}$ are equicontinuous at t_0 on bounded subsets $Y \in \overline{D}$.

(by Pavel's result [13]). A possible generalization of Pavel's result would be the characterization of the compactness of $U(t,s)$. In this direction, one condition is (I). Also we point out that Theorem 3 can be adapted to handle the nonlinear time-dependent Voltera integrodifferential equations as in [1].

Finally we formulate two results concerning the continuation of solutions of the problem (E). Since the proofs follow in the standard arguments, we shall not prove in details.

DEFINITION 2. A function $u : [s, T_{\max}) \to X$ is called a maximal integral solution of the problem (E) if

(i) u is an integral solution of the problem (E) on each interval $[s,T]$ with $T < T_{\max}$,

(ii) there is no integral solution $v : [s, T'_{\max}) \to X$ of the problem (E) with $T_{\max} < T'_{\max}$ satisfying $u(t) = v(t)$ for all $s \le t < T_{\max}$.

THEOREM 4. *Assume that* $\{A(t)\}$ *satisfies the conditions* (C1), (C2) *and* (C3) *on* R_+ *(in place of* $[0,T]$*) and that* (H1) *and* (H2) *are satisfied on* R_+. *Let the resolvent operator* $J_\lambda(t) = (I + \lambda A(t))^{-1}$ *from* X *into* $D(A(t))$ *be compact for each* $t \ge 0$ *and* $\lambda > 0$. *Let* $x_0 \in \overline{D} \cap U$ *and* $h \in L^1_{loc}(R_+; X)$ *and let* $v : [s,T) \to X$ *be an integral solution of the problem* (E) *for each* $s \ge 0$. *Then either* v *is maximal or if* v *is not maximal, then there exists at least one maximal integral solution* $u : [s, T_{\max}) \to X$ *of the problem* (E) *with* $s < T < T_{\max}$.

Proof. By the standard arguments, we can easily show that if u_1 and u_2 are integral solutions of the problem (E) on $[s, T_1]$ and $[T_1, T_2]$ with $u_1(T_1) = u_2(T_1)$, respectively, and if $u : [s, T_2] \to X$ is defined by $u(t) = u_1(t)$ on $[s, T_1]$ and $u(t) = u_2(t)$ on $[T_1, T_2]$, then u is an integral solution of the problem (E) on $[s, T_2]$. Thus the proof follows from Zorn's Lemma, and so we omit it.

THEOREM 5. *Assume that* $\{A(t)\}$ *satisfies the conditions* (C1), (C2) *and* (C3) *on* R_+ *and that* (H1) *and* (H2) *are satisfied on* R_+. *Let the resolvent operator* $J_\lambda(t) = (I + \lambda A(t))^{-1}$ *from* X *into* $D(A(t))$ *be compact for each* $t \geq 0$ *and* $\lambda > 0$ *and* $h \in L^1_{loc}(R_+; X)$. *Then for each* $x_0 \in \overline{D} \cap U$ *and* $s \geq 0$, *there exists at least one maximal integral solution* $u : [s, T_{\max}) \to X$ *of the problem* (E), *and either* i) $T_{\max} = \infty$ *or* ii) $T_{\max} < \infty$ *and* $\lim_{T \uparrow T_{\max}} \|u(t)\| = \infty$.

Proof. By Theorem 4, there exists a maximal integral solution u of the problem (E) on a maximal interval of existence $[s, T_{\max})$.

The proof of the fact that if $T_{\max} < \infty$, then $\lim_{T \uparrow T_{\max}} \|u(t)\| = \infty$ follows in the same way as in [12, Theorem 1.2], and so we omit it. Here we only note that F maps bounded subsets into bounded subsets by Remark 1 with (H2) and that the same inequalities as in proof of [12, Theorem 1.2] can be obtained with the slight modifications by using (2.2) and (2.3) in Section 2.

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