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ON THE BCI-G PART OF BCI-ALGEBRAS (III)

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This paper is a continuation of [1] and [3]. In [3], the notion of BCI-G parts of BCI-algebras was introduced and various properties were investigated. In this paper, we consider the inverse of [3; Theorem 15], and define a KG-union BCI-algebra and investigate their properties.

We first recall some definitions and properties in BCI-algebras. By a BCI-algebra we mean a nonempty set X with a binary operation $*$ and a constant 0 satisfying the following conditions:

BCI-1 $(x * y) * (x * z) \leq z * y$,

BCI-2 $x * (x * y) \leq y$,

BCI-3 $x \leq x$,

BCI-4 $x \leq y$ and $y \leq x$ imply $x = y$,

where $x \leq y$ is defined by $x * y = 0$.

A BCI-algebra X satisfying

BCK-5 $0 \leq x$ for all $x \in X$

is called a BCK-algebra.

The following identities hold for any BCI-algebra X:

(1) $x * 0 = x$,

(2) $(x * y) * z = (x * z) * y$,

(3) $0 * (x * y) = (0 * x) * (0 * y)$,

(4) $0 * (0 * (0 * x)) = 0 * x$,

(5) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$.

A nonempty subset I of a BCI-algebra X is called an ideal of X if it satisfies

(i) $0 \in I$,

(ii) $x * y, y \in I$ imply $x \in I$.

For any BCI-algebra X, the BCK-part of X is $X_+ = \{x \in X | 0 \leq x\}$. This is also called the p-radical of X and is an ideal of X.

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DEFINITION 1 ([3]). Let X be a BCI-algebra. For any subset S of X, we define

$$G(S) = \{x \in S | 0 * x = x\}.$$

In particular, if $S = X$ then we say that $G(X)$ is the BCI-G part of X.

The following property is obvious.

(6) $G(X) \cap X_+ = \{0\}$.

PROPOSITION 1 ([3]). *If S is a subalgebra of a BCI-algebra X, then $G(S)$ is a subalgebra of X.*

LEMMA 1 ([5]). *For a subalgebra S of a BCI-algebra X, $G(S)$ is an ideal of X if and only if for any $x \in X$ and any $a \in G(S)$,*

$$x = (x * a) * (0 * a).$$

For any subalgebra S of a BCI-algebra X and any element a in S, we use a_r denote the selfmap of S defined by $a_r(x) = x * a$ for all $x \in S$.

In [3], we proved the following.

PROPOSITION 2 ([3]). *Let S be a subalgebra of a BCI-algebra X. If $G(S)$ is an ideal of X, then a_r is bijective for any $a \in G(S)$.*

Now we consider the inverse of Proposition 2 above.

THEOREM 1. *Let S be a subalgebra of a BCI-algebra X. If for all $a \in G(S)$, the map $a_r : S \to S, x \mapsto x * a$ for any $x \in S$, is injective. then $G(S)$ is an ideal of X.*

Proof. Let a_r be injective for any $a \in G(S)$. Then we have

$$((x * a) * (0 * a)) * x = ((x * x) * a) * (0 * a) = 0,$$

and so $(x * a) * (0 * a) \leq x$, which implies from (5) that

$$0 * a \leq (x * ((x * a) * (0 * a))) * a.$$

Moreover,

$$(x * ((x * a) * (0 * a))) * a = (x * a) * ((x * a) * (0 * a)) \leq 0 * a.$$

Hence $(x*((x*a)*(0*a)))*a = 0*a$, i.e., $a_r(x*((x*a)*(0*a))) = a_r(0)$. Since a_r is injective, we have that

$$x * ((x * a) * (0 * a)) = 0, \text{ i.e., } x \leq (x * a) * (0 * a).$$

By Lemma 1, $G(S)$ is an ideal of X. The proof is complete.

COROLLARY 1. *Let S be a subalgebra of a BCI-algebra X. Then $G(S)$ is an ideal of X if and only if a_r is injective for any $a \in G(S)$.*

PROPOSITION 3. *Let X be a BCI-algebra. Then for all $x \in X$ and for all $y \in G(X)$, we have $x * (x * y) = y$.*

Proof. Let $x \in X$ and $y \in G(X)$. Then we have

$$\begin{aligned} y * (x * (x * y)) &= (0 * y) * (x * (x * y)) \\ &= (0 * (x * (x * y))) * y \\ &= (0 * (x * (x * y))) * (0 * y) \\ &= 0 * ((x * (x * y)) * y) \\ &= 0 * 0 \\ &= 0, \end{aligned}$$

that is, $y \leq x*(x*y)$. Combining BCI-2 and BCI-4, we get $x*(x*y) = y$. The proof is complete.

THEOREM 2. *Let S be a subalgebra of a BCI-algebra X. Then $G(S)$ is an ideal of X if and only if X satisfies the following condition:*
(7) *$b * a \neq 0 * a$ for all $b \in X_+ - \{0\}$ and all $a \in G(S)$.*

Proof. Let $G(S)$ be an ideal of X. Suppose that there exist $b \in X_+ - \{0\}$ and $a \in G(S)$ such that $b * a = 0 * a$. It follows from Lemma 1 that $b = (b * a) * (0 * a) = 0$, a contradiction.

Conversely, we prove that the condition (7) implies the condition in Lemma 1. Suppose that $x \neq (x*a)*(0*a)$ for some $x \in X$ and $a \in G(S)$. Since $((x*a)*(0*a))*x = 0$, therefore $x*((x*a)*(0*a)) \neq 0$. Moreover,

$$\begin{aligned} 0 * (x * ((x * a) * (0 * a))) &= (0 * x) * (0 * ((x * a) * a)) \\ &= (0 * x) * (((0 * x) * (0 * a)) * (0 * a)) \\ &= (0 * x) * (((0 * x) * (0 * a)) * a) \\ &= (0 * x) * (((0 * a) * (0 * a)) * x) \\ &= (0 * x) * (0 * x) \\ &= 0. \end{aligned}$$

Thus we have $x * ((x * a) * (0 * a)) \in X_+ - \{0\}$. But

$$(x * ((x * a) * (0 * a))) * a = 0 * a,$$

which contradicts to the condition (7). The proof is complete.

DEFINITION 2. A BCI-algebra X is said to be KG-union if X is represented by the union of BCK-part and BCI-G part, that is $X = X_+ \cup G(X)$.

EXAMPLES. 1. Every BCK-algebra is KG-union.
2. Let $X = \{0, 1, 2, 3\}$ in which $*$ is defined by

*	0	1	2	3
0	0	0	0	3
1	1	0	0	3
2	2	2	0	3
3	3	3	3	0

It is easy to check that $(X; *, 0)$ is a KG-union BCI-algebra.

THEOREM 3. *Let X be a BCI-algebra. If the identity $x*(0*(0*x)) = x$ holds for all $x \in X$, then X is a BCK-algebra.*

Proof. Assume that $x * (0 * (0 * x)) = x$ for all $x \in X$. Then we have

$$\begin{aligned} 0 * x &= 0 * (x * (0 * (0 * x))) \\ &= (0 * x) * (0 * (0 * (0 * x))) \\ &= (0 * x) * (0 * x) \\ &= 0. \end{aligned}$$

Hence $x \in X_+$. This shows that X is a BCK-algebra. The proof is complete.

COROLLARY 2. *Let X be a BCI-algebra. If the identity $x * (0 * (0 * x)) = x$ holds for all $x \in X$, then X is KG-union.*

THEOREM 4. *If X is a KG-union BCI-algebra, then $x * a = 0 * a$ for all $x \in X_+$ and all $a \in G(X) - \{0\}$.*

Proof. Let $X = X_+ \cup G(X), x \in X_+$ and $a \in G(X) - \{0\}$. Then we have

$$0 * (x * a) = (0 * x) * (0 * a) = 0 * a = a \neq 0.$$

It follows that $x * a \notin X_+$, so that $x * a \in G(X)$. Since $G(X)$ is a subalgebra of X, therefore $(x * a) * (0 * a) \in G(X)$. Now

$$\begin{aligned} 0 * ((x * a) * (0 * a)) &= (0 * (x * a)) * (0 * (0 * a)) \\ &= ((0 * x) * (0 * a)) * (0 * (0 * a)) \\ &= (0 * (0 * a)) * (0 * (0 * a)) = 0, \end{aligned}$$

which means that $(x * a) * (0 * a) \in X_+$. Hence $(x * a) * (0 * a) \in X_+ \cap G(X) = \{0\}$, and so $(x * a) * (0 * a) = 0$. On the other hand, noticing $(0 * a) * (x * a) \in G(X)$; we get $(0 * a) * (x * a) = 0$. Consequently, $x * a = 0 * a$. The proof is complete.

The converse of Theorem 4 is not true as shown in the following example.

EXAMPLE. Let $X = \{0, x, y, z\}$ in which $*$ is defined by

$*$	0	x	y	z
0	0	z	y	x
x	x	0	z	y
y	y	x	0	z
z	z	y	x	0

It is easy to check that $(X; *, 0)$ is a BCI-algebra, $X_+ = \{0\}$ and $G(X) = \{0, y\}$. Thus X is not KG-union, but X satisfies $x * a = 0 * a$ for all $x \in X_+$ and all $a \in G(X) - \{0\}$.

LEMMA 2 ([8]). *Let X be an abstract algebra of type (2,0) with a binary operation $*$ and a constant 0. X is a BCI-algebra if and only if it satisfies the conditions BCI-1, BCI-4 and (1).*

THEOREM 5. *Let X be a BCK-algebra, Y a BCI-algebra with $X \cap G(Y) = \{0\}$. Then there exists a KG-union BCI-algebra.*

Proof. Both the operation on X and the operation on $G(Y)$ are denoted by $*$. Consider $Z = X \cup G(Y)$ and define a binary operation $\circ$ on Z as follows:

$$x \circ y = \begin{cases} x * y, & \text{if } x, y \in X \text{ or } x, y \in G(Y); \\ 0 * y, & \text{if } x \in X \text{ and } y \in G(Y) - \{0\}; \\ x, & \text{if } x \in G(Y) - \{0\} \text{ and } y \in X. \end{cases}$$

We show that $(Z; \circ, 0)$ satisfies the conditions BCI-1, BCI-4 and (1). By the definition of the operation $\circ$, these conditions need only checking for the elements which are not all in X and not all in $G(Y)$. The condition (1) is directly holds.

Let $x \in X$ and $y \in G(Y)$. If $x \circ y = 0 = y \circ x$, then $y = y \circ x = 0$. This means that $y, x \in X$, so that $x = y = 0$. Thus the condition BCI-4 holds.

To prove the condition BCI-1 hold, it is sufficient to consider the following cases:

1. $x, y \in X$ and $z \in G(Y)$,
2. $x, z \in X$ and $y \in G(Y)$,
3. $x \in X$ and $y, z \in G(Y)$,
4. $x \in G(Y)$ and $y, z \in X$,
5. $x, z \in G(Y)$ and $y \in X$,
6. $x, y \in G(Y)$ and $z \in X$.

In each case, if the one or two of elements of $G(Y)$ is equal to 0, then BCI-1 is clearly true. Thus we now assume that every element of $G(Y)$ is nonzero.

If $x, y \in X$ and $z \in G(Y)$, then $((x \circ y) \circ (x \circ z)) \circ (z \circ y) = ((x * y) \circ (0 * z)) \circ z = ((x * y) \circ z) \circ z = (0 * z) \circ z = z * z = 0$;

If $x, z \in X$ and $y \in G(Y)$, then $((x \circ y) \circ (x \circ z)) \circ (z \circ y) = ((0 * y) \circ (x * z)) \circ (0 * y) = (y \circ (x * z)) \circ y = y \circ y = y * y = 0$;

If $x \in X$ and $y, z \in G(Y)$, then $((x \circ y) \circ (x \circ z)) \circ (z \circ y) = ((0 * y) \circ (0 * z)) \circ (z * y) = ((0 * y) * (0 * z)) * (z * y) = 0$;

If $x \in G(Y)$ and $y, z \in X$, then $((x \circ y) \circ (x \circ z)) \circ (z \circ y) = (x \circ x) \circ (z * y) = 0 * (z * y) = 0$;

If $x, z \in G(y)$ and $y \in X$, then $((x \circ y) \circ (x \circ z)) \circ (z \circ y) = (x \circ (x * z)) \circ z = (x * (x * z)) * z = 0$;

If $x, y \in G(Y)$ and $z \in X$, then $((x \circ y) \circ (x \circ z)) \circ (z \circ y) = ((x * y) \circ x) \circ (0 * y) = ((x * y) * x) * y = (0 * y) * y = y * y = 0$.

Therefore $(Z; \circ, 0)$ is a BCI-algebra.

Since $0 \circ x = 0 * x$ for all $x \in Z$, we have

$$0 \circ x = 0 \quad \text{if and only if} \quad 0 * x = 0,$$
$$0 \circ x = x \quad \text{if and only if} \quad 0 * x = x.$$

Thus we have

$$\begin{aligned} Z_+ &= \{z \in Z | 0 \circ z = 0\} \\ &= \{z \in Z | 0 * z = 0\} \\ &= \{z \in X \cup G(Y) | 0 * z = 0\} \\ &= \{z \in X | 0 * z = 0\} \cup \{z \in G(Y) | 0 * z = 0\} \\ &= \{z \in X | 0 * z = 0\} \cup \{0\} \\ &= \{z \in X | 0 * z = 0\} \\ &= X, \end{aligned}$$

and

$$\begin{aligned} G(Z) &= \{z \in Z | 0 \circ z = z\} \\ &= \{z \in Z | 0 * z = z\} \\ &= \{z \in X \cup G(Y) | 0 * z = z\} \\ &= \{z \in X | 0 * z = z\} \cup \{z \in G(Y) | 0 * z = z\} \\ &= \{0\} \cup \{z \in G(Y) | 0 * z = z\} \\ &= \{z \in G(Y) | 0 * z = z\} \\ &= G(Y). \end{aligned}$$

This completes the proof.

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