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UNITARY SERIES OF $GL_2(\mathbb{R})$ AND $GL_2(\mathbb{C})$

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1. Introduction

This paper studies the realization of irreducible unitary representations of $GL_2(\mathbb{R})$ and $GL_2(\mathbb{C})$ by Bargmann's classification [1]. Since the representations of general matrix groups can be obtained by the extensions of characters of a special linear group, we shall follow to a large extent the pattern of the results in [5], [6], and [8]. This article is divided into two sections. In the first section we describe the realization of principal series and discrete series and complementary series of $GL_2(\mathbb{R})$. The last section is devoted to the derivation of principal series and complementary series of $GL_2(\mathbb{C})$.

2. Unitary series of $GL_2(\mathbb{R})$

First we find the unitary principal series of $GL_2(\mathbb{R})$.

PROPOSITION 2.1. *An irreducible unitary representation of* $GL_2(\mathbb{R})$ *is the family of induced representations* Π_S, *where* S *is a pure imaginary number and* $\Pi_S(y)f(x) = \operatorname{sgn}(bx+d)|bx+d|^{-1-S} f\left(\frac{ax+c}{bx+d}\right)$.

Moreover, Π_S *is induced by the character* $\mu_S : P \to \mathbb{C}^\times$ *and* P *is the parabolic subgroup of* $GL_2(\mathbb{R})$ *and is therefore an upper or lower triangular group of matrices.*

Proof. Let $G = GL_2(\mathbb{R})$ and consider an Iwasawa decomposition of G. Then $G = PK = MNK$, where $\rho = MN$, and M, N are closed unimodular subgroups, and M normalizes N and K is a maximal compact subgroup and has measure 1. Therefore, M is the group of diagonal matrices, and N is the group of matrices of the form $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$. Let Δ be the modular function on ρ such that $\Delta(\rho) = \Delta(mn) = \alpha(m)$,

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where α is a continuous homomorphism $\alpha : M \to \mathbb{R}^{\times}$. We also require $\int_N f(mnm^{-1})dn = \alpha(m)^{\gamma} \int_N f(n)dn$ where f is a continuous function on G with compact support and dn is a Haar measure. Let $\rho(m) = \alpha(m)^{\frac{1}{2}}$, and let $\rho_S(x) = \rho_S(mnk) = \rho(m)^{S+1}$, $S \in \mathbb{C}$. A continuous homomorphism $\mu_S : \rho \to \mathbb{C}^{\times}$ is a character such that $\mu_S(mn) = \rho(m)^S$.

Let $H(S)$ be the space of the representations Π_S induced by μ_S, i.e., $\Pi_S(x)f(y) = f(yx)$, $f \in H(S)$, and $f(\rho y) = \Delta(\rho)^{\frac{1}{2}}\mu_S(\rho)f(y)$. Then $f(mny) = \rho(m)^{S+1}f(y)$ and $f|_K$ is in $L^2(K)$ and $H(S)$ is a Hilbert space under the $L^2(K)$-norm, and

$$
\begin{aligned}
\langle \Pi(x)\rho_S, \rho_S\rangle &= \int_K \Pi(x)\rho_S(k)\overline{\rho_S(k)}dk \\
&= \int K\rho_S(kx)dk.
\end{aligned}
$$

Thus μ_S is unitary, i.e., $|\mu_S| = 1$ iff S is pure imaginary. Consequently, $\{\Pi_S\} = \{\rho(m)^{-S-1}f(any)\}$ where S is pure imaginary, is an irreducible unitary series of G. Since $GL_2(\mathbb{R})$ operates on the upper or lower half plane with $g(z) = \frac{az+c}{bz+d}$ where

$$
g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \; g \in G \quad \text{and} \quad z = x + iy, \; y \neq 0.
$$

Therefore we can construct the realizations of this series as the above result.

Remarks. (1) The representation of the above Proposition is called the unitary principal series of $GL_2(\mathbb{R})$.

(2) Since the spherical function $\varphi_S(x) = \int_K \rho(kx)^{S+1}dk$ and

$$
\varphi_S(m) \approx \frac{1}{\sqrt{n}}\rho(m)^{S-1}\frac{\Gamma\left(\frac{S}{2}\right)}{\Gamma\left(\frac{S+1}{2}\right)},
$$

we can rewrite the unitary principal series of $GL_2(\mathbb{R})$ by the means of Gamma function [5].

THEOREM 2.2. *There is a natural representation Π_n^+ of $GL_2(\mathbb{R})$ on $L^2(R^2)$ which contains each member of the holomorphic discrete series. This can be called the discrete series representation and its realization is as follows:*
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$$\Pi_n^+ = \Big\{ f \text{ holomorphic for } Im\, z > 0 : \|f\|^2 = \iint_{Im\, z>0} |f(z)|^2 (1-r^2)^{n-2} r\, dr\, d\theta < \infty \Big\}$$

$$\Pi_n^+(g) f(z) = (bz+d)^{-n} f\left(\frac{az+c}{bz+d}\right), \quad n \geq 2.$$

The representation Π_n^- is obtained by using complex conjugates.

Proof. Let $\mathcal{F}$ be the Lie algebra of G. Then we can introduce the following elements of $\mathcal{F}$ which are identified with the Lie algebra of 2×2 matrices

$$U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$V_+ = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \qquad V_- = \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$$

$$X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad X_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

as well as

$$D = X_+X_- + X_-X_+ + \frac{Z^2}{2}.$$

These elements can be thought of as belonging to the universal enveloping algebra of $\mathcal{F} \otimes_{\mathbb{R}} \mathbb{C}$. Thus they generate a representation of $\tilde{G}$, the universal covering group of $GL_2(\mathbb{R})$. Note that the kernel of the covering map $\tilde{G} \to GL_2(\mathbb{R})$ consists of $\exp(2\pi k u)$, $k \in Z$. The spectrum of $(\frac{1}{i})U$ consists of all positive integers, since the harmonic oscillator hamiltonian has as its spectrum all even positive integers. Thus $\exp(2\pi k u)$ is represented by the identity operator on $L^2(\mathbb{R}^2)$, so the representation of

$\tilde{G}$ actually comes from a representation of $GL_2(\mathbb{R})$. Let us denote this last representation by γ. Then we can decompose γ by

$$L^2(R^2) = \bigoplus_{k=-\infty}^{\infty} H_k$$

where $H_k = \{e^{ik\theta}g(r)GL^2(R^2),\ g \in GL_2(\mathbb{R})\}$. The $ax+b$ group acts irreducibly on each H_k, this representation being intertwined with a "standard model" via a Hankel transform. It follows *a fortiori* that $GL_2(\mathbb{R})$ acts irreducibly on each K_k, via γ. Since spec$(\frac{1}{i})U$ consists of positive integers, it is clear that each irreducible component γ_k must be a holomorphic discrete series representation.

We claim that on H_k, the smallest element of spec$(\frac{1}{i})U$ is $|k|+1$. In fact, consider our operators on Bargmann-Fock space $\mathcal{H}$, a Hilbert space of entire functions on $\mathbb{C}^2$ with orthonormal basis $(2/\alpha!)^{1/2}Z^\alpha$, $\alpha = (\alpha_1, \alpha_2) \geq 0$, on which $(\frac{1}{i})U$ acts as multiplication by $|\alpha|+1$. Then the rotation group action $h(x) \mapsto h(kx)$, $k \in O(2)$, $f \in L^2(R^2)$, is intertwined to a subgroup of the natural $U(2)$ action on $\mathcal{H}$, $u(z) \mapsto u(\sigma z)$, $\sigma \in u(2)$. As we know, each irreducible representation of $U(2)$ is contained exactly once in $\mathcal{H}$, on one of the spaces $E_j =$ span of $\{Z^\alpha : |\alpha| = j\}$. On the space E_j, the operator $(\frac{1}{i})U$ is the scalar $j+1$, and the skew adjoint generator of the $O(2)$ action on E_j takes the values $-j_1 - j + 1, \ldots, j-1, j_1$, multiplied by i. This proves that the smallest eigenvalue of $(\frac{1}{i})U$ on the $\pm ij$ eigenspace of the generator of the $O(2)$ action is $j+1$, as asserted. It follows that $\gamma_k \approx \Pi^+_{|k|+1}$. Thus γ contains Π^+_n exactly twice, for each $n \geq 2$, and it contains Π^+_1 once. Therefore, the realization of the holomorphic discrete series is given on

$$\mathcal{H}_n = \{f(z) \text{ holomorphic for } \operatorname{Im} z > 0 : |f(z)|^2(1-r^2)^{n-2} r\,dr\,d\theta < \infty\}, \quad n \geq 2.$$

We can write $\Pi^+_n(g)f(z) = (bz+d)^{-r} f\left(\frac{az+c}{bz+d}\right)$.

Remark. $\Pi^+_n(k_\theta)f(z) = e^{in\theta}f(e^{2i\theta}z)$, where $k_\theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & i^{-i\theta} \end{pmatrix}$, $\theta \in R/2\pi Z$. Thus spec$(\frac{1}{i})U$ has n as its smallest element. The Hilbert space

$\mathcal{H}_n$ is nontrivial precisely for $n \geq 2$, so we obtain all the holomorphic discrete series representations Π_n^+ with $n \geq 2$. We do not get Π_1^+ in this fashion. For any f, f' in the representation space of Π_n^+, $(\Pi_n^+(g)f, f')$ belongs to $L^2(GL_2(\mathbb{R}))$ if $n \geq 2$, but not if $n = 1$. This implies that $\Pi_1^{\pm}$ does not have such an appearance.

PROPOSITION 2.3. *There is a unitary representation on the interval* $-1 < S < 1 (S \neq 0)$.

Proof. Let Π be an irreducible admissible representation of $GL_2(\mathbb{R})$ on a space H and let φ_n be defined on $H(S)$ by

$$\begin{aligned}
&\varphi_n \text{ (Iwasawa decomposition of } GL_2(\mathbb{R}))\\
&= \varphi_n\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right)\\
&= \mu_1(a_1)\mu_2(a_2)\left|\frac{a_1}{a_2}\right|^{\frac{1}{2}} e^{in\theta}\\
&= y\frac{s+1}{2}e^{in\theta},
\end{aligned}$$

where $\mu_i(t) = |t|^{S_i}\left(\frac{t}{|t|}\right)^{m_i}$ is a quasi-character on $R^\times$ and S_i is complex number and $S = S_1 - S_2$ and $m = |m_1 - m_2|$ such that $\mu_1\mu_2^{-1}(t) = |t|^s\left(\frac{t}{|t|}\right)^m$ and m and n have the same parity, and y is as in Theorem 2.1. For any differentiable function f on $GL_2(\mathbb{R})$ and any compactly supported distribution μ we define $\lambda(\mu)f$ by

$$\lambda(\mu)f(g) = \mu^\vee(\rho(g)f) \quad \text{and} \quad \rho(\mu)f \ \text{ by } \ \rho(\mu)f(g) = \mu(\lambda(g^{-1})f).$$

Then we can get the following relations;

$$\begin{aligned}
\rho(V_+)\varphi_n &= (S + 1 + n)\varphi_{n+2},\\
\rho(V_-)\varphi_n &= (S + 1 - n)\varphi_{n+2},
\end{aligned}$$

and

$$\rho(D)\varphi_n = \frac{S^2 - 1}{2}\varphi_n$$

where V_+ and V_- and D are as in the proof of Theorem 2.2.

On the other hand, we can obviously choose the length of v_0 arbitrarily, say $a_0 > 0$, i.e., $\langle v_0, v_0\rangle = {a_0}^2$ where $a_0 > 0$.

From $(\rho(V_+)\rho(V_-) - \rho(V_-)\rho(V_+))v_n = -4i\varphi(U) = 4nv_n$, we get the condition $C_n - C_{n+2} = 4n$. From the skew-Hermitian conditions $\langle\rho(V_+)v_n, v_{n+2}\rangle = \langle v_n, -\rho(V_-)v_{n+2}\rangle$, we get the relation $a_{n+2}^2 = -\overline{C}_{n+2}a_n^2$. This shows that C_n is necessarily real and negative, and a choice of C_0 completely determines C_n for all n. The possible unitarization therefore depends uniquely on the choice of a negative number C_0. A similar discussion can be carried out for the case when the parity is odd. Consequently, we see that the condition for C_0 to be negative amounts to $(S^2 - 1) = (S+1)(S-1) < 0$. This amounts to S being pure imaginary or S real and $-1 < S < 1$. The case when $-1 < S < 1 (S \neq 0)$ can be unitarized by completing $H(K)$.

COROLLARY 2.4. *We can get the complementary series of $GL_2(\mathbb{R})$ arising from the intervals $0 < S < 1$ and $-1 < S < 0$ as follows:*

$$H(S) = \left\{ f : \mathbb{R} \to \mathbb{C} \middle| \|f\|^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{f(x)\overline{f(y)}dx\,dy}{|x-y|^{1\pm S}} < \infty \right\}$$

and set $\Pi_S(g)f(x) = |bx+d|^{S\mp 1} f\left(\frac{ax+c}{bx+d}\right)$.

3. Unitary series of $GL_2(\mathbb{C})$

In this section we induce two different unitary irreducible representations of $GL_2(\mathbb{C})$. They are unitary principal series and unitary complementary series [8]. $GL_2(\mathbb{C})$ does not have a compact Cartan subgroup, so it does not have a discrete series representation [8].

EXAMPLE 3.1. Consider the finite dimensional irreducible representation of $GL_2(\mathbb{C})$. The matrix $\begin{pmatrix} 1 & 0 \\ z_{21} & 1 \end{pmatrix}$ is identified by a single complex parameter $x = z_{21}$, and we may write $f\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = f(x)$, if f is continuous complex valued function on $GL_2(\mathbb{C})$. Then, $f\left(\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\overline{g}\right) =$

$f\left(\frac{ax+c}{bx+d}\right)$, where $g=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in GL_2(\mathbb{C})$. We conclude that $\lambda_2+k_{22}=bx+d$,

$$\lambda_1=k_{11}=\frac{\det\left[\begin{pmatrix} 1 & 0 \\ x & 1\end{pmatrix}\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right]}{bx+d}=\frac{\det\begin{pmatrix} a & b \\ c & d\end{pmatrix}}{bx+d},$$

where λ_i is the entry of the diagonal matrix group $D=\left\{\delta=\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2\end{pmatrix}\right\}$ and k_{ii} is the entry of the upper triangular matrix group

$$K=\left\{k=\begin{pmatrix} k_{11} & k_{12} \\ 0 & k_{22}\end{pmatrix}\right\}.$$

The character $\alpha(\delta)$ is a continuous character of D. Then we can have $\alpha(\delta)=\lambda_1^{p_1}\,\overline{\lambda}_1^{q_1}\,\lambda_2^{p_2}\,\overline{\lambda}_2^{q_2}$ where p_1,q_1,p_2,q_2 are any complex numbers for which p_1-q_1 and p_2-q_2 are integers. Additionally, the necessary and sufficient condition to be an inductive character $\alpha(\delta)$ is that $r=p_2-p_1$ and $s=q_2-q_1$ are nonnegative integers, and $\alpha\left(\begin{pmatrix} 1 & 0 \\ x & 1\end{pmatrix}\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right)=\left(\frac{\det g}{bx+d}\right)^{p_1}\left(\frac{\overline{\det g}}{ax+c}\right)^{q_1}(ax+c)^{p_2}(\overline{bx}+\overline{c})^{q_2}=(\det g)^{p_1}(\overline{\det g})^{q_1}(bx+d)^{p_2-p_1}(\overline{bx}+\overline{d})^{q_2-q_1}$. Therefore, the finite dimensional irreducible representation of $GL_2(\mathbb{C})$ is $F^n_\alpha(g)f(z)=\alpha\left(\begin{pmatrix} 1 & 0 \\ x & 1\end{pmatrix}\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right)\cdot f\left(\frac{ax+c}{bx+d}\right)$.

THEOREM 3.2. *The nontrivial unitary principal series of $GL_2(\mathbb{C})$ is realized as follows:*

$$P_{(\lambda+1,\mu+1)}(g)f(z)=(bz+d)^{\lambda}(\overline{bz}+\overline{d})^{\mu}f(\frac{az+c}{bz+d})$$

where

$$g=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in GL_2(\mathbb{C}).$$

Proof. Let $R(g)f(z)=f(g^tz)$, $g\in GL_2(\mathbb{C})$, $z\in\mathbb{C}^2$. In this case, $R(g)$ commutes with the group of complex dilations: $D(a)f(z)=|a|^2f(az)$,

$a \in \mathbb{C}^\times$. We expect to decompose R into irreducible via the spectral decomposition of D.

Since $\mathbb{C}^\times = S^1 \times R^+$, this decomposition is accomplished by combining Fourier series and the Mellin transform. Thus given $\beta(\theta, t)$ on $S^1 \times R^+$, set $S\beta(n, S) = \int_0^\infty \int_{S^1} \beta(\theta, t) e^{in\theta} t^{-iS} d\theta\, dt$ for $n \in Z$, $x \in R$. Then we get the inversion formula

$$\beta(\theta, t) = (2\pi)^{-2} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} S\beta(n, S) e^{in\theta} t^{iS-1} dS$$

and the Plancherel formula

$$\int_{R^+} \int_{S^1} |\beta(\theta, t)|^2 t\, d\theta\, dt = (2\pi)^{-2} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} |S\beta(n, S)|^2 dS.$$

Thus, if we define

$$P_{n,S} f(z) = \int_0^\infty \int_{S^1} f(re^{i\theta} z) e^{-in\theta} r^{-iS} d\theta\, dr,$$

we see that, for $f \in C_0^\infty(R^4)$, $P_{n,S} f$ belongs to the space

$$\mathcal{H}_{n,S} = \{g \in L^2(R^4 \backslash 0) : g(re^{i\theta} z) = r^{iS-1} e^{in\theta} g(z),$$
$$r > 0,\ e^{k\theta} \in S^1, \text{ and } \int |g|^2 < \infty\}$$

We make $\mathcal{H}_{n,S}$ into a Hilbert space, with norm square $\int_{R^4\backslash 0} |g|^2$. Therefore we get a representation on $\mathcal{H}_{n,S}$ given by $\pi_{n,S}(g) f(z) = f(g^t z)$, $f \in \mathcal{H}_{n,S}$ and then we can induce the homogeneity condition

$$g(az) = a^\lambda \overline{a}^\mu g(z), \quad a \in \mathbb{C}^\times$$

with $\lambda = \frac{1}{2}(iS - 1 + n)$, $\mu = \frac{1}{2}(iS - 1 - n)$. By restricting the argument of an element of $\mathcal{H}_{n,S}$ to the hyperplane $z_2 = 1$ and applying the result of Example 3.1, e can make $\mathcal{H}_{n,S}$ unitarily equivalent to a representation as given in the statement of the theorem.

COROLLARY 3.3. *The complementary series of $GL_2(\mathbb{C})$ is given by the formula*

$$\mathcal{C}^S(g)f(z) = (bz+d)^{t-1}(\bar{b}\bar{z}+\bar{d})^{t-1} f\left(\frac{az+c}{bz+d}\right)$$

with the inner product

$$(f_1, f_2) = (i/2)^2 \int_{\mathbb{C}^2} |z_1 - z_2|^{-2t-2} f_1(z_1)\overline{f_2(z_2)}\, dz_1\, \overline{dz_1}\, dz_2\, \overline{dz_2}$$

and $\lambda + 1 = \mu + 1 = t$.

Proof. Applying Theorem 3.2 and using the results of [5] and [8], we can get the above complementary series.

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