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ON ENDOMORPHISM RINGS OF CS-MODULES

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Endomorphism rings of Artinian modules need not be semiperfect by a result of Camps and Menal [4], which answers in the negative to a question of Crawley and Jónsson [5]. However it was shown by Camps and Dicks [3] that endomorphism rings of Artinian modules are semilocal. After observing some endomorphism rings of a certain class of Artinian modules, we investigate some interesting structures of endomorphism rings of CS-modules.

For a ring R with an ideal I, we say that an idempotent f of the ring R/I can be *lifted* to an idempotent of R if there is an idempotent e in R such that $e + I = f$ in R/I. For example, it is a standard fact that if I is nil then every idempotent of the ring R/I can be lifted to an idempotent of R. Recall that a ring R is called *semilocal* if $R/J(R)$ is a right Artinian ring, where $J(R)$ is the Jacobson radical of R. And a semilocal ring R is called *semiperfect* if every idempotent of the ring $R/J(R)$ can be lifted to an idempotent of R. Furthermore, a semilocal ring R is called *semiprimary* if $J(R)$ is nilpotent.

A set of orthogonal idempotents $e_1, e_2, \ldots, e_n$ in a ring is *complete* if $e_1 + e_2 + \cdots + e_n = 1$. By [1, Corollary 27.7, p. 304], it is a standard fact that a ring R with a set of complete orthogonal idempotents is semiperfect if and only if e_iRe_i is semiperfect for every $i = 1, 2, \ldots, n$. From this fact, we obtain the following fact by noting that the endomorphism ring of a *uniform* Artinian module is local.

PROPOSITION 1. *The endomorphism ring of an Artinian module which is a direct sum of uniform modules is a semiperfect ring.*

An R-module is called a *CS-module* if every submodule of M is essential in a direct summand of M. For examples, all completely reducible, injective, quasi-injective, and continuous modules are of course CS-modules. For more detail for CS-modules, see [2 and 10].

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Recall that a module is called *uniform* if every non-zero submodule is essential. Every Artinian CS-module is a finite direct sum of uniform modules. For, if M is an Artinian CS-module then $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ with M_i an indecomposable module. By [2], each summand M_i is also a CS-module and hence M_i is a uniform module.

COROLLARY 2. *The endomorphism ring of an Artinian CS-module is a semiperfect ring.*

For an R-module M with the endomorphism ring $S = \mathrm{End}_R(M)$, let $N(S)$ be the set of endomorphisms of M whose kernels are essential in M. Then, if M is Artinian, we have $N(S) \subseteq J(S)$ where $J(S)$ is the Jacobson radical of S. In fact, letting s be in $N(S)$, then $\mathrm{Ker}(s)$ is essential in M. Thus, from the fact that $\mathrm{Ker}(s) \cap \mathrm{Ker}(1-s) = 0$, we have $\mathrm{Ker}(1-s) = 0$. Thus $1-s$ is an isomorphism by observing that a one to one R-module homomorphism of an Artinian module is onto. Therefore $1-s$ is invertible in S for any s in $N(S)$ and hence $N(S) \subseteq J(S)$.

Also there is an Artinian module which is a direct sum of uniform modules, but it is not a CS-module. For example, by [8, Corollary 2, p.207], $\mathbb{Z}$-module $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ is not a CS-module because $S = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Soc}(\mathbb{Z}/8\mathbb{Z}), \mathrm{Soc}(\mathbb{Z}/4\mathbb{Z}))$ is not extended to $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/8\mathbb{Z}, \mathbb{Z}/4\mathbb{Z})$, where $\mathbb{Z}$ is the ring of integers and $\mathrm{Soc}(-)$ is the socle of a module. Actually this extending property is related to the equality $N(S) = J(S)$.

For the observation of $N(S) = J(S)$, when an Artinian module M is a direct sum of uniform modules, say $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$. Now, for our convenience, we may write off $S = \mathrm{End}_R(M)$ as $S = (\mathrm{Hom}_R(M_j, M_i))_{i,j=1}^n$ in the matrix form. Therefore when $s \in S$, we write $s = (s_{ij})$ with s_{ij} in $\mathrm{Hom}_R(M_j, M_i)$. Also in our situation, $\mathrm{Soc}(M) = U_1 \oplus U_2 \oplus \cdots \oplus U_n$ with $U_i = \mathrm{Soc}(M_i)$ for $i = 1, 2, \ldots, n$ and each U_i a simple R-module. Thus for $s = (s_{ij})$ in S, it follows that $s \in N(S)$ if and only if $s_{ij}(U_j) = 0$ for all $i, j = 1, 2, \ldots, n$. Also when $f \in \mathrm{Hom}_R(M_j, M_i)$, denote fE_{ij} by the matrix in S with f in (i,j)-position and 0 elsewhere.

THEOREM 3. *Assume that M is an artinian R-module with a decomposition $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$, where M_i is a uniform module for $i = 1, 2, \ldots, n$. Then the following statements are equivalent:*

(1) $N(S) = J(S)$.

(2) *Every R-monomorphism from M_i to M_j is an isomorphism for $i, j = 1, 2, \ldots, n$.*

Proof. (1) implies (2). Assume that $N(S) = J(S)$. Let $e_1 = E_{11} + N(S)$, $e_2 = E_{22} + N(S)$, ... , $e_n = E_{nn} + N(S)$ in the ring $S/N(S)$ which is a semisimple Artinian ring by Proposition 1. Then $e_1, e_2, \ldots, e_n$ are complete orthogonal idempotents.

Now for proving that every R-monomorphism from M_i to M_j is an isomorphism, we just need to check when $i = 1$ and $j = 2$. When $i = j$, since M_i is Artinian, every R-monomorphism from M_i to itself is of course an isomorphism. For the other case, our method for $i = 1$ and $j = 2$ can also be applied.

Let $f \in \mathrm{Hom}_R(M_2, M_1)$ be an R-monomorphism. Then $f(U_2) \neq 0$ and so we have $f(U_2) = U_1$. In considering $x = fE_{12} + N(S)$ in $e_1(S/N(S))e_2$, there is $y = gE_{21} + N(S)$ in $e_2(S/N(S))e_1$ such that $x = xyx$. Thus $(f - fgf)E_{12}$ is in $N(S)$ and hence $(f - fgf)(U_2) = 0$. But since $f(U_2) = U_1$, $g(U_1) \neq 0$ and so $g(U_1) = U_2$. Therefore $fg(U_1) = U_1$ and $gf(U_2) = U_2$. Thus fg and gf are isomorphisms and so f is an isomorphism.

(2) implies (1). Conversely, assume that every R-monomorphism from M_i to M_j is an isomorphism for $i, j = 1, 2 \ldots, n$. To prove that $N(S) = J(S)$, it is enough to show that the ring $S/N(S)$ is a von Neumann regular ring. So we need to check that for any i, j and for any x in $e_i(S/N(S))e_j$, there is y in $e_j(S/N(S))e_i$ such that $x = xyx$ by [7, Lemma 1.6, p.3].

When $i = j$, it is obvious. For the other case, say $i = 1$ and $j = 2$. Let $x = fE_{12} + N(S) \in e_1(S/N(S))e_2$ with f in $\mathrm{Hom}_R(M_2, M_1)$. If $f(U_2) = 0$, then $x = 0$ and so there is nothing to prove. If $f(U_2) \neq 0$, then $f(U_2) = U_1$ and so f is a monomorphism from M_2 to M_1. So by the given assumption, f is an isomorphism. Let g be the inverse map of f. Then $y = gE_{21} + N(S)$ is in $e_2(S/N(S))e_1$ and $(f - fgf)(U_2) = 0$; hence $(f - fgf)E_{12}$ is in $N(S)$. Now we have $x = xyx$ and the ring $S/N(S)$ is a von Neumann regular ring. So $N(S) = J(S)$.

By Theorem 3, we can easily find an Artinian module such that $N(S) \neq J(S)$, for example, if $M = \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ then we have $N(S) \neq J(S)$. But for the case of CS-modules, we have the following fact:

COROLLARY 4. *$N(S) = J(S)$ for every Artinian CS-module M.*

Proof. Let $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ be a decomposition into uniform Artinian modules M_i with $i = 1, 2, \ldots, n$. To show that $N(S) = J(S)$, let f be an R-monomorphism from M_i to M_j. Then we have $f(U_i) = U_j \neq 0$ where $U_i = \mathrm{Soc}(M_i)$ and $U_j = \mathrm{Soc}(M_j)$. Now let g be the inverse map of f from $f(M_i)$ to M_i. Then $g(\mathrm{Soc}(f(M_i))) = g(U_j) = U_i$ and so g induces the map from $\mathrm{Soc}(M_j)$ to $\mathrm{Soc}(M_i)$. But since M is an Artinian CS-module, the induced map can be extended to g_0 in $\mathrm{Hom}_R(M_j, M_i)$ by [8, Corollary 2, p.207]. So we have that $g_0 f(U_i) = U_i$ and $f g_0(U_j) = U_j$. Thus $g_0 f$ and $f g_0$ are isomorphisms. So by Theorem 3, $N(S) = J(S)$.

For CS-modules with acc or dcc on essential submodules, the following was proved Camillo and Yousif in [2].

PROPOSITION 5. *Let M be a CS-module.*

(1) If M has acc on essential submodules then $M = N \oplus B$, where N is Noetherian and B is completely reducible.

(2) If M has dcc on essential submodules then $M = A \oplus B$, where A is Artinian and B is completely reducible.

From Proposition 1, some properties of endomorphism rings of CS-modules with dcc on essential submodules can be observed. It would be interesting to compare the following result with the fact that for a right self-injective ring R the ring $R/J(R)$ is right self-injective von Neumann regular and every idempotent of the ring $R/J(R)$ can be lifted to an idempotent of R, which was shown by Utumi [11].

THEOREM 6. *For a CS-module M with dcc on essential submodules, the following statements hold, where S is the endomorphism ring of M.*

(1) $S/J(S)$ is a right self-injective von Neumann regular ring.

(2) Every idempotent of $S/J(S)$ can be lifted to an idempotent of the ring S.

Proof. By Proposition 5, $M = A \oplus B$, where A is Artinian and B is completely reducible. In this case, since B is completely reducible, we may assume that B contains all simple submodules of M. Now we claim that $\mathrm{Hom}_R(B, A) = 0$. For, assuming there is $0 \neq f \in \mathrm{Hom}_R(B, A)$, then there is a simple R-submodule B_1 of B such that $f(B_1) \neq 0$. Since M is a CS-module, so is A and thus $f(B_1)$ is essential in a direct summand A_1

of A. Now let $A = A_1 \oplus C$ and decompose C into a finite direct sum $C = A_2 \oplus A_3 \oplus \cdots \oplus A_n$ of uniform Artinian modules A_i with $i = 2, 3, \ldots, n$. So we have $A = A_1 \oplus A_2 \oplus \cdots \oplus A_n$ and $f(B_1)$ is essential in A_1. Furthermore, since $f(B_1)$ is simple, A_1 is uniform. Let g be the inverse map of f from $f(B_1)$ to B_1. Then of course g is in $\mathrm{Hom}_R(\mathrm{Soc}(A_1), B_1)$ and hence g has an extension g_0 in $\mathrm{Hom}_R(A_1, B_1)$ by [8, Corollary 2, p.207] in considering the Artinian CS-module $B_1 \oplus A_1 \oplus A_2 \oplus \cdots \oplus A_n$. Now $g_0(\mathrm{Soc}(A_1)) \neq 0$ and so $\mathrm{Ker}(g_0) = 0$. Thus g_0 is a monomorphism and consequently g_0 is an isomorphism. Hence A_1 is a simple R-submodule of M. But in the decomposition $M = A_1 \oplus A_2 \oplus \cdots \oplus A_n \oplus B$, since B contains all simple submodules of M as we noted before, A_i's are not simple. Thus we get a contradiction. Therefore $\mathrm{Hom}_R(B, A) = 0$.

By this fact, we have

$$S = \begin{pmatrix} \mathrm{End}_R(A) & 0 \\ \mathrm{Hom}_R(A, B) & \mathrm{End}_R(B) \end{pmatrix}.$$

By Proposition 1, $\mathrm{End}_R(A)$ is semiperfect. Furthermore, since B is completely reducible, $\mathrm{End}_R(B)$ is a right self-injective von Neumann regular ring. So the ring $S/J(S)$ is a ring direct sum of a semisimple Artinian ring and a right self-injective von Neumann regular ring. For more about the ring $S/J(S)$, note that

$$\begin{pmatrix} 0 & 0 \\ \mathrm{Hom}_R(A, B) & 0 \end{pmatrix}$$

is a nilpotent ideal of S. Thus every idempotent of the ring $S/J(S)$ can be lifted to an idempotent of S by this fact together with Proposition 1.

COROLLARY 7. *The endomorphism ring of a CS-module with dcc on essential submodules has the exchange ring property.*

Recall that an R-module M is called *locally projective* if for all diagrams

$$\begin{array}{ccccc} 0 & \longrightarrow & F & \longrightarrow & M \\ & & & & \big\downarrow g \\ & & A & \xrightarrow{\ f\ } & B & \longrightarrow & 0 \end{array}$$

with exact upper and lower rows, and F a finitely generated submodule of M, there is g' in $\mathrm{Hom}_R(M, A)$ such that $g \,|_F = f g' \,|_F$.

For more detail about locally projective modules, see [12].

PROPOSITION 8. *For a locally projective or a quasi-projective CS-module M with dcc on essential submodules, the endomorphism ring S of M is a ring direct sum of a semiperfect ring and a right self-injective von Neumann regular ring.*

Proof. Proposition 5, $M = A \oplus B$, where A is Artinian and B is completely reducible. In this situation, we have $\mathrm{Hom}_R(B, A) = 0$ as in the proof of Theorem 6. Now we claim that $\mathrm{Hom}_R(A, B) = 0$. For doing this, we decompose A into a direct sum of uniform modules;

$$A = A_1 \oplus A_2 \oplus \cdots \oplus A_n$$

with A_i uniform module, $i = 1, 2, \ldots, n$. So to claim $\mathrm{Hom}_R(A, B) = 0$, we may assume that A is uniform and B is simple.

First, assume that M is locally projective. If $0 \neq f$ in $\mathrm{Hom}_R(A, B)$, then f is onto. Let π_B be the projection from M onto B. Then there exists g in $\mathrm{Hom}_R(M, A)$ such that $fg|_B = \pi_B|_B = 1_B$ and thus $(fg)(B) = B$. Therefore $g(B) \neq 0$ and hence $g(B) = \mathrm{Soc}(A)$. Wherefore $f(\mathrm{Soc}(A)) = B$ and $f^{-1} : B \to \mathrm{Soc}(A) \subseteq A$. But this is impossible because $\mathrm{Hom}_R(B, A) = 0$. So we have that $\mathrm{Hom}_R(A, B) = 0$.

On the other hand, when M is quasi-projective, there is g in $\mathrm{End}_R(M)$ such that $f\pi_A g = \pi_B$, where π_A is the projection of M onto A. So we have $(f\pi_A g)(B) = B$ and hence $f(\pi_A(g(B))) = B$ with $\pi_A(g(B)) = \mathrm{Soc}(A)$. Therefore $f(\mathrm{Soc}(A)) = B$. But this is also impossible and so $\mathrm{Hom}_R(A, B) = 0$.

Consequently, in either case, we have $\mathrm{Hom}_R(A, B) = 0$ and so we have $S = \mathrm{End}_R(A) \oplus \mathrm{End}_R(B)$, where $\mathrm{End}_R(A)$ is a semiperfect ring by Proposition 1 and $\mathrm{End}_R(B)$ is a right self-injective von Neumann regular ring.

REMARK 9. (1) We can compare Proposition 8 with a result of Camillo and Yousif [2, Proposition 8].

(2) Furthermore in Proposition 8, when M is quasi-projective, $\mathrm{End}_R(A)$ is semiprimary by Fisher [6] or Harada [8]. Therefore the ring S is a ring direct sum of a semiprimary ring and a right self-injective von Neumann regular ring.

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