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OPTIMAL BAYESIAN DESIGN FOR DISCRIMINATION OF ACCELERATION MODELS IN THE EXPONENTIAL DISTRIBUTION

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1. Introduction

The aim of the study is a powerful test for the discrimination and therefore an optimal design for that purpose. This problem is studied by Chernoff([5]) and used in Chernoff([6]) for accelerated life tests using the exponential distribution for life times. The approach used here is similar to that suggested by Läuter([10]) and used in Chaloner([3]) and Chaloner and Larntz([4]) where it is motivated using Bayesian arguments. The approach taken in this paper the loss function $L(\cdot)$ evaluating a test procedure and a design d.

Throughout this paper, we assume $O - R_0$, R_1, loss, representing the loss coming from a wrong decision. Two probabilities P_0, $P_1 = 1 - P_0$ are assigned to the two models M_0 and M_1. Each design d corresponds to a distribution in M_0 and M_1 respectively. The optimal test procedure is the Bayes–test determining the model M_0 if

$$P_0 f_0(x|d) > P_1 f_1(x|d), \tag{1-1}$$

where $f_k(\cdot|d)$ are the probability density function (pdf) with respect to a measure μ of the model $k = 0$ or 1 for fixed design d.

This choice criteria yields the Bayes–risk $r(d)$ that is the weighted probability of an incorrect decision. That is,

$$r(d) = K_1 \cdot P_0 \cdot P(X \in A^c|M_0, d) + K_0 \cdot P_1 \cdot P(X \in A|M_1, d),$$

where the set A consists of all points fulfilling (1-1). Now we denote a design d^* to be optimal, if it minimizes the Bayes risk.

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Since we suppose that pdf exist, the Bayes-risk for 0-1 loss reads

$$r(d) = P_1 - \int_R \max[0, P_1 f_1(x|d) - P_0 f_0(x|d)]d\mu(x).$$

To avoid trivial cases let the prior probabilities P_0, P_1 be different from 0 or 1. In general the optimal design depends upon the prior belief P_0, P_1.

2. Selection parameters

The pdf $f_0(\cdot)$, $f_1(\cdot)$ belong now to a parametrized class and at least some of the parameters are functions of a controlled variable. The Bayes–test should clarify which function describes the influence of the controlled variable correctly.

First we deal with unknown parameters which are independent of the adjustable variable d and determine the pdf. Let θ_0 and θ_1 be the nuisance parameters of the two models. Therefore we denote the pdf $f_0(\cdot|\theta_0, d)$ and $f_1(\cdot|\theta_1, d)$. The dimension of the parameters θ_0 and θ_1 need not be the same, thus the specification of two quite different prior pdf is requested. We introduce an additional parameter $k \in \{0, 1\}$ indicating the 'true' model. The combination of k and θ_k is treated as the model parameter $r = (k, \theta_k)$ with prior density

$$P(r) = P(\theta_k|k)P_k.$$

For fixed (and hence omitted) d the observations X lead to the posterior distribution of r which can be written as

$$P(r|X) = P(\theta_2|X, k)P(k|X).$$

$P(\theta_k|X, k)$ stands for the posterior density of θ_k and the posterior probability of the model k is

$$P(k|X) = \frac{P_k m_k(X)}{P_0 m_0(X) + P_1 m_1(X)}$$

$m_k(X)$ is the marginal density of X in the state k, i.e

$$m_k(\chi) = \int_R f(\chi|\theta_k)P(\theta_k|k)d\mu(\theta_k).$$

Despite the fact that the decision $(\widehat{k}, \widehat{\theta})$ consists of both choosing the state k and estimating the parameter θ, again an optimal selection of the model lies in our interest only. Therefore the loss stays the 0, 1–loss

$$L(k, \theta, \widehat{k}, \widehat{\theta}, d) = \begin{cases} 1, & \text{if } k = k \\ 0, & \text{otherwise.} \end{cases}$$

The Bayes–test according to that 0, 1–loss accepts the hypothesis $k = 0$ if

$$P_0 m_0(X) > P_1 m_1(X)$$

The likelihood ratio is replaced by the ratio of the marginal pdf. The principle of the best test is the same as in the case where all parameters are known.

Let us assume that our class of distributions constitutes an 1–dimensional location family of continuous distributions. The two models differ only through different acceleration models $\theta_0(d)$ and $\theta_1(d)$ for the location parameter θ. The test should distinguish the suitable acceleration models. Therefore we write

$$f_k(\chi|d) = f(\chi - \theta_k(d)), \quad k = 0, 1.$$

In general the optimal design which minimizes the Bayes-risk of the discrimination test depends upon the prior and upon the pdf $f(\cdot)$, but under special assumptions about $f(\cdot)$ we derive conditions which assure the Bayes–risk to be independent of the concrete distribution and the prior. Hence, we have the following theorem.

THEOREM 1. *If the pdf* $f : R \to R^+$ *is increasing on* R_0^- *and decreasing on* R_0^+, *then the design* d^* *is optimal if and only if the distance*

$$|\theta_1(d) - \theta_0(d)|$$

is maximum at d^*.

Proof. The Bayes–risk

$$r(d) = P_1 - P_0 \int_R \max[0, \alpha \cdot f(\chi - \theta_1(d)) - f(\chi - \theta_0(d))] d\chi, \quad (2\text{-}1)$$

where $\alpha = P_1/P_0$, is without any loss of generality less than 1. From (2-1) it can be seen that $r(\cdot)$ is a function of $\theta_1 - \theta_0$ only,

$$r(d) = P_1 - P_0 R(\theta_1 - \theta_0),$$

with

$$R(\theta) = \int_R \max[0, \alpha \cdot f(\chi - \theta) - f(\chi)] d\chi. \tag{2-2}$$

We assume that $\theta_1(d) > \theta_0(d)$ for fixed d and $\theta > 0$ in (2-1) and (2-2), respectively, otherwise we replace $f(\chi)$ by $f(-\chi)$. If suffices to show that $R(\theta)$ is not decreasing in θ.

If $x < 0$ then, $\max[0, \alpha f(x-\theta) - f(x)] = 0$ because of the monotony of $f(\cdot)$ on R_o^-, so

$$R(\theta) = \int_0^\infty \max[0, \alpha f(x - \theta) - f(x)]\, dx.$$

For a $t > 0$ we get

$$R(\theta + t) = \int_{-t}^\infty \max[0, \alpha f(x - \theta) - f(x - t)]\, dx.$$

The monotony of $f(\cdot)$ for nonnegative reals gives

$$R(\theta + t) \leq R(\theta)$$

and d cannot be optimal if the distance of the acceleration models is not as large as possible. This completes the proof of theorem.

In many applications, a sufficient statistic can be obtained which belongs to a scale parameter family. Again we restrict to a 1-dimensional pdf $f(\cdot)$. Since the parameter is now scale parameter we write

$$f_k(x \mid d) = f(\theta_k(d)) f(\theta_k(d)x), \ (k = 0, 1) \tag{2-3}$$

with positive functions $\theta_0(\cdot)$, $\theta_1(\cdot)$. In order to derive a criterion for a design to be optimal, we define the two 'parts' $f_1(x) = f(x)$ and $f_2(x) = f(x)$ both on R_0^+. For each θ and α we define the sets

$$F_{\theta,\alpha}^{(i)} = \{x \geq 0 \mid \alpha\theta f_i(\theta x) > f_i(x)\}, \quad i = 1, 2,$$

which are assumed to be intervals for our result. Therefore we especially consider Gamm adencities $\gamma(n, \theta)$. In this case $f_2 \equiv 0$ and the upper limit of the interval is $s_1(\theta) = (\ln \alpha + n \ln \theta)/(\theta - 1)$ with

$$c = \begin{cases} 1, & \text{if } \alpha \geq 1 \\ \alpha^{-1/n}, & \text{if } \alpha < 1 \end{cases}$$

using the general notation of Theorem 2. we have the following theorem

THEOREM 2. *If the pdf $f(\cdot)$ in (2-3) fulfills following conditions* (1)-(3):

(1) *$f(\cdot)$ is piecewise continuous;*

(2) *For any $c > 0$, there exist c, with $1 \leq c \leq \infty$ such that*

$$1 \leq \theta < c \Rightarrow F^{(i)}_{\theta,\alpha} = \theta \qquad F^{(i)}_{\theta,\alpha} = R^+$$

and if c is finite then

$$c < \theta < \infty \Rightarrow F^{(i)}_{\theta,\alpha} = [0, s_i(\theta));$$

(3) *$s_i(\theta)$ is differentiable on (c, ∞);*
and if (4) or (5) holds;

(4) *$s_i(\theta)$ is the unique solution of $\alpha\theta f_i(\theta x) = f_i(x) > 0$;*

(5) *$s_i(\theta)$ is decreasing;*

then a design that maximizes $|\ln(\theta_1(d)) - \ln(\theta_0(d))|$ is optimal.

Proof. The Bayes-risk is

$$r(d) = P_1 - P_0 \int_R \max\left[0, \alpha\frac{\theta_1(d)}{\theta_0(d)} \cdot f\left[\frac{\theta_1(d)}{\theta_0(d)}x\right] - f(x)\right] dx,$$

and therefore a function of the quotient $\theta_1(d)/\theta_0(d)$ only. We assume $\theta_1 > \theta_0$ for fixed d and define

$$R(\theta) = \int_R \max[0, \alpha\theta f(\theta) - f(x)]\, dx,$$

for $\theta \geq 1$. Since $R(\theta) = R_1(\theta) = R_2(\theta)$ where

$$R_i(\theta) = \int_0^\infty \max[0, \alpha\theta f_i(\theta) - f_i(x)]\, dx,$$

the assertion is proved if $R_i(\cdot)$, $i = 1, 2$ are both increasing in θ. If $c \geq 1$ then only $F^i_{\theta,\alpha} = R^+$ and if $c < 1$ then only $F^i_{\theta,\alpha} = 0$ is possible for $1 \leq \theta \leq c$ and in both cases the Bayes–risk takes its maximum value. Without loss of generality, we assume that $f_i(\cdot)$ are continuous on the right. Then assumptions about $f(\cdot)$ and $s_i(\theta)$ imply that R_i has for $c < \theta$ a derivate on the right

$$R_i'^+(\theta) = \alpha f_i(\theta s_i(\theta))s_i(\theta) + [\alpha\theta f_i(\theta s_i(\theta))\theta - f_i(s_i(\theta))]s_i'(\theta).$$

Since $\alpha f_i(\theta s_i(\theta)) \leq f_i(s_i(\theta))$ we have by virtue of the assumptions on $s_i(\theta)$

$$R_i'^+(\theta) \geq 0, \ i = 1, 2$$

This completes the proof of theorem.

3. Conclusion

As a principal application of the previous section, we consider the discrimination of acceleration models concerning life data coming form an exponential distribution with failure rate θ. Instead of design we call the (positive) controllable variable stress S under which the observations X_i, $(i = 1, \ldots, n)$ of life time are taken. Since for fixed stress $\sum_{i=1}^n X_i$ is a sufficient statistic which belongs to a scale parameter family with all the properties requested for application of Theorem 2.

It is independent of the prior pdf for the two models and the number of observations. The optimal stress maximizes the symmetric quotient $Q(s)$ of the two completely specified acceleration models $Q_0(s)$ and $Q_1(s)$ in the exponential models.

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