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ON THE RESTRICTIONS OF BMO

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Since John and Nirenberg introduced the BMO in early 1960 [JN], it has been one of the most significant function spaces. The significance of BMO lies in the fact that BMO is a limiting space of L^p $(p \to \infty)$, or a proper substitute of L^∞. A dual statement of this would be that the Hardy space H^1 is a proper substitute of L^1.

Even if BMO is a limiting class of L^p, it does not share some properties which functions in L^p satisfy trivially. It is because being a BMO function is not merely a condition on the size but a condition on its mean oscillation as its name stands for. For example, $L^\infty \cdot \mathrm{BMO} \not\subset \mathrm{BMO}$. In this paper, we consider one of such properties-the restriction.

For clarity, let us consider functions in $\mathrm{BMO}(\mathbb{R}^2)$. Let $f \in \mathrm{BMO}(\mathbb{R}^2)$. *Does it follow that $f(\cdot, y) \in BMO(\mathbb{R}^1)$ for almost all y?* It seems proper to conjecture that the answer to this question is negative by considering its dual. In fact, the restriction of a 2-dimensional atom may not be an atom in $\mathbb{R}^1$. In this paper, we consider a special case of the above question. We show that, for functions which can be separated, the answer to the above question is positive. Putting it in more general fashion, we have the following theorem.

THEOREM 1. *Let $h(x,y)$ be a nonzero function in $BMO(\mathbb{R}^{m+n})$. If h can be separated as $h(x,y) = f(x)g(y)$, then $f \in BMO(\mathbb{R}^m)$ and $g \in BMO(\mathbb{R}^n)$.*

It turns out that for $h(x,y) = f(x)g(y)$ being in BMO puts a severe restriction not only on the mean oscillation but also on the means of f and g (see the identity (1) below). This fact bears some interesting information on the condition for $h(x,y) = f(x)g(y)$ to be in BMO. For instance, we prove the following theorem.

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THEOREM 2. *Let g be a locally integrable function on $\mathbb{R}^1$. If* $\log|x|$ $g(y) \in BMO(\mathbb{R}^2)$, *then g must be constant.*

Proof of Theorem 1. Let us first fix some notations. for a cube I in $\mathbb{R}^m$, we denote by $l(I)$ and $|I|$ the side length and the volume of the cube I, respectively. We also define f_I for $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ by

$$f_I = \frac{1}{|I|}\int_I f(x)dx.$$

Note that

$$\frac{1}{|I|}\int_I |f(x) - f_I|^2 dx = \frac{1}{|I|}\int_I |f(x)|^2 dx - |f_I|^2.$$

Let I and J be cubes in $\mathbb{R}^m$ and $\mathbb{R}^n$ with $l(I) = l(J)$. Then, $I \times J$ is a cube in $\mathbb{R}^{m+n}$ and it is easy to see that

$$h_{I\times J} = f_I g_J.$$

Then, a straight forward computation shows that

$$\begin{aligned} &\frac{1}{|I||J|}\int_{I\times J} |h(x,y) - h_{I\times J}|^2 dxdy \qquad (1)\\ &= \left(\frac{1}{|I|}\int_I |f(x) - f_I|^2 dx\right)\left(\frac{1}{|J|}\int_J |g(x) - g_J|^2 dy\right)\\ &\quad + |f_I|^2\left(\frac{1}{|J|}\int_J |g(y) - g_J|^2 dy\right) + \left(\frac{1}{|I|}\int_I |f(x) - f_I|^2 dx\right)|g_J|^2. \end{aligned}$$

Since $h(x,y) \in \text{BMO}(\mathbb{R}^{m+n})$, the left hand side of (1) is bounded by a constant, say C. And hence

$$\left(\frac{1}{|I|}\int_I |f(x) - f_I|^2 dx\right)\left(\frac{1}{|J|}\int_J |g(x) - g_J|^2 dy + |g_J|^2\right) \le C \qquad (2)$$

$$|f_I|^2\left(\frac{1}{|J|}\int_J |g(y) - g_J|^2 dy\right) \le C. \qquad (3)$$

Suppose that $f \notin BMO(\mathbb{R}^m)$. Then there exists a sequence of cubes $\{I_k\}$ in $\mathbb{R}^n$ such that

$$\frac{1}{|I_k|}\int_{I_k} |f(x) - f_{I_k}|dx > k. \tag{4}$$

Then, by Jensen's inequality, we have

$$\frac{1}{|I_k|}\int_{I_k} |f(x) - f_{I_k}|^2 dx > k^2. \tag{5}$$

We may assume that $\lim_{k\to\infty} |I_k|$ exists (possibly infinite) by taking a subsequence if necessary. Put $A = \lim_{k\to\infty} |I_k|$. We deal with various possibilities of $A = 0$, $0 < A < \infty$, and $A = \infty$.

Suppose first that $A = 0$. Then, it follows from (2) and (5) that

$$|g_J|^2 \le \frac{C}{k^2} \qquad \text{for all } J \text{ with } |J| = |I_k|$$

for each k. Then, by the Lebesgue differentiation theorem, we get $g = 0$ which contradicts to our hypothesis.

Suppose now that $0 < A < \infty$. Then, it follows again from (2) and (5) that

$$\frac{1}{|J|}\int_J |g(x) - g_J|^2 dy \le \frac{C}{k^2} \qquad \text{for all } J \text{ with } |J| = |I_k|$$

for each k. This implies that

$$\frac{1}{|J|}\int_J |g(x) - g_J|^2 dy = 0 \qquad \text{for all } J \text{ with } |J| = A,$$

which in turn implies that $g =$ constant. It then follows from (1) that

$$\frac{1}{|I||J|}\int_{I\times J} |h(x,y) - h_{I\times J}|^2 dxdy = g^2 \frac{1}{|I|}\int_I |f(x) - f_I|^2 dx$$

and hence $f \in \mathrm{BMO}(\mathbb{R}^m)$. This leads us to a contradiction.

Finally, we suppose that $A = \infty$. We claim that, for each k, one can choose a cube $\tilde{I}_k$ with $\tilde{I}_k \subset I_k$ so that

$$2^{-n} < |\tilde{I}_k| \leq 1,$$
$$\frac{1}{|\tilde{I}_k|}\int_{\tilde{I}_k} |f(x)|dx \geq \frac{1}{|I_k|}\int_{I_k} |f(x)|dx. \tag{6}$$

To see this, we subdivide I_k into 2^n nonoverlapping subcubes with equal sidelengths, and call them $\{I_k^j\}$. Then, we have

$$\frac{1}{|I_k|}\int_{I_k} |f(x)|dx = 2^{-n}\sum_{j=1}^{2^n} \frac{1}{|I_k^j|}\int_{I_k^j} |f(x)|dx.$$

Hence, for at least one j, $|f|_{I_k} \leq |f|_{I_k^j}$. Choose such a j and apply the same process to I_k^j. We continue this process until we get the desired cube. Note that $|f|_{I_k} > k/2$ and hence $|f|_{\tilde{I}_k} > k/2$ by (4).

We again assume that $|\tilde{I}_k|$ converges by taking a subsequence if necessary. Suppose that

$$\limsup_{k\to\infty} \frac{|f_{\tilde{I}_k}|}{k} \leq \frac{1}{4}.$$

We then assume that $|f_{\tilde{I}_k}| \leq \frac{k}{3}$ for all k by again taking a subsequence if necessary. Then, we have

$$\frac{1}{|\tilde{I}_k|}\int_{\tilde{I}_k} |f(x) - f_{\tilde{I}_k}|dx \geq \frac{1}{|\tilde{I}_k|}\int_{\tilde{I}_k} |f(x)|dx - |f_{\tilde{I}_k}| \geq \frac{k}{6}$$

and

$$2^{-n} \leq \lim_{k\to\infty} |\tilde{I}_k| \leq 1.$$

Therefore, the case is reduced to the second of previous cases and hence leads us to a contradiction.

If

$$\limsup_{k\to\infty} \frac{|f_{\tilde{I}_k}|}{k} \geq \frac{1}{4},$$

then we assume that $|f_{\tilde{I}_k}| \geq \frac{k}{5}$ for all k by passing to the subsequence if necessary. It then follows from (3) that

$$\frac{1}{|J|}\int_J |g(y) - g_J|^2 dy \leq C\frac{25}{k^2} \quad \text{for all } J \text{ with } |J| = |\tilde{I}_k|$$

for each k. Hence, we again obtain g = constant. The proof is complete.

Proof of Theorem 2. Suppose that $\log|x|g(y) \in \mathrm{BMO}(\mathbb{R}^2)$. Since the average of $\log|x|$ can be arbitrary large, we have from (3)

$$\frac{1}{|J|}\int_J |g(y) - g_J|^2 dy = 0$$

for any interval J. Therefore, g must be constant.

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