J. Korean Math. Soc. **31** (1994), No. 4, pp. 691–702

THE ANNIHILATORS AND THE HAHN–BANACH EXTENSION PROPERTY

SUNG HO PARK

1. Introduction

Let X be a normed linear space, M a subspace of X, and V a subspace of the dual space X^*. In [3], we studied the Hahn-Banach extension property in V. Here we give the definition and a characterization of the Hahn-Banach extension property in V.

DEFINITION 1.1.[3]. Let M be a subspace of a normed linear space X, and let V be a subspace of X^*. We say that M has the Hahn-Banach extension property in V if for each $f \in V$ there exists $f_\circ \in V$ such that

(1) $f_\circ(x) = f(x)$ for each $x \in M$, and
(2) $\|f_\circ\| = \|f|_M\|$.

THEOREM 1.2.[3]. *Let M be a subspace of a normed linear space X, and let V be a subspace of the dual space X^*. Then the following are equivalent:*

(a) *M has the Hahn-Banach extension property in V.*
(b) *i) $M_V^\perp$ is proximinal in V.*
ii) for each $f \in V$, $d(f, M_V^\perp) = \|f|_M\|$.

In this paper, we want to study some topological properties of the Hahn-Banach extension property in V, describe the spaces of continuous linear functonals on a subspace which has the Hahn-Banach extension property in V and the quotient spaces by using the concept of an annihilator, and find a relationship between the Hahn-Banach extension property in V and the V-Hahn-Banach extension property.

In section 2, we give some basic properties of $\sigma(X, V)$.

Received August 9, 1993.
This study was supported by the Sogang University Faculty Research Grant and BSRIP, MOE, 1993.

In section 3, we give the definitions of the annihilator $M_V^{\perp}$ and the property (Sep) in V and study some topological properties.

In section 4, we describe the space of continuous linear functionals on the subspace and quotient spaces of normed linear spaces.

In section 5, we study the relation between the Hahn-Banach extension property in V and the V-Hahn-Banach extension property.

2. Preliminary

Our goal in this section is to recall how one can introduce a topology into a linear space X by means of families of linear functionals on X and to examine some of the basic properties of such topologies.

To be more precise, suppose that X is a linear space over Φ and let $V \subset X'$ be a family of linear functionals on X, where X' is the space of all linear functionals on X. We wish to define a topology $\sigma(X, V)$ on X such that $(X, \sigma(X, V))$ is a locally convex topological linear space for which a net $\{x_\alpha\} \subset X$ converges to $x \in X$ in the topology $\sigma(X, V)$ if and only if $lim_\alpha x'(x_\alpha) = x'(x), x' \in V$ and for which the functionals in V are continuous linear functionals on $(X, \sigma(X, V))$. Defining a topology $\sigma(X, V)$ on X that satisfies the last two requirement is not difficult.

Indeed, recalling the discussion of topologies in seminormed linear spaces, we define a family $P = \{p_{x'} | x' \in V\}$ of seminorms on X by setting $p_{x'}(x) = |x'(x)|, x \in X$ and $x' \in V$, if V separates points. It is easy to see that each $p_{x'}$ is a seminorm. Then the topology $\sigma(X, V)$ generated by P, that is, the topology whose a neighborhood base at a point $x \in X$ consists of sets of the form

$$U(x; \varepsilon; p_{x'_1}, p_{x'_2}, \cdots, p_{x'_n}) = \{y : y \in X, p_{x'_k}(x - y) < \varepsilon, k = 1, 2, \cdots, n\},$$

where $\varepsilon > 0, n \in \mathbb{Z}, n > 0$, and the choice of $x'_1, x'_2, \cdots, x'_n$ in V is arbitrary, is such that $\{x_\alpha\}$ converges to x in $\sigma(X, V)$ if and only if $lim_\alpha x'(x_\alpha) = x'(x), x' \in V$. Moreover, it is apparent that the elements of V are continuous with respect to $\sigma(X, V)$.

However, the pair $(X, \sigma(X, V))$ may not be a seminormed linear space-equivalently, not a locally convex topological linear space,since we have made no requirement that $|x'(x)| = p_{x'}(x) = 0, x' \in V$ should imply $x = 0$.

Now we will summarize some basic well-known properties of $\sigma(X, V)$ -topology on X.

THEOREM 2.1. *Let X be a linear space and suppose that $V \subset X'$ separates points. If $P = \{p_{x'} | x' \in V\}$ where*

$$p_{x'}(x) = |x'(x)|,$$

for any $x \in X$ and $x' \in V$, then P is a family of seminorms on X such that (X, P) is a seminormed linear space. Moreover, if $\sigma(X, V)$ is the topology on X determined by the family P, then

(i) *$(X, \sigma(X, V))$ is a locally convex topological linear space.*
(ii) *A net $\{x_\alpha\} \in X$ converges to $x \in X$ in $\sigma(X, V)$ if and only if $lim_\alpha x'(x_\alpha) = x'(x)$, for any $x' \in V$.*
(iii) *If $x' \in V$, then x' is a continuous linear functional on $(X, \sigma(X, V))$.*
(iv) *$\sigma(X, V)$ is the weakest topology on X for which the elements of V are continuous.*

THEOREM 2.2. *Let X be a linear space and suppose $V \subset X'$ separates points. If V is countable, then the topology $\sigma(X, V)$ is metrizable.*

PROPOSITION 2.3. *Let X be a linear space and suppose that $V_k \subset X'$ separates points, k = 1, 2. If $V_1 \subset V_2$, then $\sigma(X. V_1) \subset \sigma(X, V_2)$.*

PROPOSITION 2.4. *Let (X, τ) be a locally convex topological linear space. If $V \subset X^*$ separates points, then $\sigma(X, V) \subset \tau$.*

We noted in Theorem 2.1 that each $x' \in V$ is a continuous linear functional on $(X, \sigma(X, V))$. Actually V is precisely the set of continuous linear functionals on $(X, \sigma(X, V))$, provided V is a linear space.

THEOREM 2.5. *Let X be a linear space and suppose that $V \subset X'$ is a subspace that separates points. Then the following are equivalent:*

(i) *$x^* \in V$.*
(ii) *x^* is a continuous linear functional on $(X, \sigma(X, V))$.*

This result, combined with the Hahn-Banach theorem, immediately yields the following corollary:

COROLLARY 2.6. *Let X be a linear space and suppose that $V \subset X'$ is a subspace that separates points. If $M \subset X$ is a linear subspace, then the following are equivalent:*

(i) *M is a proper closed subspace of $(X, \sigma(X, V))$.*
(ii) *If $x_\circ \notin M$, then there exists some $x' \in V$ such that $x'(x_\circ) = 1$ and $x'(x) = 0$, for any $x \in M$.*

3. Annihilators

In this section, we want to study the slightly generalized annihilators and their basic properties.

DEFINITION 3.1. Let X be a normed linear space and let V be a subspace of X^*. If $M \subset X$, then

$$M_V^{\perp} = \{x^* \in V : x^*(x) = 0 \, for \, any \, x \in M\}$$

is called the annihilator of M in V. If $W \subset X^*$, then

$$W_{\perp} = \{x \in X : x^*(x) = 0 \, for \, any \, x^* \in V\}$$

is called the annihilator of W.

Thus $M_V^{\perp} \subset V$ is the set of all continuous linear functionals on X in V that vanish identically on M, and $W_{\perp} \subset X$ is the set of common zeros of the continuous linear functionals on X that belong to W.

Now we can have some basic properties of annihilators.

PROPOSITION 3.2. *Let X be a normed linear space, $M \subset X$, $W \subset X^*$, and let V be a closed subspace of X^*. Then*

(i) *$M_V^{\perp}$ is a closed subspace of V.*
(ii) *$W_{\perp}$ is a closed subspace of X.*
(iii) *$M \subset (M_V^{\perp})_{\perp}$.*

PROPOSITION 3.3. *Let M and N be subspaces of a normed linear space X and let V be a subspace of the dual space X^*. Then*

1) $(M + N)_V^{\perp} = M_V^{\perp} \cap N_V^{\perp}$,
2) $(M \cup N)_V^{\perp} \subset M_V^{\perp} + N_V^{\perp} \subset (M \cap N)_V^{\perp}$.

Proof. 1) Suppose $x^* \in (M+N)_V^\perp$. Then $x^* \in V$ and $x^*(m+n) = 0$, for any $m \in M$ and $n \in N$, and so $x^*(m) = x^*(n) = 0$. Hence $x^* \in M_V^\perp \cap N_V^\perp$. Therefore $(M+N)_V^\perp \subset M_V^\perp + N_V^\perp$.

Conversely, if $x^* \in M_V^\perp \cap N_V^\perp$, then $x^* \in V$ $x^*(m) = x^*(n) = 0$ for any $m \in M$ and $n \in N$. By the linearity of $x^*, x^*(m+n) = 0$ for any $m+n \in M+N$ and so $x^* \in (M+N)_V^\perp$. Therefore $M_V^\perp \cap N_V^\perp \subset (M+N)_V^\perp$.

2) Suppose that $x^* \in (M \cup N)_V^\perp$. Then $x^* \in V$ and $x^*(m) = 0$ for any $m \in M \cup N$. and so $\frac{x^*}{2}(m) = 0$ for any $m \in M$, $\frac{x^*}{2}(n) = 0$ for any $n \in N$ and $x^* = \frac{x^*}{2} + \frac{x^*}{2}$. Hence $x^* \in M_V^\perp + N_V^\perp$. Suppose that $x^* \in M_V^\perp + N_V^\perp$. Then $x^* \in V$ and $x^* = y^* + z^*$ for some $y^* \in M_V^\perp$ and $z^* \in N_V^\perp$ so $y^*, z^* \in V$, $y^*(m) = 0$ for any $m \in M$ and $z^*(n) = 0$ for any $n \in N$. Thus $x^*(m) = y^*(m) + z^*(m) = 0$ for any $m \in M \cap N$. Hence $x^* \in (M \cap N)_V^\perp$.

DEFINITION 3.4. Let M be a subspace of a normed linear space X and let V be a subspace of X^*. If for each $x \notin \overline{M}$, there exists $x^*(\neq 0) \in M_V^\perp$ such that $x^*(x) \neq 0$, then we say that M has the property (Sep) in V or V separates the points in $X \backslash \overline{M}$.

Next we give some simple examples which have or do not have the property (Sep) in V.

EXAMPLES 3.5. 1) Every subspace of a normed linear space X has the property (Sep) in X^*.

2) Let M be a closed hyperplane of a normed linear space X. If $x \notin M$, then there exists $x^* \in M^\perp$ such that $x^*(x) \neq 0$. Thus if we put $V = span\{x^*\}$, then M has the property (Sep) in V since for each $x \notin M$, $x^*(x) \neq 0$. In this case, $M_V^\perp = V$.

3) Let M be a closed subspace of a normed linear space X with $X = M \oplus [x, y]$ where $\{x, y\}$ is a linearly independent set with $x, y \notin M$. By Hahn-Banach Theorem, there exists $x^* \in (M \oplus [y])^\perp$ such that $x^*(x) \neq 0$ and $x^*(y) = 0$. Since $x^*(x) \neq 0$ and $x^*(y) = 0$, M does not have the property (Sep) in V if we put $V = span\{x^*\}$.

THEOREM 3.6. *Let M be a subspace of X, V a closed subspace of X^* and W a subspace of V. Then*

(i) *$(M_V^\perp)_\perp$ is the norm-closure of M in X if M has the property (Sep) in V.*

(ii) $(W_\perp)_V^\perp$ *is the weak*-closure of* W *in* V *if* $\overline{W}^{w^*}$ *has the property (Sep) in* $(W_\perp)^\wedge$, *where* $\wedge : X \to X^{**}$ *is the canonical embedding.*

Proof. (i) If $x \in M$, then $x^*(x) = 0$ for any $x^* \in M_V^\perp$ so $x \in (M_V^\perp)_\perp$. Thus $M \subset (M_V^\perp)_\perp$. Since $(M_V^\perp)_\perp$ is norm-closed, it contains the norm-closure $\overline{M}$ of M, that is, $\overline{M} \subset (M_V^\perp)_\perp$. But if $x_\circ \in (M_V^\perp)_\perp \setminus \overline{M}$, then, by the property (Sep), there exists $x^* \in M_V^\perp$ such that $x^*(x_\circ) \neq 0$. However, $x_\circ \in (M_V^\perp)_\perp$ implies that $x^*(x_\circ) = 0$. This contradict the choice of x^*. Therefore $\overline{M} = (M_V^\perp)_\perp$.

(ii) If $x^* \in W \cap V$, then $x^*(x) = 0$ for any $x \in W_\perp$ then $x^*(x) = 0$ for any $x \in W_\perp$ so $x^* \in (W_\perp)_V^\perp$. Thus $W \cap V \subset (W_\perp)_V^\perp$ so $\overline{W}_V^{w^*} \subset (W_\perp)_V^\perp$. Suppose that $x^* \in (W_\perp)_V^\perp \setminus \overline{W}_V^{w}$. Then $x^* \in V$ and $x^* \in (W_\perp^\perp) \setminus \overline{W}^{w^*}$. Since $\overline{W}^{w^*}$ has the property (Sep) in $(W_\perp)^\perp$, there exists $x \in W_\perp$ such that $\widehat{x}(x^*) = x^*(x) \neq 0$. Since $x^* \in (W_\perp)^\perp$, $x^*(x) = 0$. It is a contracdiction. Thus $(W_\perp)_V^\perp \subset \overline{W}_V^{w^*}$.

COROLLARY 3.7. *Let* M *be a subspace of a normed linear space* X *and* W *a subspace of* X^*. *Then*

(i) $(M^\perp)_\perp$ *is the norm-closure of* M *in* X, *and*
(ii) $(W_\perp)^\perp$ *is the weak*-closure of* W *in* X^*.

Proof. Since every subspace of a normed linear space has the property (Sep) in X^*, these follow from Theorem 3.6.

Observe, as a corollary, that every norm-closed subspace of X is the annihilator of its annihilator and the same is true of every *weak**-closed subspace of X^*.

Now, from the basic property of $\sigma(X, V)$-topology. we have the following results.

PROPOSITION 3.8. *Let* X *be a Banach space and let* V *be a subspace of* X^* *that separates points. Then the following are equivalent:*

i) M *is a proper closed subspace of* $(X, \sigma(X, V))$,
ii) M *has the property (Sep) in* V.

Proof. It follows from Corollary 2.6.

PROPOSITION 3.9. *Let M be a subspace of a normed linear space X and let V be a subspace of X^*. Suppose that M has the property (Sep) in V. Let $x_\circ \in X$.*

1) *Suppose that M is closed. If $x^* \in V$ and $x^*(x) = 0$ for any $x \in M$ implies $x^*(x_\circ) = 0$, then $x_\circ \in M$.*
2) *Suppose that V separates points. The following are equivalent:*
   i) $cl(M)^{\sigma(X,V)} = X$.
   ii) If $x^* \in V$ is such that $x^*(x) = 0$ for any $x \in M$, then $x^* = 0$.

Proof. 1) Suppose not, i.e., $x_\circ \notin M$. Since M has the property (Sep) in V, there exists $x^* \in M_V^\perp$ such that $x^*(x_\circ) \neq 0$. It contradicts the hypothesis. Thus $x_\circ \in M$.

2) Suppose not, i.e., there exists $x^* \in V$ with $x^*(x) = 0$ for any $x \in M$ such that $x^* \neq 0$. Since $x^* \neq 0$, there exists $x_\circ \in X \setminus M$ such that $x^*(x_\circ) \neq 0$. Since $cl(M)^{\sigma(X,V)} = X$, there exists a net $\{x_n\}$ in M such that $x_n \to x_\circ$ in $\sigma(X, V)$. Then $x^*(x_n) = 0$ and $x^*(x_n) \to x^*(x_\circ)$. It contradcits the fact that $x^*(x_\circ) \neq 0$. Thus ii) holds.

Conversely, suppose not, i.e., $cl(M)^{\sigma(X,V)} \neq X$. Then there exists $x_\circ \in X$ such that $x_\circ \notin cl(M)^{\sigma(X,V)}$. Since $\overline{M}^{\|\cdot\|} \subset cl(M)^{\sigma(X,V)}$, $x_\circ \notin \overline{M}$. By the property (Sep) in V, there exists $x^* \in V$ such that $x^*(m) = 0$ for any $m \in M$ and $x^*(x_\circ) \neq 0$. By ii), $x^* = 0$ but $x^*(x_\circ) \neq 0$. It is a contradiction. Hence $cl(M)^{\sigma(X,V)} = X$.

4. Duals of subspaces and of quotient spaces

Let M be a subspace of a normed linear space X and let V be a subspace of X^*. Define

$$M_V^* = \{x^*|_M : x^* \in V\}$$

and

$$M_V^\perp = \{x^* \in V : x^*(x) = 0 \text{ for any } x \in M\}.$$

Now we shall use the concept of an annihilator to describe the spaces of continuous linear functionals on the subspaces and quotient spaces of normed linear spaces.

THEOREM 4.1. *Let M be a closed subspace of a Banach space X and let V be a subspace of X^**

(a) Suppose that M has the Hahn-Banach extension property in V. Then the Hahn-Banach extension property in V extends each $m^* \in M_V^*$ to a functional $x^* \in V$. Define

$$\sigma(m^*) = x^* + M_V^\perp.$$

Then σ is an isometric isomorphism of M_V^* onto $V/M_V^\perp$.

(b) Let V be closed and let $\pi : X \to X/M$ be the quotient map. Put $Y = X/M$ and $Y_V^* = \{y^* \in Y^* | y^*\pi \in V\}$. Then Y_V^* is a closed subspace of Y^*. For each $y^* \in Y_V^*$ define

$$\tau y^* = y^*\pi.$$

Then τ is an isometric isomorphism of Y_V^* onto $M_V^\perp$.

Proof. (a) Let $m^* \in M_V^*$. If x^* and z^* in V are extensions of m^*, then $x^* - z^*$ is in $M_V^\perp$. Hence $x^* + M_V^\perp = z^* + M_V^\perp$. Thus σ is well-defined. Let $m_1^*, m_2^* \in M_V^\perp$. Since M has the Hahn-Banach extension property in V, there exist $x_1^*, x_2^* \in V$ such that $x_1^*|_M = m_1^*$, $x_2^*|_M = m_2^*$, $\|m_1^*\| = \|x_1^*\|$ and $\|m_2^*\| = \|x_2^*\|$. Thus, for any $\alpha, \beta \in \mathbb{R}$,

$$\begin{aligned}\sigma(\alpha m_1^* + \beta m_2^*) &= \alpha x_1^* + \beta x_2^* + M_V^\perp \\ &= \alpha(x_1^* + M_V^\perp) + \beta(x_2^* + M_V^\perp) \\ &= \alpha\sigma(m_1^*) + \beta\sigma(m_2^*)\end{aligned}$$

so σ is linear. Since the restriction of every $x^* \in V$ to M is a member of M_V^*, the range of σ is all of $V/M_V^\perp$. Fix $m^* \in M_V^*$. If $x^* \in V$ extends m^*, it is clear that $\|m^*\| \leqq \|x^*\|$ so

$$\begin{aligned}\|x^* + M_V^\perp\| &= \inf_{z^* \in M_V^\perp} \|x^* + z^*\| \\ &= \inf_{\substack{y^* \in V \\ y^*|_M = m^*}} \|y^*\|.\end{aligned}$$

Hence

$$\|m^*\| \leqq \|\sigma(m^*)\| \leqq \|x^*\|.$$

By the Hahn-Banach extension property in V, there exists $x^* \in V$ such that $x^*|_M = m^*$ and $\|x^*\| = \|m^*\|$. Thus $\|\sigma(m^*)\| = \|m^*\|$. This completes (a).

(b) If $x \in X$ and $y^* \in Y_V^*$, then $\pi x \in Y$, and hence $x \longmapsto y^*\pi x$ is a continuous linear functional on X which is in V and which vanishes for each $x \in M$. Thus $\tau y^* \in M_V^\perp$. Clearly τ is linear.

Fix $x^* \in M_V^\perp$. Let K be the null space of x^*. Since $M \subset K$, there is a linear functional Λ on Y such that $\Lambda\pi = x^*$ The null space of Λ is $\pi(K)$, a closed subspace of Y, by the definition of the quotient topology in $Y = X/M$, and so Λ is closed. By the closed graph theorem, $\Lambda \in Y^*$. Hence $\tau\Lambda = \Lambda\pi = x^*$. Therefore the range of τ is all of $M_V^\perp$.

Fix $y^* \in Y_V^*$. If $y \in Y$, $\|y\| = 1$, and $r > 1$, the definition of the quotient norm in X/M shows that there is an $x_\circ \in X$ with $\|x_\circ\| < r$ such that $\pi x_\circ = y$. Then

$$\begin{aligned} |y^*(y)| &= |y^*(\pi(x_\circ))| \\ &= |\tau y^*(x_\circ)| \\ &\leqq \|\tau y^*\|\|x_\circ\| < r\|\tau y^*\|, \end{aligned}$$

so

$$\|y^*\| \leqq \|\tau y^*\|.$$

On the other hand, $\|\pi x\| \leqq \|x\|$ for every $x \in X$. Then

$$\begin{aligned} |(\tau y^*)(x)| &= |y^*(\pi(x))| \\ &\leqq \|y^*\|\|\pi(x)\| \\ &\leqq \|y^*\|\|x\|, \end{aligned}$$

so $\|\tau y^*\| \leqq \|y^*\|$. Therefore $\|\tau y^*\| = \|y^*\|$. This completes the proof.

By the Hahn-Banach Theorem, every subspace of a normed linear space X has the Hahn-Banach extension property in X^*. Moreover, X^* is closed. From the above theorem, we obtain the following corollary.

COROLLARY 4.2.[2,5]. *Let M be a closed subspace of a Banach space X. Then*

(a) The Hahn-Banach theorem extends each $m^* \in M^*$ to a functional $x^* \in X^*$. Define

$$\sigma m^* = x^* + M^\perp.$$

Then σ is an isometric isomorphism of M^* onto $X^*/M^\perp$.

(b) Let $\pi : X \to X/M$ be the quotient map. Put $Y = X/M$. For each $y^* \in Y^*$, define

$$\tau y^* = y^* \pi.$$

Then τ is an isometric isomorphism of Y^* onto $M^\perp$.

Next, we have the following property that, if M has the Hahn-Banach extension property in V, then an extreme point of $B(M_V^*)$ can be extended an extreme point of $B(V)$, where $B(V)$ is the closed unit ball of V.

THEOREM 4.3. *Let M be a subspace of a normed linear space X and let V be a closed subspace of X^*. If M has the Hahn-Banach extension property in V and $f \in ext(B(M_V^*))$, then f has an extension $x^* \in extB(V)$, i.e., $extB(M_V^*) \subset \{x^*|_M : x^* \in extB(V)\}$, where $ext(B(V))$ is the set of all extreme points of $B(V)$.*

Proof. Let E be the set of all norm preserving extensions of f to X in V, i.e., $E = \{x^* \in V : x^*|_M = f, \|x^*\| = 1 = \|f\|\}$. Since M has the Hahn-Banach extension property in V, $E \neq \phi$. Let $x^*, y^* \in B(V)$, $0 < \lambda < 1$ and $z^* = \lambda x^* + (1-\lambda)y^*$. Since $\|z^*\| = 1$, it follows that $\|x^*\| = \|y^*\| = 1$. Further, $f = z^*|_M = \lambda x^*|_M + (1-\lambda)y^*|_M$. Since $f \in extB(M_V^*)$, $x^*|_M = y^*|_M = f$. Thus $x^*, y^* \in E$, and E is an extremal subset of $B(V)$.

Let $\{x_\delta^*\}$ in E and $x_\delta^* \to x^*$ in the *weak** topology. Then for all $y \in M$, $f(y) = x_\delta^*(y) \to x^*(y)$ so $f = x^*|_M$. Note that if $x_\delta^* \to x^*$ in the *weak** topology, then $\|x^*\| \leqq lim\,inf\|x_\delta^*\|$. Thus $\|x^*\| \leqq 1$. Since x^* is an extension of f and V is closed, $\|x^*\| = 1$ and $x^* \in V$. Thus $x^* \in E$ and E is a *weak**-closed in V and hence in $B(V)$. Since $B(X^*)$ is *weak** compact, $B(V)$ is *weak** compact. Hence E has an extreme point x^* so $x^* \in extB(V)$. [Note that if E is an extremal subset of K, then $extE = E \cap extK$.]

Since every subspace has the Hahn-Banach extension property in X^*, we have the following well-known result:

COROLLARY 4.4. *Let M be a subspace of a normed linear space X and let $f \in extB(M^*)$. Then f has an extension $x^* \in extB(X^*)$, i.e.,*

$$extB(M^*) \subset \{x^*|_M : x^* \in extB(X^*)\}.$$

5. Hahn-Banach extension property

In this section we give the definition of V-Hahn-Banach extension property and the relation between the Hahn-Banach extension property in V and the V-Hahn-Banach extension property.

DEFINITION 5.1. Let M be a closed subspace of a normed linear space X and let V be a subspace of X^*. If for each $m^* \in M^*$, there exists $f \in V$ such that

i) $f|_M = m^*$,

ii) $\|f\| = \|m^*\|$,

we say that M has the V-Hahn-Banach extension property.

PROPOSITION 5.2. *If M has the V-Hahn-Banach extension property, then M has the Hahn-Banach extension property in V, but the converse is not true.*

Proof. By the definitions, the V-Hahn-Banach extension property implies the Hahn-Banach extension property in V. Next we will give an example that the converse is not true.

EXAMPLE 5.3. Let $X = \mathbb{R}^3$ be the Euclidean 3-space, $M = span\{(1,0,0)\}$ and $V = [(0,1,0),(0,0,1)]$. Then M has the Hahn-Banach extension property in V, but M does not have the V-Hahn-Banach extension property.

Indeed, let $m^* = (1,0,0)$. Then there does not exist $f \in V$ such that

(i) $f|_M = m^*$, (ii) $\|f\| = \|m^*\|$.

Even though there does not exist an extension of $m^* = (1,0,0)$ in V, M does not have the V-Hahn-Banach extension property but M has the Hahn-Banach extension proeprty in V.

THEOREM 5.4. *The following are equivalent:*

(1) M has the V-Hahn-Banach extension property.

(2) $M^* \subset V|_M$ and M has the Hahn-Banach extension property in V.

Proof. Suppose that M has the V-Hahn-Banach extension property. Then for each $m^* \in M^*$, there exists $f \in V$ such that $f|_M = m^*$ and $\|f\| = \|m^*\|$. Since for each $f \in V$, $f|_M \in M^*$, there exists $f_\circ \in V$ such that $\|f_\circ\| = \|f|_M\|$. Hence (2) holds.

Conversely, suppose (2) holds. Since $M^* \subset V|_M$, for each $m^* \in M^*$ there exists $f \in V$ such that $f|_M = m^*$. Since M has the Hahn-Banach extension property in V, M has the V-Hahn-Banach extension property. Hence (1) holds.

References

1. A. S. Holland and B. N. Sahney, *The general problem of approximation and spline functions*, Robert E. Krieger Publishing Company, Huntington, New York (1979).
2. R. Larsen, *Functional analysis*, Marcel Dekker, INC. New York (1973).
3. S. H. Park, *A little generalization of Hahn-Banach extension property*, J. Korean Math. Soc. **30(1)** (1993), 139-150.
4. R. R Phelps, *Uniqueness of Hahn-Banach extensions and unique best approximation*, Trans. Amer. Math. Soc. **95** (1960), 238-255.
5. W. Rudin, *Functional Analysis*, McGraw-Hill Book Company (1973).
6. I. Singer, *Some extensions of a dual of the Hahn-Banach theorem, with applications to separation and Helly type theorems*, Bull. Austral. Math. Soc. . **15** (1976), 277-291.

Department of Mathematics
Sogang University
CPO 1142.
Seoul, 121 - 742, Korea